Lecture : a problem in Physics and duality as solution

Consider QCD as theory of "nuclear Physics" $\rightarrow$

$$\mathcal{L} = -\frac{1}{4} \vec{F}_{\mu\nu}^{\,2} + i\, \bar{\Psi}_K \not{D} \Psi_K + m_K \bar{\Psi}_K \Psi_K \qquad K = \left\{ \begin{array}{l} \text{up } 1 \\ \text{down } 2 \\ \text{Strange } 3 \\ \text{Charm } 4 \\ \text{bottom}^5, \text{top}^6 \end{array} \right\}$$

gauge group $SU(3)$

$$\Psi_K^{\alpha} \qquad \alpha = 1,2,3$$

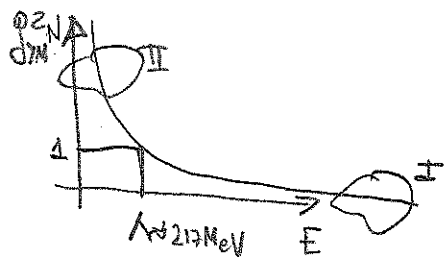
$$F_{\mu\nu}^{a} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} [A_\mu^b A_\nu^c]$$

$$A_\mu^{\alpha\beta} \equiv A_\mu^a \qquad a = 1, 2, \ldots, 8$$

$$D_\mu \Psi_K = \partial_\mu \Psi_K^\alpha + g A_\mu^{\alpha\beta} \Psi_K^\beta$$

[in general $SU(N)$ gauge group $\quad a: 1, \ldots, N^2-1$
$\alpha: 1, \ldots N.$]

in 1973 Gross, Wilczek and Politzer showed that the gauge coupling changes with energy as



The "problem of Physics" is then to get observational consequences from this theory.
$\rightarrow$ compute correlation functions of gauge invariant operators.

Example compute $\langle 0 | F_{\mu\nu}^a F^{\mu\nu a} | 0 \rangle$

In the region I this is "easy" $\rightarrow$
- just set the operators
- do Wick contractions $\rightarrow$ loops
- few loops give correct result

In the region II is very difficult as one doesn't have a natural expansion parameter.

So, this problem is solved by, for example (and for some particular observables) by Lattice $\longrightarrow$ non-perturbative definition of the QFT

Also other methods were developed

- Schwinger Dyson eq
- Instantons, and other field theory elaborations

$\Big\}$ partial results

An interesting idea is to propose a Duality

that is propose a field theory such that is weakly coupled when QCD is strongly coupled.

[Let me now depart a bit from QCD [we will keep it as our "inspiring" problem only]]

Let us study one example of duality that is very well understood

d = 1+1

System ①

ϕ: real scalar

System ②

ψ: Dirac fermion

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{\alpha}{\beta^2}\cos\beta\phi \quad (\beta^2 < 8\pi)$$

$$\mathcal{L} = i\bar{\Psi}\not{\partial}\Psi - \frac{g}{2}(\bar{\Psi}\gamma_\mu\Psi)^2$$

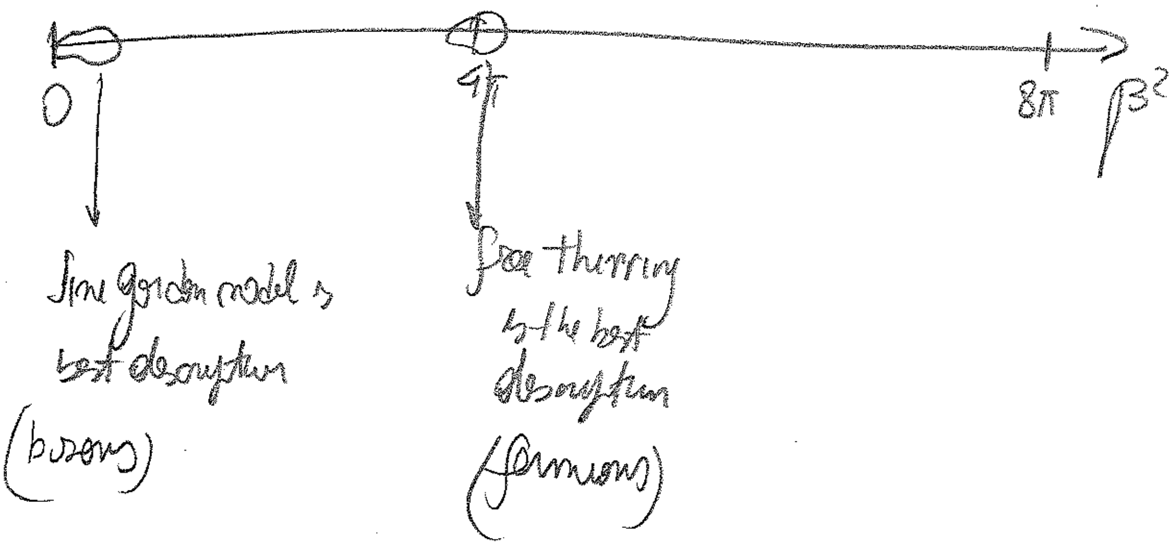
Coleman and Mandelstam Phys Rev D 1975 showed that both "Systems" (both theories) are actually the same $\rightsquigarrow$ describe the same Physics $\rightarrow$ Bosonization

Soliton $\longleftrightarrow$ fermion

$J_\mu = \varepsilon_{\mu\nu}\partial_\nu\phi \longleftrightarrow J_\mu = \bar{\Psi}\gamma_\mu\Psi$ $\rightsquigarrow$ $\boxed{\text{some global symm}}$

