BRST Comology 2006-2007

Tutorial Sheet 6

BRST cohomology of the bosonic string

In this tutorial, we will compute some of the low-lying physical states of the critical bosonic string by hand. In fact, we will be computing only those states in the holomorphic sector.

The matter content of the bosonic string is given by 26 free bosons $J^\mu(z) = i\partial X^\mu(z)$ with operator product algebra

$$[J^\mu, J^\nu]_2 = \eta^{\mu\nu}\mathbf{1} \,, \tag{1}$$

where $\eta^{\mu\nu} = \mathrm{diag}[-++\cdots+]$ is (the inverse of) the 26-dimensional Minkowski metric. The currents $J^\mu(z)$ have the following mode expansions $J^\mu(z) = \sum_n a_n^\mu z^{-n-1}$.

Problem 6.1. *Canonical commutation relations*

Prove that the operator product algebra (1) implies the following commutation relations:

$$[a_n^\mu, a_m^\nu] = n\eta^{\mu\nu}\delta_{m+n,0} \tag{2}$$

The energy-momentum tensor of the currents is given by $T(z) = \frac{1}{2}\eta_{\mu\nu}(J^\mu J^\nu)(z)$. Expanding in modes we find $L_n = \frac{1}{2}\sum_{\ell\in\mathbb{Z}}\eta_{\mu\nu} : a_\ell^\mu a_{n-\ell}^\nu :$. Let d denote the BRST operator defined in Problem 5.3, but with the above expression for the L_n. We will now proceed to compute some of the low-lying states.

The BRST operator acts on the space $\bigwedge_{\infty/2}\otimes\mathscr{F}(p)$ spanned by monomials of the form

$$b_{-n_1}b_{-n_2}\cdots b_{-n_B}c_{-m_1}c_{-m_2}\cdots c_{-m_C}|0\rangle\otimes a_{-k_1}^{\mu_1}a_{-k_2}^{\mu_2}\cdots a_{-n_A}^{\mu_A}|p\rangle \,, \tag{3}$$

where $2 < n_1 < n_2 < \cdots < n_B$, $-1 < m_1 < m_2 < \cdots < m_C$, and $1 \le k_1 \le k_2 \le \cdots \le k_A$, and where $|p\rangle$ is the Fock vacuum of momentum p and $|0\rangle$ is the $SL(2,\mathbb{C})$-invariant vacuum of the ghost system.

The space $\bigwedge_{\infty/2}\otimes\mathscr{F}(p)$ is naturally bigraded.

Problem 6.2. *Ghost number*

One grading is the **ghost number**, which is the eigenvalue of the zero mode of the ghost current $j_{gh}(z) = -(bc)$. Prove that the ghost number of the monomial in (3) is given simply by $C - B$.

Problem 6.3. *Conformal weight*

The other important grading is the **conformal weight**, defined as the eigenvalue of $L_0^{tot} = [d, b_0]$.

Prove that the monomial in (3) has conformal weight given by $\sum_{i=1}^B n_i + \sum_{i=1}^C m_i + \sum_{i=1}^A k_i + \frac{1}{2}p^2$.

Prove that the cohomology of the BRST operator acting on the space $\bigwedge_{\infty/2}\otimes\mathscr{F}(p)$ is isomorphic to the one where d acts on the smaller subspace of states having conformal weight zero.

(*Hint:* use the fact that L_0^{tot} acts diagonally and that it is BRST exact.)

This result has several implications. First of all, since $\sum_{i=1}^B n_i \ge 0$, $\sum_{i=1}^C m_i \ge -1$, and $\sum_{i=1}^A k_i \ge 0$ it follows that $p^2 \le 2$ and moreover, since the above sums are all integers, it follows that p^2 has to be an even integer. Hence the allowed values of p^2

are $2, 0, -2, -4, \ldots$. Moreover, for a fixed allowed p there are only a finite number of states with zero conformal weight. Hence the BRST complex for a fixed p is quasi-isomorphic to a finite complex. We will now compute the cocycles and coboundaries for the first few levels.

Notice that momenta which are related by Lorentz transformations give rise to isomorphic cohomology. This follows because d only depends on the action of the Virasoro algebra and the expression of $T(z)$ is manifestly Lorentz invariant. Hence we are free to choose a convenient p in the mass shell to do the calculations.

Problem 6.4. *Tachyonic states: $p^2 = 2$*

One convenient choice of p is $p = (0, 0, \cdots, 0, \sqrt{2})$. Prove that the only states with $p^2 = 2$ and zero L_0^{tot} are $c_1 |0\rangle \otimes |p\rangle$ and $c_1 c_0 |0\rangle \otimes |p\rangle$. These states are created by the fields $c e^{ip\cdot X}$ and $c\partial c e^{ip\cdot X}$. Prove that they are both cocycles. Hence there are no coboundaries, and the BRST cohomology at $p^2 = 2$ is given by

$$H^1(p^2 = 2) \cong H^2(p^2 = 2) \cong \mathbb{R}$$

The physical states here are therefore nothing but the tachyon vertex operators.

