

Obstructions to Gauging WZ Terms: a Symplectic Curiosity

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Dedicated to the memory of Arnaldo Ferrer Andreu (1916-1994)

0. As children we are taught to expect that behind any number of continuous symmetries of a dynamical system, there always lurk an equal number of conserved quantities. However at some point in our lives we find out that this is not necessarily the case. The correspondence between symmetries and conservation laws—equivalently, the existence of a moment mapping associated to a symplectic group action—must overcome a homological obstruction. That is, this obstruction takes the form of a class in a suitably defined cohomology theory which must vanish for the correspondence to go through. The purpose of this talk is to point out a curious coincidence. In my joint work with Sonia Stanciu trying to understand the gauging of nonreductive Wess-Zumino-Witten models, I came across the fact that the obstructions to gauging the Wess-Zumino term of a (toy) one-dimensional σ -model are none other than the obstructions for the existence of the moment mapping. Of course, as a physical system this σ -model is not very interesting, but I hope that this symplectic curiosity serves to bring a little *divertimento* to fit the occasion.

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1. Let (M, ω) be a symplectic manifold; that is, the two-form ω is closed and is nondegenerate when thought of as a section of $\text{Hom}(TM, T^*M)$. We say that a vector field ξ on M is **symplectic** if its flow fixes ω :

$$\mathcal{L}_\xi \omega = 0 .$$

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Since $d\omega = 0$, this means that the one-form $\iota(\xi)\omega$ is closed. If $\iota(\xi)\omega$ is actually exact—so that there is a function f such that $\iota(\xi)\omega = df$ —then ξ is called **hamiltonian**. We see then that in a symplectic manifold one has the following interpretation of the first de Rham cohomology:

$$H^1(M) = \frac{\text{closed one-forms}}{\text{exact one-forms}} = \frac{\text{symplectic vector fields}}{\text{hamiltonian vector fields}} .$$

In other words, we have an exact sequence of vector spaces

$$0 \longrightarrow \text{Ham}(M) \longrightarrow \text{Sym}(M) \longrightarrow H^1(M) \longrightarrow 0 , \quad (1.1)$$

where $\text{Ham}(M)$ and $\text{Sym}(M)$ denote the hamiltonian and symplectic vector fields, respectively. It is clear from its definition as the stabilizer of ω , that $\text{Sym}(M)$ is a Lie algebra. Moreover, $\text{Ham}(M)$ is a Lie subalgebra. Indeed, if ξ_f and ξ_g are hamiltonian vector fields associated to the functions f and g , then

$$[\xi_f, \xi_g] = \xi_{\{f, g\}} \quad (1.2)$$

where $\{f, g\}$ is the Poisson bracket. More is true, however, and $\text{Ham}(M)$ is actually an ideal of $\text{Sym}(M)$; for if η is a symplectic vector field

$$[\eta, \xi_f] = \xi_{\eta \cdot f} .$$

In other words, the exact sequence (1.1) is actually an exact sequence of Lie algebras. The induced Lie bracket on $H^1(M)$ is zero, however, because of the fact that $\text{Ham}(M)$ contains the first derived ideal $\text{Sym}(M)' \equiv [\text{Sym}(M), \text{Sym}(M)]$.

The assignment of a hamiltonian vector field to a function defines a map

$$\begin{aligned} C^\infty(M) &\rightarrow \text{Ham}(M) \\ f &\mapsto \omega^{-1}(df) \end{aligned}$$

which by (1.2) is a Lie algebra morphism. Its kernel consists of the locally constant functions $df = 0$; that is, $H^0(M)$. This gives rise to another exact sequence of Lie algebras

$$0 \longrightarrow H^0(M) \longrightarrow C^\infty(M) \longrightarrow \text{Ham}(M) \longrightarrow 0 , \quad (1.3)$$

where $H^0(M)$ is the center of $C^\infty(M)$ and hence abelian. Putting this sequence together with (1.1) we find the following 4-term exact sequence of Lie algebras interpolating between $H^0(M)$ and $H^1(M)$:

$$0 \longrightarrow H^0(M) \longrightarrow C^\infty(M) \longrightarrow \text{Sym}(M) \longrightarrow H^1(M) \longrightarrow 0 . \quad (1.4)$$

