Lecture 3: The BRST complex for a group action

In this lecture we will start our discussion of the homological approach to coisotropic reduction by studying the case where the coisotropic submanifold is the zero locus of an equivariant moment map coming from a group action.

(Co)homology is algebraic by its very nature, whereas coisotropic reduction as described above is geometric. This means that before we can relate them, they must be phrased in a common language. In this case, and as much as it may hurt one's sensibilities, the simplest thing to do is to translate geometry into algebra.

3.1 An algebraic interlude

The natural algebraic structure associated to a smooth manifold M is its algebra $C^\infty(M)$ of smooth functions. It is a commutative associative unital algebra which encodes a lot of information on M and from which in many cases on can reconstruct M. A symplectic structure on M lends $C^\infty(M)$ additional structure. The Poisson bracket turns $C^\infty(M)$ into a Lie algebra and moreover, for any $f \in C^\infty(M)$, $\{f,-\}$ is a derivation over the commutative multiplication. This turns $C^\infty(M)$ into a **Poisson algebra**.

Any closed embedded submanifold M_0 of M defines an ideal $\mathscr{I} \subset C^\infty(M)$ consisting of those functions which vanish on M_0. We call this the **vanishing ideal of** M_0. If $M_0 = \Phi^{-1}(0)$ is the zero locus of a smooth function $\Phi : M \to \mathbb{R}^k$ where $0 \in \mathbb{R}^k$ is a regular value, then the ideal $\mathscr{I}$ is precisely the ideal generated by the components ϕ_i of Φ relative to any basis for $\mathbb{R}^k$.

Every smooth function on M restricts to a smooth function on M_0 and two such functions restrict to the same function if and only if their difference belongs to the ideal $\mathscr{I}$. Conversely every smooth function on M_0 can be extended (not uniquely) to a smooth function on M. In other words, there is an isomorphism

$$C^\infty(M_0) \cong C^\infty(M)/\mathscr{I} \ . \tag{5}$$

We must now algebraize the fact that M_0 is coisotropic. We start by recalling that vector fields are derivations of the algebra of functions: $\mathscr{X}(M) = \operatorname{Der} C^\infty(M)$. From the isomorphism in (5), a derivation of $C^\infty(M)$ gives rise to a derivation of $C^\infty(M_0)$ if and only if it preserves the ideal $\mathscr{I}$. Indeed, it is not hard to show that

$$\operatorname{Der} C^\infty(M_0) = \left\{\xi \in \operatorname{Der} C^\infty(M) \middle| \xi(\mathscr{I}) \subset \mathscr{I}\right\} .$$

As we saw above, vector fields in $TM_0^\perp$ are precisely the hamiltonian vector fields which arise from functions in $\mathscr{I}$, whence the coisotropy condition $TM_0^\perp \subset TM_0$

becomes the condition that the vanishing ideal be closed under the Poisson bracket: $\{\mathscr{I},\mathscr{I}\}\subset\mathscr{I}$. Such ideals are called **coisotropic** for good reason.

Finally, the functions on $\widetilde{M}$ are those functions on M_0 which are constant on the leaves of the foliation. Since the leaves are connected and the tangent vectors to the leaves are the hamiltonian vector fields of functions in $\mathscr{I}$, we have an isomorphism

$$C^\infty(\widetilde{M}) = \left\{f\in C^\infty(M_0)\middle|\{f,\mathscr{I}\}=0\right\} ,$$

where $\{f,\mathscr{I}\}=0$ on M_0. Extending f to a function on M, the isomorphism becomes

$$C^\infty(\widetilde{M}) = \left\{f\in C^\infty(M)\middle|\{f,\mathscr{I}\}\subset\mathscr{I}\right\}/\mathscr{I} ,$$

which does not depend on the extension because $\mathscr{I}$ is closed under the Poisson bracket. In other words,

$$C^\infty(\widetilde{M}) = N(\mathscr{I})/\mathscr{I} ,$$

where $N(\mathscr{I})$ is the Poisson normalizer of $\mathscr{I}$ in $C^\infty(M)$.