$4\pi/\beta^2 \longleftrightarrow 1+g$

So a picture to describe this duality is



Another duality — Seiberg 1994

$\mathcal{N}=1$ SQCD

QCD	SQCD
A_μ	A^a_μ, λ^a
ψ_i	$\psi^\alpha_i \; \phi^\alpha$
$\bar\psi_i$	$\tilde\psi^\alpha_i, \tilde\phi^\alpha_i$

Consider $\mathcal{N}=1$ SQCD with

$$SU(N_c) \quad \times \quad SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_R$$

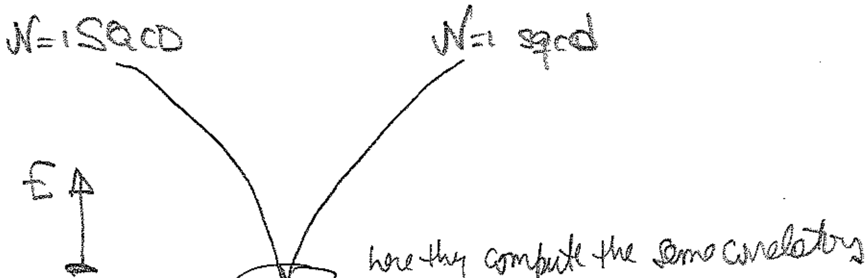
and consider the case in which $\underline{N_f > N_c + 1}$

Then Seiberg proposed that **another** version of SQCD in "magnetic" variables

$$\left.\begin{array}{l} a^a_\mu, \lambda^a \\ \psi, \varphi \\ \tilde\psi, \tilde\varphi \\ (m, \psi_m) \end{array}\right\} \begin{array}{l}\mathcal{N}=1 \\ \text{SQCD}\end{array} \qquad \cancel{SU(\ )} \times \quad SU(N_f - N_c) \times SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_R$$

and a particular interaction between (m, ψ_m) and the other fields $(\psi, \varphi), (\tilde\psi, \tilde\varphi)$

are dual to each other in the IR



Picture of Seiberg duality

- Notice that the global group is the same but the gauge group is not
- If $N_c + 2 \le N_f \le \frac{3}{2} N_c$ the magnetic theory is weakly coupled

The idea now is to present the Maldacena Conjecture / AdS-CFT correspondence as just another duality

The conjecture proposes the equivalence (duality) between a QFT ($\mathcal{N}=4$ SYM) and a string theory on a particular spacetime $AdS_5 \times S^5$.

Let us study both bits separately.

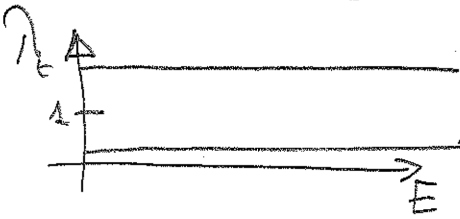
$\mathcal{N}=4$ SYM

A^a_μ ; $6\times\phi^a$, $4\times\lambda^a$ $a=1,\dots,N_c^2-1$ adjoint $SU(N_c)$ gauge group

$d=4=(3+1)$dim

global symmetries:
- conformal theory $SO(1,3) \longrightarrow SO(2,4)$ conformal group in 3+1
- 16 SUSY Poincare; 16 SUSY conformal [a bit on conformal algebra]
- Flavor-like symmetry = R-symmetry $SO(6)_R \simeq SU(4)_R$ ($SO(6)_R$ rotates scalars; $SU(4)_R$ rotates fermions)

't Hooft coupling $\lambda_t = g_{YM}^2 N_c$ → perturbative $\lambda < 1$; → non-perturbative $\lambda \gg 1$



→ need "dual" description

→ correlator is computed with Feynman diagrams

Let us see the string side

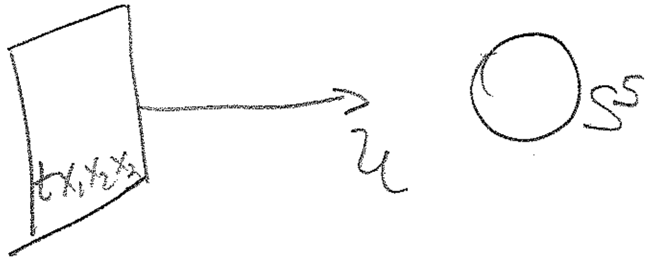
IIB string theory on $AdS_5 \times S^5$.

IIB string $\left\{ g_{\mu\nu}, \phi, \cancel{B_{\mu\nu}}, F_3, F_5 \right.$ truncate ⇝ $\begin{pmatrix} g_{\mu\nu} \\ F_5 \end{pmatrix}$

$$ds^2 = \alpha' \left[\underbrace{\frac{u^2}{R^2}(-dt^2+dx_1^2+dx_2^2+dx_3^2) + \frac{R^2}{u^2}du^2}_{AdS_5} + R^2 \underbrace{d\Omega_5}_{S^5} \right]$$

$\alpha' = \ell_s^2$ = length

coordinates $t, x_1 x_2 x_3$, u, $\Omega_5(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)$



There is also a generalized Maxwell field $\underline{F_{\mu\nu\rho\sigma\tau} = F_5}$

In order for this to be a solution it must have legs in

$F_{t x_1 x_2 x_3 u}$ and $F_{\theta_1\theta_2\theta_3\theta_4\theta_5}$.

Using a Gauss' law

$$\boxed{\int_{S^5} F_5 = N_c} ⇝ \boxed{R^4 = 4\pi g_s N_c = g_{YM}^2 N_c}$$

$R^2 = \sqrt{\lambda_t}$

Implies that there are N_c objects generating the solution

What are the Isometries here?