Problem 6.5. *Generic massless states: $p^2 = 0$, $p \neq 0$*

All $p^2 = 0$ momenta with $p \neq 0$ are Lorentz-related, so we can choose one convenient value $p = (1, 0, \cdots, 0, 1)$. Prove that the allowed states are now:

$$\begin{array}{ll} |0\rangle \otimes |p\rangle & c_0 |0\rangle \otimes |p\rangle \\ c_1 c_{-1} |0\rangle \otimes |p\rangle & c_1 c_0 c_{-1} |0\rangle \otimes |p\rangle \\ c_1 |0\rangle \otimes a^{\mu}_{-1} |p\rangle & c_1 c_0 |0\rangle \otimes a^{\mu}_{-1} |p\rangle \end{array}$$

Prove that the cohomology is now given by

$$H^1(p^2 = 0, p \neq 0) \cong H^2(p^2 = 0, p \neq 0) \cong \mathbb{R}^{24} .$$

As representative cocycles we can choose (for the above choice of momentum) the states created by the fields $c\partial X^I e^{ip\cdot X}$ and $c\partial c\partial X^I e^{ip\cdot X}$, where $I = 1, \ldots, 24$ index the transverse directions.

Problem 6.6. *Exceptional massless states: $p = 0$*

The allowed states are the same, but now there is a minor explosion of cohomology. States which before were not cocycles become cocycles at $p = 0$, and hence some cocycles which before were coboundaries are no longer coboundaries. Prove that the cohomology is now given by

$$H^0(p = 0) \cong H^3(p = 0) \cong \mathbb{R}, \qquad H^1(p = 0) \cong H^2(p = 0) \cong \mathbb{R}^{26} .$$

Write down representative cocycles and the BRST invariant fields which create them.

Notice that 24 of the 26 states in H^1 and H^2 are already present in the generic case. The new states present at $p = 0$ are the critical bosonic string analogues of the discrete states in the noncritical string. In particular the unique physical state in H^0 (the $SL(2, \mathbb{C})$-invariant vacuum $|0\rangle \otimes |0\rangle$) now generates the "ground ring" which only has one element.

Problem 6.7. *The first massive level: $p^2 = -2$*

All such momenta are once again Lorentz-related, and we are free to choose, say,

$p = (\sqrt{2}, 0, \dots, 0)$. Verify that the allowed states are now given by

$$
\begin{array}{ll}
c_{-1}\,|0\rangle \otimes |p\rangle & c_0 c_{-1}\,|0\rangle \otimes |p\rangle \\
c_1 c_{-2}\,|0\rangle \otimes |p\rangle & c_1 c_0 c_{-2}\,|0\rangle \otimes |p\rangle \\
b_{-2} c_1\,|0\rangle \otimes |p\rangle & b_{-2} c_1 c_0\,|0\rangle \otimes |p\rangle \\
|0\rangle \otimes a^{\mu}_{-1}\,|p\rangle & c_0\,|0\rangle \otimes a^{\mu}_{-1}\,|p\rangle \\
c_1\,|0\rangle \otimes a^{\mu}_{-2}\,|p\rangle & c_1 c_0\,|0\rangle \otimes a^{\mu}_{-2}\,|p\rangle \\
c_1\,|0\rangle \otimes a^{\mu}_{-1} a^{\nu}_{-1}\,|p\rangle & c_1 c_0\,|0\rangle \otimes a^{\mu}_{-1} a^{\nu}_{-1}\,|p\rangle
\end{array}
$$

Prove that the cohomology is now given by

$$\mathrm{H}^1(p^2 = -2) \cong \mathrm{H}^2(p^2 = -2) \cong \mathbb{R}^{24} \oplus \mathrm{S}^2\mathbb{R}^{24} \,.$$

Find representative cocycles and BRST invariant states for the cohomology.

Observe the following patterns: except for the exceptional case ($p = 0$), we find that $\mathrm{H}^1 \cong \mathrm{H}^2$ is the only cohomology and that the count of physical states is identical with the number of states one can make out of the transverse oscillators a^{I}_{-n}. This is of course precisely the counting of states coming from the light-cone quantisation and in fact this remains true at all higher levels.

The difference between light-cone quantisation and the BRST quantisation occurs precisely at $p = 0$, where light-cone quantisation breaks down and where the BRST quantisation shows new physical states. In the critical bosonic string there are very few such states, but in noncritical strings they can be many such states and they are responsible for some interesting algebraic structures in the physical spectrum.