2. Now let G be a connected Lie group acting on M in such a way that ω is G -invariant. Let \mathfrak{g} denote the Lie algebra of G . Every $X \in \mathfrak{g}$ gives

rise to a Killing vector field on M which we denote ξ_X . The map $X \mapsto \xi_X$ is a Lie algebra morphism. Since ω is G -invariant, ξ_X is symplectic. In other words, a symplectic G -action on M gives rise to a Lie algebra morphism $\mathfrak{g} \rightarrow \text{Sym}(M)$. There will be conserved charges associated to these continuous symmetries if and only if this map lifts to a Lie algebra morphism $\mathfrak{g} \rightarrow C^\infty(M)$ in such a way that the resulting diagram

$$\begin{array}{ccccccccc}
 0 & \rightarrow & H^0(M) & \rightarrow & C^\infty(M) & \rightarrow & \text{Sym}(M) & \rightarrow & H^1(M) & \rightarrow & 0 \\
 & & & & \swarrow & & \nearrow & & & & \\
 & & & & & \mathfrak{g} & & & & &
 \end{array}$$

commutes. The obstruction to the existence of such a lift follow easily from the exactness of (1.4). First of all, the image of \mathfrak{g} in $\text{Sym}(M)$ will come from $C^\infty(M)$ if it is sent to zero in $H^1(M)$. That is, if there exists functions ϕ_X such that $\iota(\xi_X)\omega = d\phi_X$. This is not enough because we want the map $X \mapsto \phi_X$ to be a Lie algebra morphism. Because the map $X \mapsto \xi_X$ is a Lie algebra morphism, the map $X \mapsto \phi_X$ is at most a projective representation characterized by the $H^0(M)$ -valued cocycle $c(X, Y) \equiv \{\phi_X, \phi_Y\} - \phi_{[X, Y]}$. If and only if this cocycle is a coboundary is the representation an honest representation. Indeed, if there exists some map $X \mapsto b_X \in H^0(M)$ such that $c(X, Y) = -b_{[X, Y]}$, then one straightens the map $X \mapsto \phi'_X = \phi_X - b_X$ and the resulting map $\mathfrak{g} \rightarrow C^\infty(M)$ is a morphism.

If this is case then one can define the **moment(um) mapping**

$$\Phi : M \longrightarrow \mathfrak{g}^*$$

by $\langle \Phi(m), X \rangle = \phi'_X(m)$ for all $m \in M$. This map is equivariant in that it intertwines between the G -action on M and the coadjoint action on \mathfrak{g}^* .

3. We can understand the conditions

$$\iota(\xi_X)\omega = d\phi_X \tag{3.1a}$$

$$\{\phi_X, \phi_Y\} = \phi_{[X, Y]} \tag{3.1b}$$

purely in terms of cohomology as follows. First of all notice that the map $\mathfrak{g} \rightarrow H^1(M)$ defined by $X \mapsto [\iota(\xi_X)\omega]$ annihilates the first derived ideal \mathfrak{g}' , since $[\text{Sym}(M), \text{Sym}(M)] \subset \text{Ham}(M)$. Therefore it induces a map $\mathfrak{g}/\mathfrak{g}' \rightarrow H^1(M)$; or, in other words, it defines an element in

$$(\mathfrak{g}/\mathfrak{g}')^* \otimes H^1(M) \cong H^1(\mathfrak{g}) \otimes H^1(M) .$$

Then (3.1a) simply says that this element is zero. Similarly the cocycle $c : \bigwedge^2 \mathfrak{g} \rightarrow H^0(M)$ defined above defines a class in $H^2(\mathfrak{g}) \otimes H^0(M)$. Then (3.1b) says that this class should be zero. In other words, the obstruction to the existence of a moment mapping defines a class

$$[O] \in (H^1(\mathfrak{g}) \otimes H^1(M)) \oplus (H^2(\mathfrak{g}) \otimes H^0(M)) . \tag{3.2}$$