Notice that $N(\mathscr{I})$ is a Poisson subalgebra of $C^\infty(M)$ having $\mathscr{I}$ as a Poisson ideal. This means that the quotient $N(\mathscr{I})/\mathscr{I}$ inherits the structure of a Poisson algebra.

The power of the algebraic formalism is that it continues to make sense in situations where the geometry might be singular. Indeed, it is possible now to rephrase this reduction purely in the category of Poisson algebras. Let P be a Poisson algebra and $I\subset P$ a coisotropic ideal. Then the normalizer $N(I)\subset P$ of I in P is a Poisson subalgebra containing I as a Poisson ideal, and hence the quotient $N(I)/I$ is a Poisson algebra, which we can think of as the reduced Poisson algebra of P by I. The aim of the BRST construction is to construct a complex of Poisson (super)algebras and a differential which is a Poisson (super)derivation so that its cohomology (at least in zero degree) is isomorphic as a Poisson algebra to $N(I)/I$. This turns out to be possible in huge generality, but the details of the construction depend on the 'regularity' of the ideal I. To keep things simple we will make a regularity assumption along the way.

3.2 The BRST complex of a group action

As a warmup we will construct the BRST complex for the case of a group action with an equivariant moment map $\Phi: M\to\mathfrak{g}^*$ which has $0\in\mathfrak{g}^*$ as a regular value. Let $M_0=\Phi^{-1}(0)$ be the coisotropic submanifold of zero momentum and let $\mathscr{I}$ denote its vanishing ideal. Let $\pi: M_0\to\widetilde{M}$ denote the projection onto the quotient $\widetilde{M}=M_0/G$. The pull-back $\pi^*: C^\infty(\widetilde{M})\to C^\infty(M_0)$ allows us to view functions on the quotient as functions on M_0. Indeed, a function on M_0 comes from a function on $\widetilde{M}$ if and only if it is constant on the fibres, which are the G-orbits. Since G is connected, this is the same thing as being constant along the flows of the vector fields ξ_X, which is the same thing as Poisson-commuting with (the restriction to M_0 of) ϕ_X, or in more algebraic terms,

$$C^\infty(\widetilde{M}) \cong C^\infty(M_0)^{\mathfrak{g}} = H^0\left(\mathfrak{g}; C^\infty(M_0)\right) .$$

This is not satisfactory because $C^\infty(M_0)$ is a quotient of $C^\infty(M)$, whereas we would like to work directly with $C^\infty(M)$. There is a standard construction in homological algebra which will help us achieve our goal: a "resolution".

3.2.1 The Koszul resolution

Consider the graded vector space $K^\bullet = \Lambda^\bullet\mathfrak{g} \otimes C^\infty(M)$, which can be interpreted as smooth functions on M with values in the exterior algebra of $\mathfrak{g}$. The moment map allows us to define a differential $\delta : K^\bullet \to K^{\bullet-1}$ as follows:

$$\begin{aligned} \delta f &= 0 \quad \forall f \in C^\infty(M) , \\ \delta X &= \phi_X \quad \forall X \in \mathfrak{g} , \end{aligned}$$

and extending it as an odd derivation:

$$\delta(\eta \wedge \zeta \otimes f) = \delta\eta \wedge \zeta \otimes f + (-1)^{|\eta|}\eta \wedge \delta\zeta \otimes f .$$

It is clear that on generators $\delta^2 f = 0$ and that $\delta^2 X = 0$, whence by the derivation property δ is a differential. The resulting differential graded complex $(K^\bullet, \delta)$,

$$\cdots \longrightarrow \Lambda^2\mathfrak{g} \otimes C^\infty(M) \xrightarrow{\delta} \mathfrak{g} \otimes C^\infty(M) \xrightarrow{\delta} C^\infty(M) \longrightarrow 0$$

is called the **Koszul complex**. Let us calculate its homology. In dimension 0, we have $Z^0 = K^0 = C^\infty(M)$, whereas $B^0 = \delta K^1 = I[\Phi]$, the ideal generated by the components of the moment map.