- $AdS_5 \to SO(2,4)$

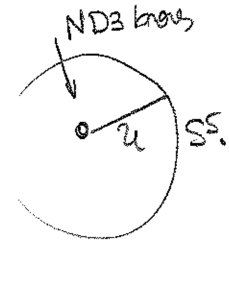
In general AdS_{p+1} ~~has~~ $SO(2,p)$

⇝ duals to conformal theories in $(p-1)$ dimensions

[for example 3-d CFT have AdS_4 duals]

S^5 has $SO(6) \to$ R-symmetry.

The solution is Maximally SUSY ⇝ 32 SUSY



$$\int_{S^5} F_5 = N_c$$

Notice, by looking at units that $[u]$ = Energy = $\frac{1}{\text{Length}}$ (R has no units). ($R^4 = \lambda_t$)

the radial coordinate is associated with the energy in the dual QFT.

In order to see this more intuitively; let us take AdS_5 in these coordinates

$$\frac{u^2}{R^2}\left(-dt^2+dx_1^2+dx_2^2+dx_3^2\right)+R^2\frac{du^2}{u^2}$$

and change $u \to \xi u$ then $\frac{du}{u}$ = invariant

(ξ: constant) $(t, X) \to \left(t, \frac{X}{\xi}\right)$ $u^2 d\vec{x}^2$ = invariant

This is scale invariance [part of the conformal group].

but notice that $X \to \frac{X}{\xi}$ implies that for large ξ X: decreases, u: increases

ξ large $\to$ going to smaller distances $\to$ higher Energy
going to larger radius

$\Rightarrow$ u large $\to$ large Energy in the QFT
u small $\to$ small " " "

Of course, in the <u>conformal theory</u> this argument is meaningless but as soon as we break conformality this will play a role.

Now, let us get a more pictoric view of this conjecture

Notice that

$\left(g_s \to 0 \text{ suppress string loops} \right)$

if we take $g_s \to 0$ but $N_c \to \infty$

$$\lambda = g_s N_c = g_{YM}^2 N_c = \text{fixed} \begin{cases} \to \text{large} \\ \to \text{small} \end{cases}$$

$\alpha' \to 0$ (suppress sigma model corrections)

does the decoupling if $\frac{r}{\alpha'} = \text{fixed}$

Notice that

$$\text{if } g_s N_c = g_{YM}^2 N_c = \lambda_{tH} \begin{cases} \to \text{large} \\ \quad \text{compared to "1"} \\ \to \text{small} \end{cases}$$

then the Radius of AdS or the Radius of S^5

$$\frac{R_{S^5}^4}{\alpha'^2} = \frac{R_{AdS}^4}{\alpha'^2} = g_s N_c = \lambda \begin{cases} \to \text{large} \\ \to \text{small} \end{cases} \text{compared to a string}$$

(do not confuse the Radius with the curvature $\alpha' R \sim \frac{1}{\sqrt{\lambda}}$)

the picture is

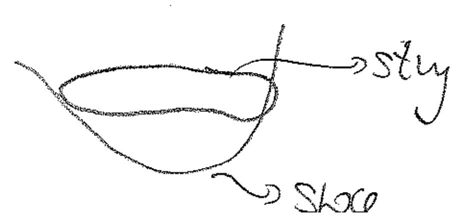
$\lambda \to \infty$

gauge theory is strongly coupled

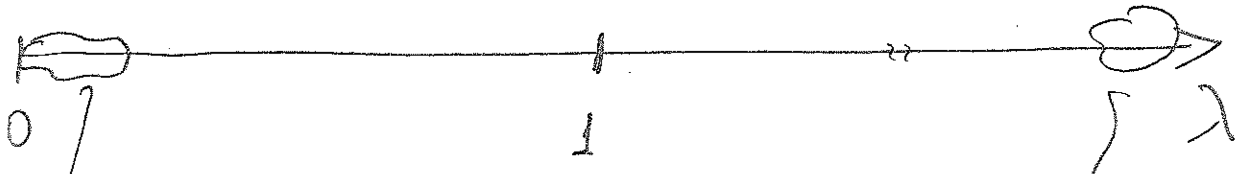
the string theory can be approximated by Supergravity (point particle limit)

$\lambda \to 0$

gauge theory is weakly coupled



So the picture of the duality is



best description in terms of $\mathcal{N}=4$ degrees of freedom

A_μ^a, $6\times\phi^a$, $4\times\lambda^a$

best description in terms of supergravity fields $\oplus$ semiclassical strings

This takes us almost immediately to the "operative" version of this duality Gubser-Klebanov-Polyakov $\oplus$ Witten 1998

$$= \int \exp\left[-\int J(x)\,\mathcal{O}(x)\,d^4x\right] \overset{\text{conjecture}}{\equiv} Z_{\text{IIB on } AdS_5\times S^5}\left[\Phi_{\text{bod}}(\text{boundary}) = J(x)\right]$$

generator of correlators in $\mathcal{N}=4$ SYM

$J \rightsquigarrow$ external current

$\mathcal{O} \rightsquigarrow$ any gauge invariant operators

partition function of IIB string theory on $AdS_5\times S^5$ with boundary conditions

$$\Phi\Big|_{\underbrace{\text{boundary}}_{R^{1,3}}} = J(x)$$

field of IIB string

If we are in the limit $\lambda \xrightarrow{\alpha' \to 0} \infty$

$$\underbrace{\left\langle e^{-\int d^4x\, J(x)\mathcal{O}(x)} \right\rangle}_{\text{difficult to compute}} \simeq e^{-S_{\text{IIB supergravity}}[\Phi \to J]}$$

In the next lecture we see an application of this.