In fact, we can understand this class as a single class in a different cohomology theory. Let us start by considering the G -action on M as a map

$$\alpha : G \times M \longrightarrow M$$

and let us define a G -action on $G \times M$ to make α equivariant. One convenient way to do so is

$$\beta : G \times G \times M \longrightarrow G \times M$$

where $\beta(g, h, m) = (gh, m)$; that is, G acts via left translations on the first factor and ignores the second. Equivariance of α allows us to pull back G -invariant forms on M to G -invariant forms on $G \times M$. The G -invariant forms on $G \times M$ form a subcomplex $\Omega^*(G \times M)^G$ of the de Rham complex. Therefore $\alpha^*\omega \in \Omega^2(G \times M)^G$. Similarly if we denote by $\pi : G \times M \rightarrow M$ the Cartesian projection onto the second factor, $\pi^*\omega$ is also a G -invariant form on $G \times M$. Define then $\omega_\alpha \equiv \alpha^*\omega - \pi^*\omega$. This is a closed form in $\Omega^2(G \times M)^G$ and hence defines a class in $H^2(G \times M)^G$. The complex $\Omega^*(G \times M)^G$ is isomorphic to the double complex $\Omega^*(G)^G \otimes \Omega^*(M)$. Applying the Künneth theorem to this complex, one finds that

$$H^n(G \times M)^G \cong \bigoplus_{p+q=n} H^p(\mathfrak{g}) \otimes H^q(M) . \quad (3.3)$$

It is then an easy computational matter to prove that under this isomorphism the class of ω_α goes over to the class $[O]$ in (3.2). (The $H^0(\mathfrak{g}) \otimes H^2(M)$ component is zero precisely because in ω_α we subtract $\pi^*\omega$ from $\alpha^*\omega$.)

As an example, if $(T^*N, d\theta)$ is the phase space of some configuration space N on which G acts, the action of G lifts naturally to a symplectic action on M . In fact, the tautological one-form θ is already invariant. In this case, $\omega_\alpha = d(\alpha^*\theta - \pi^*\theta)$ and since $(\alpha^*\theta - \pi^*\theta)$ is G -invariant, the class $[\omega_\alpha]$ in $H^2(G \times M)^G$ is trivial. Our “classical” intuition on the correspondence between continuous symmetries and conservation laws is borne out of this example.

4. What does this have to do with gauging σ -models? Let B be a two-manifold with boundary $\partial B = \Sigma$. Let (M, ω) be as before except that we drop the nondegeneracy condition on ω . The Wess-Zumino term of the σ -model in question is given by the function

$$S_{\text{WZ}}[\varphi] = \int_B \varphi^*\omega \quad (4.1)$$

on the space of maps $\varphi : B \rightarrow M$; but because ω is closed, the resulting equations of motion only depend on the restriction of φ to the boundary Σ . Therefore it defines a variational problem in the space $\text{Map}(\Sigma, M)$ of maps $\Sigma \rightarrow M$ (which extend to B). The σ -model also comes with a kinetic

term defined on Σ , but since the gauging of this term is simply accomplished via minimal coupling we shall disregard it in what follows. It should also be mentioned that we are ignoring for the present purposes the topological obstructions concerning the well-definedness of the WZ term itself. Similarly we will consider only gauging the algebra: demanding invariance under “large” gauge transformations invariably brings about other topological obstructions.

Let G be a connected Lie group, acting on M in such a way that it fixes ω . The action of G on M induces an action of G on $\text{Map}(B, M)$ under which the action (4.1) is invariant. For our purposes, gauging the WZ term will consist in promoting (4.1) to an action which is invariant under $\text{Map}(\Sigma, \mathfrak{g})$ via the addition of further terms involving a gauge field. We do this in steps following the Noether procedure.