Lemma 3.1. *The ideal* $I[\Phi]$ *generated by the components of the moment map is precisely the vanishing ideal* $\mathscr{I}$ *of* M_0.

Proof. Since the components of the moment map vanish on M_0, it is clear that $I[\Phi] \subseteq \mathscr{I}$. What we have to show is that if a function vanishes on M_0 it is contained in the ideal generated by the components of the moment map. We only prove this locally, leaving the globalisation to a standard argument using partitions of unity.

Let $N = \dim M$ and $k = \dim \mathfrak{g}$ for definiteness. We will choose a basis $\mathbf{X}_i$ for $\mathfrak{g}$ and let $\phi_i = \Phi(\mathbf{X}_i)$ be the components of the moment map relative to this basis. Since M_0 is an embedded submanifold, around every point of M_0 there is an open set $U \subset M$ and local coordinates $(x, y) : U \to \mathbb{R}^k \times \mathbb{R}^{N-k}$ where $x^i = \phi_i$ for $i = 1, \ldots, k$.

Suppose now that a function f vanishes on M_0. Restricting to U, this means that the $f(0,y)=0$ for all $y^1,\ldots,y^{N-k}$. Then

$$\begin{aligned} f(x,y) &= \int_0^1 \tfrac{d}{dt} f(tx,y)\,dt \\ &= \int_0^1 \sum_{i=1}^k x^i (D_i f)(tx,y)\,dt \\ &= \sum_{i=1}^k \phi_i \int_0^1 (D_i f)(tx,y)\,dt\,, \end{aligned}$$

whence the restriction of f to U belongs to the ideal generated by the ϕ_i. In other words, there are functions $h^i_U \in C^\infty(U)$ such that $f|_U = \sum_i h^i_U \phi_i|_U$. We now cover M with such charts and patch things up with a partition of unity subordinate to this cover, which ensures that $f = \sum_i h^i \phi_i$ for some $h^i \in C^\infty(M)$. □

Using the isomorphism (5), we have that $H^0(K^\bullet) \cong C^\infty(M_0)$. The rest of the homology vanishes, as the following result shows.

Proposition 3.2. *The homology of the Koszul complex is given by*

$$H^p(K^\bullet) \cong \begin{cases} C^\infty(M_0)\,, & p=0 \\ 0\,, & \textit{otherwise.} \end{cases}$$

Proof. We sketch the proof. A sequence of functions $(\phi_1,\ldots,\phi_k)$ is said to be **regular** if the following property holds for every j: if for some function f, $\phi_j f$ belongs to the ideal I_{j-1} generated by $(\phi_1,\ldots,\phi_{j-1})$, then $f \in I_{j-1}$ already. Letting $I_0 = 0$, this implies that ϕ_1 is not identically zero. Now from the Lemma it follows that the sequence $(\phi_1,\ldots,\phi_k)$, where $\phi_i = \Phi(\mathbf{X}_i)$, is regular. Finally it is a straight-forward result in homological algebra (see, for example, [Lan84, Ch.XIV,§10,Theorem 10.5]) that the Koszul complex of a regular sequence is acyclic in positive dimension. □

Augmenting the complex by the homology, we obtain an exact sequence

$$\cdots \longrightarrow K^2 \xrightarrow{\delta} K^1 \xrightarrow{\delta} C^\infty(M) \longrightarrow C^\infty(M_0) \longrightarrow 0\,,$$

which is a (projective) resolution of $C^\infty(M_0)$ in terms of (free) $C^\infty(M)$-modules, called the **Koszul resolution**.