<u>Brief mention</u>

$AdS_5 \times S^5 \xrightarrow{\text{reduced on } S^5}$ 5d Supergravity with a $SO(6)$ gauge field

Typically any global symmetry in the QFT is associated with a conserved current. $\underline{J_\mu} \rightsquigarrow$ any massless gauge field in the bulk is associated with a <u>conserved</u> <u>current</u> in the QFT

IIB supergravity and D3 branes

• Bosonic : $g_{\mu\nu}, \phi, \chi, F_3, F_5, H_3$

The Lagrangian in 10 dimensions contains various fields • fermionic : ψ_μ, λ.

$$\mathcal{L} = \sqrt{g}\left[R - \frac{1}{2}(\partial_\mu\phi)^2 - \frac{e^{2\phi}}{2}(\partial_\mu\chi)^2 - \frac{e^{-\phi}}{12}H_3^2 - \frac{e^{\phi}}{12}F_3^2 - \frac{1}{240}F_5^2\right] + C_4 \wedge F_3 \wedge H_3 + \begin{pmatrix}\text{fermionic} \\ \text{terms}\end{pmatrix}$$

Eqs of motion : $\boxed{F_5 = * F_5}$

$$\left.\begin{aligned} & R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = T_{\mu\nu} \\ & \text{with} \\ & T_{\mu\nu} = \tfrac{1}{2}(\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2} g_{\mu\nu}(\partial\phi)^2) + \tfrac{e^{2\phi}}{2}(\partial_\mu\chi\partial_\nu\chi - \tfrac{1}{2}g_{\mu\nu}(\partial\chi)^2) + \tfrac{e^{\phi}}{12}\left(F_{\mu\cdot\cdot}F_\nu{}^{\cdot\cdot} - \tfrac{g_{\mu\nu}}{12}F_3^2\right) \\ & \quad + \tfrac{e^{-\phi}}{12}\left(H_{\mu\cdot\cdot}H_\nu{}^{\cdot\cdot} - \tfrac{1}{12}g_{\mu\nu}H_3^2\right) + \tfrac{1}{240}F_{\mu\cdot\cdot\cdot\cdot}F_\nu{}^{\cdot\cdot\cdot\cdot} \end{aligned}\right\} \text{Einstein eqs.}$$

$$\left.\begin{aligned} & \partial_\mu[\sqrt{g}\, e^{\phi} F^{\mu\nu\rho}] = *[F_5 \wedge H_3] \\ & \partial_\mu[\sqrt{g}\, e^{-\phi} H^{\mu\nu\rho}] = e^{\phi}\partial_\mu\chi F^{\mu\nu\rho} - *(F_5 \wedge F_3) \\ & \partial_\mu[\sqrt{g}\, e^{2\phi}\partial^\mu\chi] = -e^{\phi} H_{\mu\nu\rho}F^{\mu\nu\rho} \\ & \partial_\mu[\sqrt{g}\, F^{\mu\nu_1\nu_2\nu_3\nu_4}] = -*(F_3 \wedge H_3) \end{aligned}\right\} \text{"Maxwell" eqs}$$

$$\left.\begin{aligned} & dF_1 = 0 \\ & dF_3 = H \wedge F_1 \\ & dH_3 = 0 \\ & dF_5 = H_3 \wedge F_3 \end{aligned}\right\} \text{Bianchi identities}$$

$$\boxed{\Box\phi = -e^{2\phi}(\partial\chi)^2 + \frac{e^{-\phi}}{12}H_3^2 - \frac{e^{\phi}}{12}F_3^2} \rightsquigarrow \text{dilaton}$$

Now, let us propose a <u>consistent</u> <u>truncation</u> [Pick a set of fields, the rest $\equiv 0$]

Pick $g_{\mu\nu}$, F_5 nonzero. The eqs reduce to

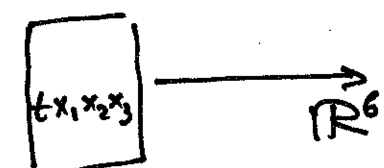
$$F_5 = {}^*F_5$$] Self-duality condition

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = \frac{1}{240} F_{\mu \circ\circ\circ\circ} F_{\nu}{}^{\circ\circ\circ\circ}$$] Einstein eqs.

$$\partial_\mu \left[\sqrt{g}\, F^{\mu \nu_1 \nu_2 \nu_3 \nu_4} \right] = 0 \quad , \quad dF_5 = 0$$] Maxwell and Bianchi

We will <u>propose</u> a solution where the $SO(1,9)$ invariance is broken to

$$SO(1,9) \longrightarrow SO(1,3) \times SO(6)$$



$$ds_{10}^2 = H_1(r)\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + H_2(r)\left(dr^2 + r^2 d\Omega_5\right)$$

$$F_5 = N_c H_3 \,\mathrm{vol}\,\Omega_5 \,(1 + *)$$

here we impose isotropy: things depend only on the distance to the <u>D3 branes</u> "r" and not on angular position

