BRST Comology 2006-2007

Tutorial Sheet 5

BRST differential in conformal field theory

Problem 5.1. *BC systems*
Let $\boldsymbol{b}(z)$ and $\boldsymbol{c}(z)$ be the fields of a BC system. This means that they obey the following operator product algebra:

$$[\boldsymbol{b}, \boldsymbol{c}]_1 = \mathbf{1} = -\varepsilon[\boldsymbol{c}, \boldsymbol{b}]_1 \tag{1}$$

where $\varepsilon = +1$ (resp. -1) if we are dealing with a bosonic (resp. fermionic) BC system.[1] We say that the BC system has weight λ if $\boldsymbol{b}$ is a conformal field of weight λ.

a. Prove that $\boldsymbol{c}$ must have weight $1-\lambda$. (*Hint:* apply $[\mathrm{T}, -]_2$ to both sides of the operator product algebra (1).)

b. Prove that the energy-momentum tensor can be written as

$$\mathrm{T}_{\boldsymbol{bc}} := \varepsilon\left(\lambda \boldsymbol{b}\partial\boldsymbol{c} + (\lambda-1)\partial\boldsymbol{b}\boldsymbol{c}\right) \tag{2}$$

and verify that $\mathrm{T}_{\boldsymbol{bc}}$ obeys a Virasoro algebra with central charge $c = 2\varepsilon(6\lambda^2 - 6\lambda+1)$; and relative to which both $\boldsymbol{b}$ and $\boldsymbol{c}$ are indeed primary fields of weights λ and $1-\lambda$, respectively.

c. Let $|q\rangle$ denote the vacuum of charge q, defined by the following conditions:

$$\begin{aligned} \boldsymbol{b}_n |q\rangle &= 0, \qquad n > \varepsilon q - \lambda \\ \boldsymbol{c}_n |q\rangle &= 0, \qquad n \geq -\varepsilon q + \lambda\,. \end{aligned}$$

Prove that $|q\rangle$ is a highest weight vector of the Virasoro algebra. What is its weight?

d. Find the value of q for which $|q\rangle$ is the $\mathrm{SL}(2,\mathbb{C})$-invariant vacuum. Recall that this means that $|q\rangle$ should be annihilated by L_0 and $\mathrm{L}_{\pm 1}$.

Notice that there is a crucial difference between the fermionic and bosonic cases. In the fermionic case ($\varepsilon = -1$) different vacua $|q\rangle$, although they belong to different representations of the Virasoro algebra, are in the same irreducible representation of the BC algebra. On the other hand, different vacua of a bosonic BC system ($\varepsilon = 1$) give rise to inequivalent representations of the BC algebra. In other words, the vacuum $|q'\rangle$ is not in the subspace generated by the modes of the BC algebra acting on the vacuum $|q\rangle$ unless $q = q'$. This fact underlies the existence of different "pictures" in the BRST quantisation of fermionic strings.

Problem 5.2. *BRST differential for the bosonic string*
Let $\mathrm{T}(z)$ obey the Virasoro algebra with central charge D. Let $b(z)$ and $c(z)$ be a fermionic BC system of weight λ. Define the following field

$$j := c\mathrm{T} + \tfrac{1}{2} c\mathrm{T}_{bc}$$

where T_{bc} is as in equation (2) with $\varepsilon = -1$.

[1] By convention, when defining an operator product algebra, we give only the nonvanishing $[-,-]_{n>0}$.

a. Prove that
$$j = c\mathrm{T} - \tfrac{1}{2}\lambda b\partial c c\,.$$

b. Prove that
$$[j,j]_1 = (2-\lambda)\mathrm{T}\partial c c + \tfrac{3}{8}\lambda^2\partial^2 c\partial c + \left(\tfrac{1}{12}\mathrm{D} - \tfrac{1}{6}\lambda^2\right)\partial^3 c c$$

c. Conclude that for this to be a total derivative (so that $d := [j,-]_1$ squares to zero) we need to impose $\lambda = 2$ and $\mathrm{D} = 26$.

d. From now on let $\lambda = 2$ and $\mathrm{D} = 26$. Show that
$$d b = \mathrm{T} + \mathrm{T}_{bc}$$
and show that $\mathrm{T}_{\text{tot}} := \mathrm{T} + \mathrm{T}_{bc}$ obeys a Virasoro algebra with zero central charge.