3.2.2 A double complex and the BRST complex

Letting $\mathfrak{g}$ act on $\Lambda\mathfrak{g}$ as exterior powers of the adjoint representation, the spaces in the Koszul complex are $\mathfrak{g}$-modules and moreover the Koszul differential δ is a $\mathfrak{g}$-map. As in Problem 1.15, it defines a chain map

$$\delta : C^\bullet(\mathfrak{g};K^q) \to C^\bullet(\mathfrak{g};K^{q-1})\,.$$

Indeed, defining $C^{p,q} := C^p(\mathfrak{g};K^q)$, we have two commuting differentials: the Chevalley–Eilenberg differential $d : C^{p,q} \to C^{p+1,q}$ and the Koszul differential $\delta : C^{p,q} \to C^{p,q-1}$. We may picture these maps as defining a **double complex**:

$$\begin{tikzcd}
\cdots \arrow[r] & C^{0,2} \arrow[r, "\delta"] \arrow[d, "d"] & C^{0,1} \arrow[r, "\delta"] \arrow[d, "d"] & C^{0,0} \arrow[d, "d"] \\
\cdots \arrow[r] & C^{1,2} \arrow[r, "\delta"] \arrow[d, "d"] & C^{1,1} \arrow[r, "\delta"] \arrow[d, "d"] & C^{1,0} \arrow[d, "d"] \\
\cdots \arrow[r] & C^{2,2} \arrow[r, "\delta"] \arrow[d, "d"] & C^{2,1} \arrow[r, "\delta"] \arrow[d, "d"] & C^{2,0} \arrow[d, "d"] \\
& \vdots & \vdots & \vdots
\end{tikzcd}$$

where the top row is the Koszul complex and the columns are Chevalley–Eilenberg complexes.

Let us 'roll up' the above double complex into the **total complex**

$$TC^n := \bigoplus_{p-q=n} C^{p,q} ,$$

and define the **total differential** $D : TC^n \to TC^{n+1}$ by $D = d + (-1)^p\delta$ on $C^{p,q}$. Then $D^2 = 0$, which justifies calling it a differential. We have the following fundamental result.

Theorem 3.3. *The cohomology of the total complex is*

$$H^p(TC^\bullet) \cong C^\infty(\widetilde{M}) \otimes H^p(\mathfrak{g}) .$$

Proof. We will only prove the result in dimension 0, by which

$$H^0(TC^\bullet) \cong N(\mathscr{I})/\mathscr{I} \cong C^\infty(\widetilde{M}) .$$

The proof uses the 'tic-tac-toe' technique in [BT82, Ch.II] and the acyclicity of the Koszul complexes $K^\bullet \otimes \Lambda\mathfrak{g}^*$ in positive dimension. These complexes are the rows of the double complex above and acyclicity in general follows from the acyclicity (in positive dimension) of the top row and the fact that the Koszul differential does not act on $\Lambda\mathfrak{g}^*$.

We will first show that a function $f \in N(\mathscr{I})$ defines a class in $H^0(TC^\bullet)$. Its Chevalley–Eilenberg differential is given by

$$d f = \sum_{i=1}^{k} c^i\{\phi_i, f\} \in C^{1,0} .$$

Because $f \in N(\mathscr{I})$, there are functions h_i^j such that $\{\phi_i, f\} = \sum_{j=1}^k h_i^j \phi_j$, whence

$$df = \sum_{i,j=1}^{k} c^i h_i^j \phi_j = \delta \sum_{i,j=1}^{k} c^i b_j h_i^j =: \delta f_1 ,$$

which defines $f_1 \in C^{1,1}$. Let $F_1 = f + f_1 \in TC^0$. We compute

$$DF_1 = df_1 ,$$

but $\delta d f_1 = d\delta f_1 = d^2 f = 0$, whence there exists some $f_2 \in C^{2,2}$ such that $\delta f_2 = -df_1$. Let $F_2 = f + f_1 + f_2 \in TC^0$. Again, we compute

$$DF_2 = df_2 ,$$

and again one can argue that, since $\delta df_2 = 0$, there exists $f_3 \in C^{3,3}$, etc. At the end of the day we have

$$F = f + f_1 + f_2 + \cdots + f_k ,$$

where $f_i \in C^{i,i}$ such that $DF = 0$, whence it defines a class in $H^0(TC^\bullet)$. It is not hard to see that making different choices along the way only change F by a D-coboundary and hence do not change the cohomology class. Similarly one can show that adding something in $\mathscr{I}$ to f only changes F by a D-coboundary and does not affect the cohomology class. Thus we have constructed a map $N(\mathscr{I})/\mathscr{I} \to H^0(TC^\bullet)$.