~~Plugging~~ Plugging this into ~~these~~ the eqs. one finds that a solution is

$$H_1 = \frac{1}{H_2} = \frac{1}{\sqrt{1+\frac{C^4}{r^4}}} \qquad C^4: \text{ integration constant.}$$

$$H_3 = \partial_r H_1$$

$$\left.\begin{array}{l} ds^2_{10} = \dfrac{1}{\sqrt{1+C^4/r^4}}\left(-dt^2+dx_1^2+dx_2^2+dx_3^2\right) + \sqrt{1+C^4/r^4}\,\left(dr^2+r^2\,d\Omega_5\right) \\[2ex] F_{tx_1x_2x_3r} = \partial_r \dfrac{1}{\sqrt{1+C^4/r^4}} \quad ; \quad F_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = * F_{tx_1x_2x_3r}. \end{array}\right\} \text{ Solves all the previous eqs.}$$

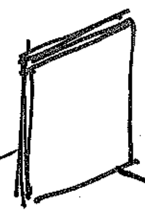
If we impose [Gauss' law]

$$\int_{S^5} F_5 \cong N_c \rightsquigarrow \text{ the integration constant turns out to be } \boxed{C^4 = 4\pi g_s N_c \alpha'^2}$$

$$\left.\begin{array}{l} \rightsquigarrow \; ds^2 = \hat{h}^{-1/2}\, dx^2_{1,3} + \hat{h}^{1/2}\left(dr^2 + r^2\, d\Omega_5\right) \; ; \; \boxed{\hat{h} = 1 + \dfrac{4\pi g_s N_c \alpha'^2}{r^4}} \\[2ex] F_5 = dt\wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge d\hat{h}^{-1}\,(1+*_{10}) \end{array}\right\} \text{ D3 brane Solution}$$

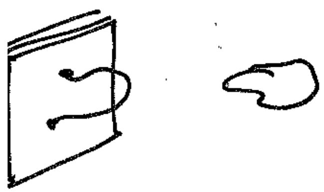
This is the string background corresponding to N_c D3 branes

Actually the picture is that N_c D3 branes in IIB string theory



N_c D3 branes

$g_s N_c$ smaller than 1



N_c D3 branes

[d]ynamics is described by open strings interacting with closed string + free closed strings in 10dim Minkowski

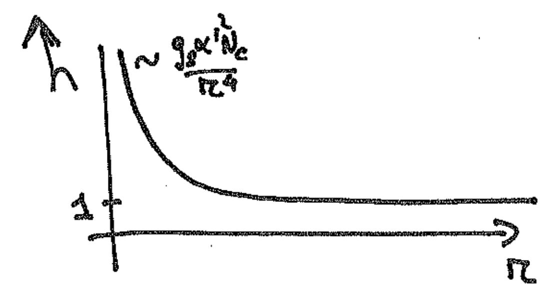
increasing $g_s N_c$ $\longrightarrow$

$g_s N_c$ bigger than 1

background solution

$ds^2 = h^{-1/2} dx_{1,3}^2 + h^{1/2} [dr^2 + r^2 d\Omega_5]$

with closed strings moving in it



There is an interesting "Low Energy Limit" [sometimes called "decoupling limit"]

that focuses these descriptions of the system into
- $\mathcal{N}=4$ SYM — $g_s N$ is small
- $AdS_5 \times S^5$ — $g_s N$ is large

Let us see how the decoupling limit acts on the "closed string side"

Suppose that we take our metric

$$ds^2 = \hat{h}^{-1/2}\, dx^2_{1,3} + \hat{h}^{1/2}\left(dr^2 + r^2\, d\Omega_5\right) \qquad \hat{h} = 1 + \frac{4\pi g_s N_c \alpha'^2}{r^4}$$

and define $\quad u = \dfrac{r}{\alpha'} \quad$ (notice $[u] = \text{Energy}$)

<u>Show that</u> this can be written as

$$\hat{h} = \frac{1}{\alpha'^2}\left(\alpha'^2 + \frac{4\pi g_s N_c}{u^4}\right)$$

$\rightsquigarrow$ now we take the limit $\alpha' \to 0$

keeping $\underline{\underline{u = \text{fixed}}}$

$$ds^2 = \alpha' \left\{ \underbrace{\frac{u^2}{\sqrt{4\pi g_s N_c}}\, dx^2_{1,3} + \sqrt{4\pi g_s N_c}\, \frac{du^2}{u^2}}_{AdS_5} + \underbrace{\sqrt{4\pi g_s N_c}\, d\Omega_5^2}_{S5} \right\}$$

with radius $R^4_{AdS} = 4\pi g_s N_c$

$$\boxed{R^4_{AdS_5} = R^4_{S5} = \lambda_{\text{t'Hooft}}}$$

Some points that I leave to be discussed

- How AdS/CFT $\rightsquigarrow$ "composite" graviton
  $\rightsquigarrow$ How this evades Weinberg-Witten No-go theorem

- Is SUSY **necessary** in AdS/CFT? $\to$ phase transitions (?)
  $\to$ Instability by particle Production

- why is large N_c needed?

$$\frac{R_{AdS}^4}{\alpha'^2} = \lambda_t \cong g_s N_c \longrightarrow \text{sigma model coupling } \frac{\alpha'}{R_{AdS}^2}$$

$$\frac{2\kappa_{10}^2}{\alpha'^4} = \frac{16\pi G_{10}}{\alpha'^4} = (2\pi)^7 g_s^2 = 8\pi^5 \cdot \frac{\lambda_t^2}{N_c^2} \rightsquigarrow G_{10} \sim \frac{1}{N_c^2} \longrightarrow \text{Quantum gravity correction} \equiv \text{non planar diagrams} \sim \frac{1}{N_c^2}$$

What is the Lagrangian for IIB?