5. Let $\lambda \in \text{Map}(\Sigma, \mathfrak{g})$. More explicitly, if we fix a basis $\{X_a\}$ for \mathfrak{g} , then $\lambda = \lambda^a X_a$ with λ_a functions on Σ . The action of λ on the pull-back of any form Ω on M , is given by

$$\delta_\lambda \varphi^* \Omega = d\lambda^a \wedge \varphi^* \iota_a \Omega + \lambda^a \varphi^* \mathcal{L}_a \Omega$$

where ι_a and \mathcal{L}_a denote respectively the contraction and Lie derivative relative to the Killing vector corresponding to X_a . In particular since ω is closed and \mathfrak{g} -invariant, we find that $\delta_\lambda \varphi^* \omega = d(\lambda^a \varphi^* \iota_a \omega)$, whence the variation of (3.1) becomes

$$\delta_\lambda S_{\text{WZ}}[\varphi] = \int_\Sigma \lambda^a \varphi^* \iota_a \omega .$$

Let us now introduce a \mathfrak{g} -valued gauge field $A = A^a X_a$ on Σ , which transforms under $\text{Map}(\Sigma, \mathfrak{g})$ as

$$\delta_\lambda A = d\lambda + [A, \lambda] .$$

The most general (polynomial) term we can add to (3.1) involving the gauge field is given by

$$S_{\text{extra}}[\varphi, A] = \int_\Sigma A^a \varphi^* \phi_a$$

for some functions $\phi_a \in C^\infty(M)$. It is then a small computational matter to work out the conditions under which the total action

$$S_{\text{GWZ}}[\varphi, A] = \int_B \varphi^* \omega + \int_\Sigma A^a \varphi^* \phi_a$$

is gauge-invariant; that is, $\delta_\lambda S_{\text{GWZ}} = 0$. Doing so one finds that the conditions are

$$\begin{aligned} \iota_a \omega &= d\phi_a \\ \mathcal{L}_a \phi_b &= f_{ab}{}^c \phi_c \end{aligned}$$

which are none other than (3.1a) and (3.1b) relative to the chosen basis for \mathfrak{g} .

We therefore conclude that, for ω a symplectic form, the WZ term (4.1) can be gauged if and only if one can define an equivariant moment mapping for the G -action.

6. Bibliography

The homological obstructions to defining a moment mapping have been well-known since at least the mid nineteen-seventies. The treatment here follows in spirit the one in Weinstein's 1976 lectures [We]. The conditions for gauging reductive (two-dimensional) WZW models were obtained independently by Hull and Spence in [HS1] and by Jack, Jones, Mohammedi and Osborne in [JJMO]. The conditions for gauging higher-dimensional σ -models with WZ term were later considered by Hull and Spence in [HS2]. The conditions for gauging nonreductive WZW models have been obtained in [FS1] as part of a general analysis of such models. The homological (re)interpretation of the obstructions to gauging a general WZ term will appear in [FS2], where they are understood in terms of the equivariant cohomology.

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7. Postscript

After the talk, J. Cariñena pointed out to me another way to understand the obstructions in (3.1) in terms of Lie algebra cohomology with coefficients in the exact one-forms (equivalently, the hamiltonian vector fields). If we think of (1.1) and (1.3) as exact sequences of \mathfrak{g} -modules, we obtain two long exact sequence in Lie algebra cohomology. The map $X \mapsto \xi_X$ defines a class in $H^1(\mathfrak{g}; \text{Sym}(M))$. By exactness of the sequence induced by (1.1), we see that it comes from $H^1(\mathfrak{g}; \text{Ham}(M))$ if and only if its image in $H^1(\mathfrak{g}; H^1(M))$ vanishes. Supposing it does and using now the exactness of the sequence induced by (1.3), we see that this class in $H^1(\mathfrak{g}; \text{Ham}(M))$ comes from $H^1(\mathfrak{g}; C^\infty(M))$ precisely when its image in $H^2(\mathfrak{g}; H^0(M))$ vanishes. These two obstructions precisely correspond to the classes in (3.2). Finally, I was informed by G. Papadopoulos, that the obstructions in (3.2) can also be understood as “anomalies” to global symmetries in the quantization of a particle interacting with a magnetic field. The details appear

in *Comm. Math. Phys.* **144** (1992) 491-508. I am grateful to them both for letting me know of their results during the conference.