OPERATOR PRODUCT TECHNOLOGY

State-operator correspondence

$$\phi(z) = \sum_{n\in\mathbb{Z}} \phi_n z^{-n-\Delta} \qquad \phi_{n>-\Delta}\,\mathbf{1} = 0 \qquad \phi = \phi_{-\Delta}\,\mathbf{1} \qquad \mathbf{1}(z) = \mathrm{id}$$

Operator product expansions

$$A(z)B(w) = \sum_{n\ll\infty} \frac{[AB]_n(w)}{(z-w)^n}$$

$$[AB]_n(w) = \oint_{C_w} \frac{dz}{2\pi i}(z-w)^{n-1}A(z)B(w)$$

Derivation

$$[(\partial A)B]_n = (1-n)[AB]_{n-1} \qquad [A(\partial B)]_n = (n-1)[AB]_{n-1} + \partial[AB]_n$$

Normal-ordered products

$$(AB)(z) \equiv [AB]_0(z) = \sum_n (AB)_n z^{-n-\Delta_A-\Delta_B}$$

$$(AB)_n = \sum_{\ell\leq-\Delta_A} A_\ell B_{n-\ell} + (-)^{|A||B|} \sum_{\ell>-\Delta_A} B_{n-\ell}A_\ell$$

$$([AB]) \equiv (AB) - (-)^{|A||B|}(BA)$$

$$(A(BC)) = (-)^{|A||B|}(B(AC)) + (([AB])C)$$

Rearrangements

$$[AB]_n - (-)^{|A||B|+n}[BA]_n = \sum_{\ell\geq 1} \frac{(-)^{1+\ell}}{\ell!}\partial^\ell [AB]_{n+\ell}$$

$$= (-)^{|A||B|} \sum_{\ell\geq 1} \frac{(-)^{n+\ell}}{\ell!}\partial^\ell [BA]_{n+\ell}$$

Jacobi-type identities

$$[A[BC]_n]_1 = [[AB]_1C]_n + (-)^{|A||B|}[B[AC]_1]_n$$

$$[A[BC]_0]_n = (-)^{|A||B|}[B[AC]_n]_0 + \sum_{\ell=0}^{n-1} \tbinom{n-1}{\ell}[[AB]_{n-\ell}C]_\ell$$

$$\begin{aligned}[A[BC]_n]_{m>0} &= \sum_{\ell=0}^{m-1} \tbinom{m-1}{\ell}[[AB]_{m-\ell}C]_{n+\ell} + (-)^{|A||B|}[B[AC]_m]_n \\ &= \sum_{\ell\geq 0} \frac{(-)^\ell}{\ell!}[[(\partial^\ell A)B]_mC]_{n+\ell} + (-)^{|A||B|}[B[AC]_m]_n\end{aligned}$$

$$[A[BC]_m]_n = \sum_{\ell\geq 0}(-)^\ell\tbinom{m-1}{\ell}\left([[AB]_{n+\ell}C]_{m-\ell} + (-)^{n+|A||B|}[B[AC]_{\ell+1}]_{m+n-\ell-1}\right)$$

$$[[AB]_pC]_q = \sum_{n\geq q}\tbinom{p-1}{n-q}(-)^{n-q}[A[BC]_n]_{p+q-n} - (-)^{|A||B|}\textstyle\sum_{n>0}\tbinom{p-1}{n-1}(-)^{n+p}[B[AC]_n]_{p+q-n}$$

$$[(AB)C]_q = \sum_{n\geq q}[A[BC]_n]_{q-n} + (-)^{|A||B|}\sum_{n>0}[B[AC]_n]_{q-n}$$