For the massless modes. $5! = \overset{1\cdot2\cdot3\cdot4}{24\cdot5} = 120$

$$\mathcal{L} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{g}\left[R - \frac{1}{2}(\partial\phi)^2 - \frac{e^{-\phi}}{\underbrace{12}_{2.3!}}H_3^2 - \frac{e^{\phi}}{\underbrace{12}_{2.3!}}F_3^2 - \frac{e^{2\phi}}{2}(\partial\chi)^2 - \frac{1}{\underbrace{240}_{2.5!}}F_5^2\right] +$$

$$\frac{1}{2\kappa_{10}^2}\int C_4\wedge F_3\wedge H_3.$$

$$+\ \boxed{F_5 = {}^*F_5} \longleftrightarrow \text{eqs}$$

given this Lagrangian $\rightsquigarrow$ eqs of motion

Einstein eqs

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi\, T_{\mu\nu}$$

$$T_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2\right) + \frac{e^{2\phi}}{2}\left(\partial_\mu\chi\,\partial_\nu\chi - \frac{g_{\mu\nu}}{2}(\partial\chi)^2\right) + \frac{e^{\phi}}{12}\left(F_{\mu\cdot\cdot}F_\nu^{\ \cdot\cdot} - \frac{1}{12}g_{\mu\nu}F_3^2\right)$$

$$+\ \frac{e^{-\phi}}{12}\left(H_{\mu\cdot\cdot}H_\nu^{\ \cdot\cdot} - \frac{1}{12}g_{\mu\nu}H_3^2\right) + \frac{1}{240}F_{\mu\cdot\cdot\cdot\cdot}F_\nu^{\ \cdot\cdot\cdot\cdot}$$

Maxwell eqs

$$\partial_\mu\left[\sqrt{g}\,e^{\phi}F^{\mu\nu\rho}\right] = {}^*(F_5\wedge H_3)$$

$$\partial_\mu\left(\sqrt{g}\,e^{-\phi}H^{\mu\nu\rho}\right] = e^{\phi}F_\mu\hat{F}^{\mu\nu\rho} - F_5\wedge F_3$$

$$\partial_\mu\left[\sqrt{g}\,e^{2\phi}F^\mu\right] = -e^{\phi}H_{\mu\nu\rho}F^{\mu\nu\rho}$$

$$\partial_\mu\left(\sqrt{g}\,F^{\mu\nu_1\nu_2\nu_3\nu_4}\right] = -{}^*(F_3\wedge H_3)$$

Bianchi Id

$$dF_1 = 0$$

$$dF_3 = H\wedge F_1$$

$$dH = 0$$

$$dF_5 = -F_3\wedge H$$

Dilaton

$$\Box\phi = \partial\mathcal{L}$$

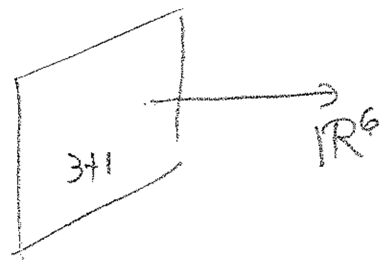
given these eqs people tried to find solutions → p branes of IIB

solutions in which $g_{\mu\nu}$ + some RR form is turned on
(essentially electric)

Examples

D3 brane $\qquad g_{\mu\nu}$, F_5

there you search for a solution that represents a (3+1)d object in 10d



$$\boxed{\mathbb{R}^6 = dr^2 + r^2 d\Omega_5^2}$$

$$ds^2 = H_1^{-1/2}(r)\, dx_{1,3}^2 + H_2^{1/2}(r)\,(dr^2 + r^2 d\Omega_5)$$

$$F_5 = N H_3 \,\mathrm{vol}\,\Omega_5 + (*)$$

motion eqs.

$$H_1 = \left(1 + \frac{a}{r^4}\right) = H_2$$

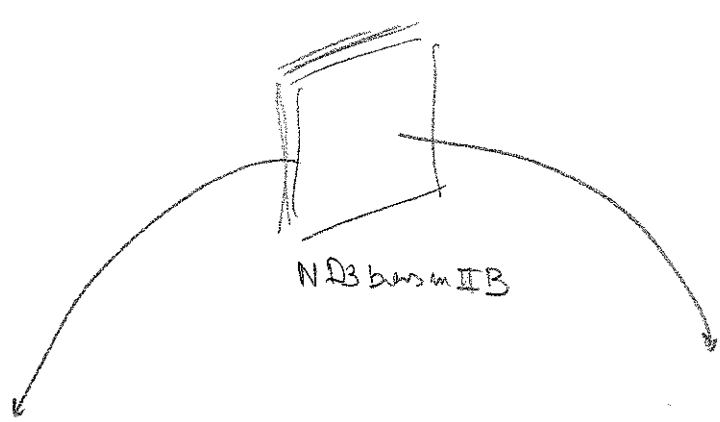
$$H_3 = H_1' = H'$$

↳ 3 brane solution

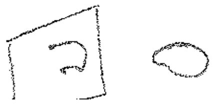
$$\boxed{a = g_s N \alpha'^2}$$ → quantization condition

$\int F = \# N_c$

Polchinski 1994 → a D brane is a p brane



$g_s N \to 0$



$g_s N \to \infty$

ds^2

F_5

So, we have <u>two descriptions</u> of the <u>same system</u> [N_c D_3 branes in IIB string theory]

g_s, α' parameters of the theory.

