BRST Comology 2006

Tutorial Sheet 3

The BRST complex

Problem 3.1. Let P be a Poisson algebra; that is, P has a commutative associative multiplication $(a,b) \mapsto ab$ and a Lie bracket $(a,b) \mapsto \{a,b\}$ satisfying the condition $\{a,bc\} = \{a,b\}c + \{a,c\}b$. Define a new multiplication on P by

$$(a,b) \mapsto a \bullet b := \frac{1}{\sqrt{2}}(ab + \{a,b\}) .$$

Show that the new operation satisfies the condition

$$(a \bullet c) \bullet b + (b \bullet c) \bullet a - (b \bullet a) \bullet c - (c \bullet a) \bullet b = 3\mathrm{A}(a,b,c) , \tag{1}$$

where A is the associator: $\mathrm{A}(a,b,c) = a \bullet (b \bullet c) - (a \bullet b) \bullet c$.
Conversely, if P is a vector space with a multiplication $(a,b) \mapsto a \bullet b$ obeying equation (1), show that

$$ab := \frac{1}{\sqrt{2}}(a \bullet b + b \bullet a) \qquad \text{and} \qquad \{a,b\} := \frac{1}{\sqrt{2}}(a \bullet b - b \bullet a)$$

turn P into a Poisson algebra.
For extra credit, formulate and prove a 'super' version of these results.

Problem 3.2. Show that a submanifold $\mathrm{M}_0 \subset \mathrm{M}$ is given by the zero locus of a smooth function $\Phi : \mathrm{M} \to \mathbb{R}^k$, where $k = \operatorname{codim} \mathrm{M}_0$, if and only if its normal bundle is trivial.

Problem 3.3. Show that the tensor product of two Poisson superalgebras is naturally a Poisson superalgebra.

Problem 3.4. Let $\mathrm{P} = \bigoplus_n \mathrm{P}^n$ be a graded Poisson superalgebra and let $\nu : \mathrm{P} \to \mathrm{P}$ denote the **degree derivation** such that $\nu(a) = pa$ if $a \in \mathrm{P}^p$. Show that if the degree derivation is inner, then so is any other Poisson derivation of nonzero degree.

Problem 3.5. Let P be a Poisson superalgebra and $\mathrm{Q} \in \mathrm{P}$ an odd element satisfying $\{\mathrm{Q},\mathrm{Q}\} = 0$. Show that $\mathrm{D} := \{\mathrm{Q}, -\}$ is a Poisson derivation and that $\mathrm{D}^2 = 0$. Then show that the kernel of D is a Poisson sub-superalgebra containing the image of D as a Poisson ideal. Conclude that the cohomology $\ker \mathrm{D} / \operatorname{Im} \mathrm{D}$ is a Poisson superalgebra.

Problem 3.6. Show that the classical BRST operator for the case of a group action,

$$\mathrm{Q} = c^i \phi_i - \tfrac{1}{2} f_{jk}^i c^j c^k b_i ,$$

satisfies $\{\mathrm{Q},\mathrm{Q}\} = 0$.

Problem 3.7. In the case of general "first-class constraints", let $\mathrm{Q} \in \mathscr{C}^1$ satisfy $\{\mathrm{Q},\mathrm{Q}\} = 0$. Let $\mathrm{Q} = \mathrm{Q}_0 + \mathrm{Q}_1 + \cdots$, with $\mathrm{Q}_i \in \mathrm{C}^{i+1,i}$, and $\mathrm{Q}_0 = c^i \phi_i$. Prove that the cohomology of the graded complex $(\mathscr{C}^\bullet, \mathrm{D} := \{\mathrm{Q}, -\})$ in degree zero is given by

$$\mathrm{H}^0(\mathscr{C}^\bullet) \cong \frac{\mathrm{N}(\mathscr{I})}{\mathscr{I}} ,$$

where $N(\mathscr{I})$ is the normalizer of $\mathscr{I}$ in $C^\infty(M)$ and the isomorphism is one of Poisson algebras.
(*Hint*: Use 'tic-tac-toe', exploiting the acyclicity of the Koszul complex in positive degree. Where is the Koszul differential in D?)

Problem 3.8. Let us try to extend the construction of the general BRST complex to the case when M_0 has nontrivial normal bundle. Cover M by open sets $\{U_\alpha\}$ such that either $M_0 \cap U_\alpha = \varnothing$ or else the normal bundle of M_0 is trivial on $U_\alpha \cap M_0$. From now on we will consider only those α for which $U_\alpha \cap M_0 \neq \varnothing$. On each such U_α, the ideal $I_\alpha \subset C^\infty(U_\alpha)$ of functions vanishing on $U_\alpha \cap M_0$ is generated by k functions ϕ_i^α. By the results in the lecture there is on U_α a local BRST operator $Q_\alpha \in \Lambda(V \oplus V^*) \otimes C^\infty(U_\alpha)$ obeying $\{Q_\alpha, Q_\alpha\} = 0$ and the BRST cohomology in zero degree is isomorphic as a Poisson algebra to $N(I_\alpha)/I_\alpha$, where $N(I_\alpha)$ is the normalizer of I_α in $C^\infty(I_\alpha)$. Now consider two overlapping open sets U_α and U_β with $U_\alpha \cap U_\beta \cap M_0 \neq \varnothing$. Show that whereas the complexes need not agree in the overlap $U_\alpha \cap U_\beta$, the BRST cohomologies are isomorphic (at least in zero degree, although it can be shown that they agree in general). Conclude that to each U_α intersecting M_0, we can assign a Poisson algebra $P_\alpha := N(I_\alpha)/I_\alpha$ and isomorphisms $\psi_{\alpha\beta} := P_\alpha\big|_{U_\alpha \cap U_\beta} \to P_\beta\big|_{U_\alpha \cap U_\beta}$. Show that this defines a sheaf of Poisson algebras, whose space of global sections is precisely $N(\mathscr{I})/\mathscr{I}$.