Conversely, if $F = f_0 + f_1 + \cdots + f_k \in TC^0$, where $f_i \in C^{i,i}$ is D-closed, we see that $df_0 = \delta f_1$, whence $f_0 \in N(\mathscr{I})$. Adding a coboundary to F only changes f_0 by something in $\mathscr{I}$, hence this defines a map $H^0(TC^\bullet) \to N(\mathscr{I})/\mathscr{I}$, which is readily seen to be the inverse of the previous map, which is thus an isomorphism. □

The isomorphism $H^0(TC^\bullet) \cong C^\infty(\tilde{M})$ of the theorem is one of associative algebras. However we know that $C^\infty(\tilde{M})$ is a Poisson algebra and it is therefore a natural question to ask whether we can strengthen this theorem by showing that the isomorphism is one of Poisson algebras. This requires defining a Poisson algebra structure on the BRST cohomology, which is the task we turn to now.

3.2.3 The classical BRST operator and the Poisson structure

We will now show that the total complex $TC^\bullet$ can be given the structure of a graded Poisson superalgebra in such a way that the total differential $D = \{Q, -\}$ is an inner derivation by an element $Q \in TC^1$ called the **classical BRST operator**. Since the differential acts by Poisson derivations, the cocycles are Poisson sub-superalgebras of which the coboundaries are Poisson ideals, thus making the cohomology into a Poisson superalgebra. In particular, the cohomology in dimension zero is a Poisson algebra.

The total complex is $\Lambda(\mathfrak{g}\oplus\mathfrak{g}^*)\otimes C^\infty(M)$. We will show that it admits the structure of a Poisson superalgebra, but first we will recall the relevant notions.

A **Poisson superalgebra** is a $\mathbb{Z}_2$-graded vector space $P = P_0 \oplus P_1$ together with two bilinear operations preserving the grading:

$$\begin{aligned} P\times P &\to P \\ (a,b) &\mapsto ab \end{aligned} \qquad \text{and} \qquad \begin{aligned} P\times P &\to P \\ (a,b) &\mapsto \{a,b\}\ , \end{aligned}$$

obeying the following properties:

- P is an associative supercommutative superalgebra under multiplication:

$$a(bc) = (ab)c \qquad \text{and} \qquad ab = (-1)^{|a||b|}ba\ ,$$

- P is a Lie superalgebra under Poisson bracket:

$$\{a,b\} = (-1)^{|a||b|}\{b,a\} \qquad \text{and} \qquad \{a,\{b,c\}\} = \{\{a,b\}\,c\} + (-1)^{|a||b|}\{b,\{a,c\}\}\ ,$$

- the Poisson bracket is a derivation over multiplication:

$$\{a,bc\} = \{a,b\}c + (-1)^{|a||b|}b\{a,c\}\ ,$$

  for all $a,b,c\in P$ and where $|a|$ equals 0 or 1 according to whether a is even or odd, respectively.

The algebra $C^\infty(M)$ is clearly an example of a Poisson superalgebra without odd part. On the other hand, the exterior algebra $\Lambda(\mathfrak{g}\oplus\mathfrak{g}^*)$ posseses a Poisson superalgebra structure. The associative multiplication is given by the wedge product and the Poisson bracket is defined for $X,Y\in\mathfrak{g}$ and $\alpha,\beta\in\mathfrak{g}^*$ by

$$\{\alpha,X\} = \alpha(X) = \{X,\alpha\} \qquad \{X,Y\} = 0 = \{\alpha,\beta\}\ .$$

We then extend it to all of $\Lambda(\mathfrak{g}\oplus\mathfrak{g}^*)$ as an odd derivation.

To show that the total complex $TC^\bullet$ is a Poisson superalgebra we need to discuss tensor products. Given two Poisson superalgebras P and Q, their tensor product $P\otimes Q$ can be given the structure of a Poisson superalgebra as follows. For $a,b\in P$ and $u,v\in Q$ we define

$$\begin{aligned} (a\otimes u)(b\otimes v) &= (-1)^{|u||b|}ab\otimes uv \\ \{a\otimes u, b\otimes v\} &= (-1)^{|u||b|}\left(\{a,b\}\otimes uv + ab\otimes\{u,v\}\right) \end{aligned}$$

One can easily show that these operations satisfy the axioms of a Poisson superalgebra.