$g_s N_c \overset{!}{<} 1$ $\qquad\qquad$ $g_s N_c \overset{!}{>} 1$

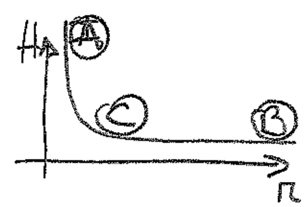
open strings attached to the D3 branes with closed strings in 10d

I

$\bar{F}_5$ (4)

a solution in IIB supergravity

$$ds^2 = H^{-1/2} dx^2_{1,3} + H^{1/2}(dr^2 + r^2 d\Omega_5)$$



$H = 1 + \frac{C^4}{r^4}$

$C^4 = g_s N_c \alpha'$

II

Let us write an "effective Lagrangian" describing these degrees of freedom

I

$$\mathcal{L} = \mathcal{L}_{4d}[\text{open}] + \mathcal{L}_{10d}[\text{closed strings}] + \mathcal{L}[\text{interaction}]$$

$$\mathcal{L}[\text{open}] = \mathcal{L}_{N=4\ SYM}[g_s N_c = \lambda] + \int \alpha' \cdot \{F^4_{\mu\nu} + \dots\}\, d^4x + \dots$$

$$\mathcal{L}[\text{int}] = \frac{\alpha'}{4}\int d^4x \left[\Phi F^2_{\mu\nu} + \chi F_{\mu\nu}\tilde{F}_{\mu\nu} + h_{\mu\nu}T_{\mu\nu} + \dots\right]$$

$$\mathcal{L}[\text{closed}] = \int d^{10}x\ \mathcal{L}[\text{closed strings in 10d Minkowski}]$$

II

$$\mathcal{L}_A = \mathcal{L}[\text{closed string on } AdS_5\times S^5]$$

$$\mathcal{L}[B] = \mathcal{L}[\text{closed strings on interpolating region}]$$

$$\mathcal{L}[C] = \mathcal{L}[\text{closed string on Minkowski 10d}]$$

Let us study a bit more closely the description II

We will take the metric

$$ds^2 = H^{-1/2}\, dx_{1,3}^2 + H^{1/2}\, dr^2 + r^2 d\Omega_5$$

up to factors of 4π's that do not affect the idea here

$$H = 1 + \frac{g_s N_c \alpha'^2}{r^4}$$

we will define a new radial coordinate

$$u = \frac{r}{\alpha'}$$ notice $[u] = $ Energy

$$H = 1 + \frac{g_s N_c}{u^4 \alpha'^2}$$

and we will take the limit

$\alpha' \longrightarrow 0$ keeping $u = $ fixed $\begin{cases}\to \text{small} \\ \to \text{large}\end{cases}$ $\to$ fixed Energy

$$H = \frac{1}{\alpha'^2}\left(\alpha'^2 + \frac{g_s N_c}{u^4}\right)$$

$$ds^2 = \alpha' \left[\alpha'^2 + \frac{g_s N_c}{u^4}\right]^{-1/2} dx_{1,3}^2 + \left(\alpha'^2 + \frac{g_s N_c}{u^4}\right)^{1/2} \frac{1}{\alpha'}\left(\alpha'^2 du^2 + u^2 \alpha'^2 d\Omega_5\right)$$

$$ds^2 = \alpha' \left\{ \left[\alpha'^2 + \frac{g_s N_c}{u^4}\right]^{-1/2} dx_{1,3}^2 + \left(\alpha'^2 + \frac{g_s N_c}{u^4}\right)^{1/2} \left[du^2 + u^2 d\Omega_5\right]. \right\}$$

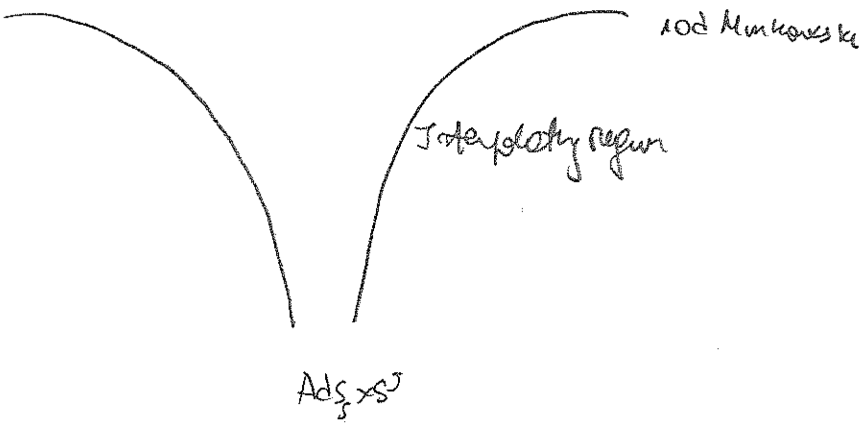
and now when $\alpha' \longrightarrow 0$

$$ds^2 = \alpha' \cdot \left\{ \underbrace{\frac{u^2}{\sqrt{g_s N_c}}\, dx_{1,3}^2 + \sqrt{g_s N_c}\,\frac{du^2}{u^2}}_{AdS} + \underbrace{\sqrt{g_s N_c}\; d\Omega_5}_{S^5} \right\}$$

$\boxed{\text{note } R_{AdS}^2 = R_{S^5}^2 = \alpha' \sqrt{g_s N_c}}$

So, taken $\alpha' \to 0$ has decoupled $AdS_5 \times S^5$, but since we keep fixed Energy we should also consider the modes in $Mink_{10}$ why is this?