Problem 5.3. *Mode expansions*
Let $\mathrm{T}(z)$ obey the Virasoro algebra with central charge 26; that is,
$$[\mathrm{T},\mathrm{T}]_4 = 13\,\mathbf{1}\,, \qquad [\mathrm{T},\mathrm{T}]_2 = 2\mathrm{T}, \qquad [\mathrm{T},\mathrm{T}]_1 = \partial\mathrm{T}\,.$$

a. Let $\mathrm{T}(z) = \sum_{n\in\mathbb{Z}} \mathrm{L}_n z^{-n-2}$ and prove that the above operator product algebra translates into the following commutation relations of the modes:
$$[\mathrm{L}_n, \mathrm{L}_m] = (n-m)\mathrm{L}_{n+m} + \frac{13}{6}n(n^2-1)\delta_{n+m,0}\,,$$
where we have written $\mathbf{1}_0$ simply as 1.

b. Let $b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-2}$ and $c(z) = \sum_{n\in\mathbb{Z}} c_n z^{-n+1}$ and prove that the modes obey the following canonical anticommutation relations:
$$[b_n, c_m] = \delta_{n+m,0}, \qquad [b_n, b_m] = [c_n, c_m] = 0\,.$$

c. Expand the BRST current $j = c\mathrm{T} + bc\partial c$ as $j(z) = \sum_{n\in\mathbb{Z}} j_n z^{-n-1}$ and show that the BRST differential $d = j_0$ is given by
$$d = \sum_{n\in\mathbb{Z}} c_{-n}\mathrm{L}_n + \sum_{\substack{n,m\in\mathbb{Z}\\ n<m}} (n-m) : b_{n+m} c_{-m} c_{-n} :\,,$$
where the triple normal-ordered product in the second term stands for
$$: b_{n+m} c_{-m} c_{-n} := \begin{cases} b_{n+m} c_{-m} c_{-n}, & \text{for } n+m \le -2 \\ c_{-m} c_{-n} b_{n+m}, & \text{for } n+m > -2\,. \end{cases}$$

d. Verify by direct computation from this expression that $d^2 = 0$; or simply contemplate what this entails and spare a moment's thought for those less fortunate who didn't know about OPEs when they first did this!

Problem 5.4. *Other critical dimensions*
As we saw for the bosonic string, as a condition for the BRST cohomology to be defined, the contribution to the central charge from the matter sector has to cancel the contribution from the ghost system. The ghost system has opposite statistics to the generators of the algebra (the Virasoro algebra in this case) but the same conformal weight. From Problem 5.1 we can read off the central charge of the ghost system ($\varepsilon = -1, \lambda = 2$) to be -26, which explains the critical dimension of the bosonic string. Let's play this game for other "string" theories, assuming that a BRST operator exists and that the same argument holds. Calculate the critical central charges for the string theories based on the following algebras:

a. $N = 1$ Virasoro algebra (NSR string)

b. $N = 2$ Virasoro algebra

c. (small) $N = 4$ Virasoro algebra

d. W_3 algebra

e. W_N algebra

f. your favourite Kac-Moody algebra

Part of this exercise consists in looking up the field content of those algebras you may be unfamiliar with.

Problem 5.5. *Fairlie, Feigin, Fuchs and Fock*
The mode algebra of a free boson $i\partial\varphi(z) = \sum_n a_n z^{-n-1}$ is the infinite-dimensional Heisenberg algebra

$$[a_n, a_m] = n\delta_{m+n,0} \ .$$

Irreducible representations are indexed by the momentum α and are spanned by monomials of the form

$$a_{-n_1} a_{-n_2} \cdots a_{-n_k} |\alpha\rangle \ , \qquad \text{where} \quad n_1 \geq n_2 \geq \cdots n_k > 0 \ , \tag{3}$$

and where $|\alpha\rangle$ obeys $a_0 |\alpha\rangle = \alpha |\alpha\rangle$ and $a_{n>0} |\alpha\rangle = 0$. We can define on this space an action of the Virasoro algebra by the Fairlie–Feigin–Fuchs construction:

$$T(z) = -\tfrac{1}{2}(\partial\varphi)^2 + i\alpha_0 \partial^2\varphi \ .$$

a. In terms of modes, prove that $T(z) = \sum_n L_n z^{-n-2}$ where

$$L_n = \tfrac{1}{2} \sum_{m\in\mathbb{Z}} : a_m a_{n-m} : - \alpha_0 (n+1) a_n \ .$$

b. Prove that $T(z)$ obeys the Virasoro algebra with central charge $c = 1 - 12\alpha_0^2$.

c. Prove that $|\alpha\rangle$ is a Virasoro highest-weight vector of weight $\frac{1}{2}\alpha(\alpha - 2\alpha_0)$.

The linear span of the monomials in equation (3) with the above action of the Virasoro algebra is called a **Feigin–Fuchs module** and is denoted $\mathscr{F}_{\alpha,\alpha_0}$. If $\alpha_0 = 0$, then the Feigin–Fuchs module is simply the **Fock module** with c=1.

d. Prove that $\mathscr{F}_{\alpha,\alpha_0} \cong \mathscr{F}_{-\alpha,-\alpha_0}$ as Virasoro modules.