Now let P be a Poisson superalgebra which, in addition, is $\mathbb{Z}$-graded, that is, $P = \bigoplus_n P^n$ and $P^nP^m \subseteq P^{m+n}$ and $\{P^n,P^m\}\subseteq P^{m+n}$; and such that the $\mathbb{Z}_2$-grading is the reduction modulo 2 of the $\mathbb{Z}$-grading, that is, $P_0 = \bigoplus_n P^{2n}$ and $P_1 = \bigoplus_n P^{2n+1}$. We call such an algebra a **graded Poisson superalgebra**. Notice that P^0 is a Poisson subalgebra of P.

For example, $TC = \Lambda(\mathfrak{g} \oplus \mathfrak{g}^*) \otimes C^\infty(M)$, with the grading described above becomes a $\mathbb{Z}$-graded Poisson superalgebra. Although the bigrading is preserved by the exterior product, the Poisson bracket does not preserve it. In fact, the Poisson bracket obeys

$$\{C^{i,j}, C^{k,l}\} \subseteq C^{i+k,j+l} \oplus C^{i+k-1,j+l-1} ,$$

but the total degree is preserved.

By a **Poisson derivation** of degree k we will mean a linear map $D : P^n \to P^{n+k}$ such that

$$D(ab) = (Da)b + (-1)^{k|a|} a(Db)$$
$$D\{a,b\} = \{Da, b\} + (-1)^{k|a|}\{a, Db\} .$$

The map $a \mapsto \{Q, a\}$ for some $Q \in P^k$ is an **inner Poisson derivation**.
Proposition 3.4. *The total differential* $D = \{Q, -\}$*, where* $Q \in TC^1$ *is given explicitly by the following expression*

$$Q = \boldsymbol{\alpha}^i \phi_i - \tfrac{1}{2} f_{jk}^i \boldsymbol{\alpha}^j \wedge \boldsymbol{\alpha}^k \wedge \mathbf{X}_i ,$$

where we have introduced a basis $(\mathbf{X}_i)$ *for* $\mathfrak{g}$*, relative to which* $[\mathbf{X}_i, \mathbf{X}_j] = f_{ij}^k \mathbf{X}_k$ *and a dual basis* $(\boldsymbol{\alpha}^i)$ *for* $\mathfrak{g}^*$ *and where we have used the summation convention.*

Proof. Being a derivation, it is enough to show that $\{Q, -\}$ acts as it should on the generators; that is, on functions $f \in C^\infty(M)$, and elements $Y \in \mathfrak{g}$ and $\beta \in \mathfrak{g}^*$. Clearly,

$$\{Q, f\} = \boldsymbol{\alpha}^i \{\phi_i, f\} \in \mathfrak{g}^* \otimes C^\infty(M)$$

agrees with the Chevalley–Eilenberg differential df. On $\beta \in \mathfrak{g}^*$,

$$\{Q, \beta\} = -\tfrac{1}{2} \beta_i f_{jk}^i \boldsymbol{\alpha}^j \wedge \boldsymbol{\alpha}^k \in \Lambda^2 \mathfrak{g}^*$$

which again agrees with $d\beta$. Finally on $Y \in \mathfrak{g}$ we have

$$\{Q, Y\} = Y^i \phi_i + f_{jk}^i Y^k \boldsymbol{\alpha}^j \wedge \mathbf{X}_i ,$$

where the first term agrees with $\delta Y = \phi_Y$ and the second term agrees with $dY \in \mathfrak{g}^* \otimes \mathfrak{g}$ defined by $dY(Z) = [Z, Y]$. □

One can show that the classical BRST operator Q satisfies $\{Q, Q\} = 0$, which is not immediate because the Poisson bracket on odd elements is symmetric.