Plot of spacetime



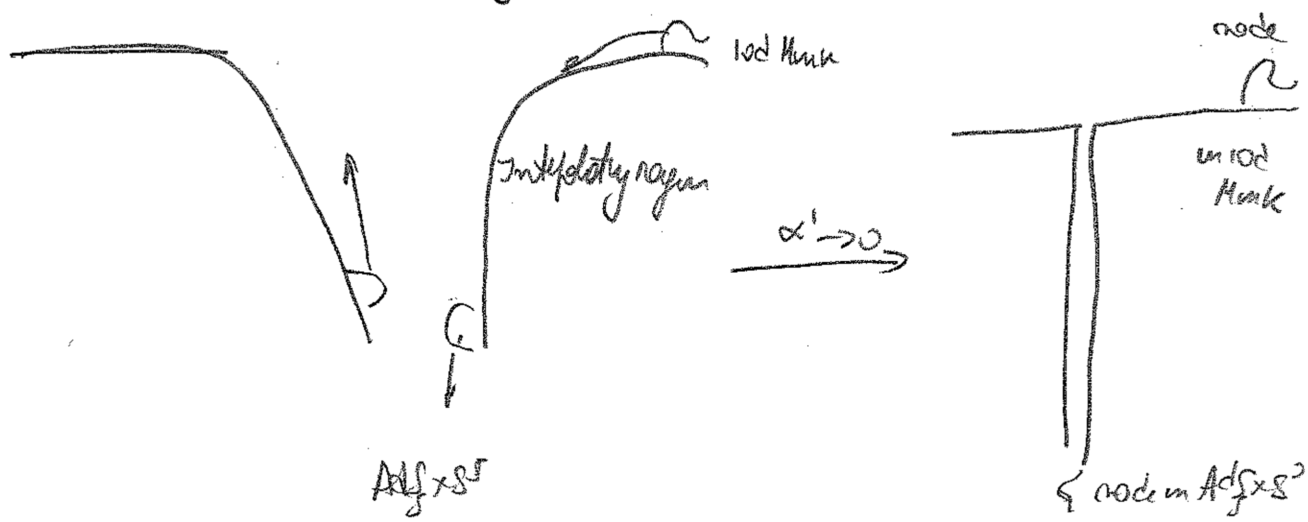
$$E = \sqrt{g_{tt}}\; \mathcal{E}_{\text{proper}}$$

measured at ∞

$$\sqrt{g_{tt}} = \frac{u}{(g_s N_c)^{1/4}}$$

even if a mode has very small Energy for large enough "u" we will need to consider it.

what the $\alpha' \to 0$ limit is doing is this



$$\mathcal{L} = \mathcal{L}\,[\text{closed string on } AdS_5 \times S^5] + \mathcal{L}_{\text{interpolating}} + \mathcal{L}\,[\text{closed string on 10d Mink}]$$

$\to \alpha' \to 0 \rightsquigarrow \mathcal{L} = \mathcal{L}\,(\text{IIB on } AdS_5 \times S^5) + \mathcal{L}\,[\text{closed string in 10d Mink}]$

$\Longrightarrow$ the two descriptions of the same system

I

$\mathcal{L} = \mathcal{L}_{4d}\,[\text{open}] + \mathcal{L}_{4d}\,[\text{int}] + \mathcal{L}_{10d}\,(\text{strings in Mink})$

II

$\mathcal{L}_{(A)} + \mathcal{L}_{(B)} + \mathcal{L}_{(C)}$

$\alpha' \to 0$

$\mathcal{L} = \mathcal{L}\,[\mathcal{N}=4 \text{ SYM}] + \mathcal{L}_{10d}\,[\text{IIB } \cancel{\text{string on Mink}_{10}}]$

$\mathcal{L} = \underline{\mathcal{L}\,[\text{IIB } \cancel{\text{closed}} \text{ string on } AdS_5 \times S^5]} + \mathcal{L}\,[\cancel{\text{IIB on 10d Mink}}]$

Esto me induce a pensar así

Versions of the conjecture

$\mathcal{N}=4$ SYM	Strings on $AdS_5 \times S^5$
g_{YM}	g_s, N, α'
N.	

Strong form

$\forall\, g_{YM}$ $\quad\quad$ $\forall\, g_s$, $\forall \alpha$, $\forall N$

$\forall\, N$

$$\boxed{\mathcal{N}=4 \text{ SYM} \longleftrightarrow \text{strings on } AdS_5 \times S^5}$$

$R^4_{AdS} = \sqrt{2 g^2_{YM} N}$

$R^4_{AdS} = \alpha'^2 g_s N$.

any value!

So, one could compute a Feynman graph by doing strings.

mild

$g_{YM} \to 0$ $\quad\quad$ $g_s \to 0$ $\to$ $\boxed{\text{no string loop}}$

$\forall N$ $\quad\quad$ $\forall N$, $\forall \alpha'$

$\mathcal{N}=4$ SYM $\longleftrightarrow$ classical strings on $AdS_5 \times S^5$

$g^2_{YM} N$ is small (J fijo)

$R^4_{AdS} = \alpha'^2 g_s N$ small $\left(\frac{\text{Radius}}{\alpha'^2}\right)$

$T_{string} = \frac{1}{\alpha'}$ fijo any value

weak

$g^2_{YM} \to 0$ $\quad\quad$ $\alpha' \to 0$ $\quad$ $\boxed{\text{No worldsheet corrections}}$

$N \to \infty$ $\quad\quad$ $N \to \infty$

$\mathcal{N}=4$ SYM at

$\lambda = g^2_{YM} N$ fixed $\longleftrightarrow$ $\frac{R^4_{AdS}}{\alpha'^2} = g_s N \to$ large

---

Nota pp waves

el límite allí es $\quad$ g_s = fixed and small $\quad$ $g^2_{YM} N \to \infty$ $\quad$ $\frac{\lambda}{J^2}$ = fixed

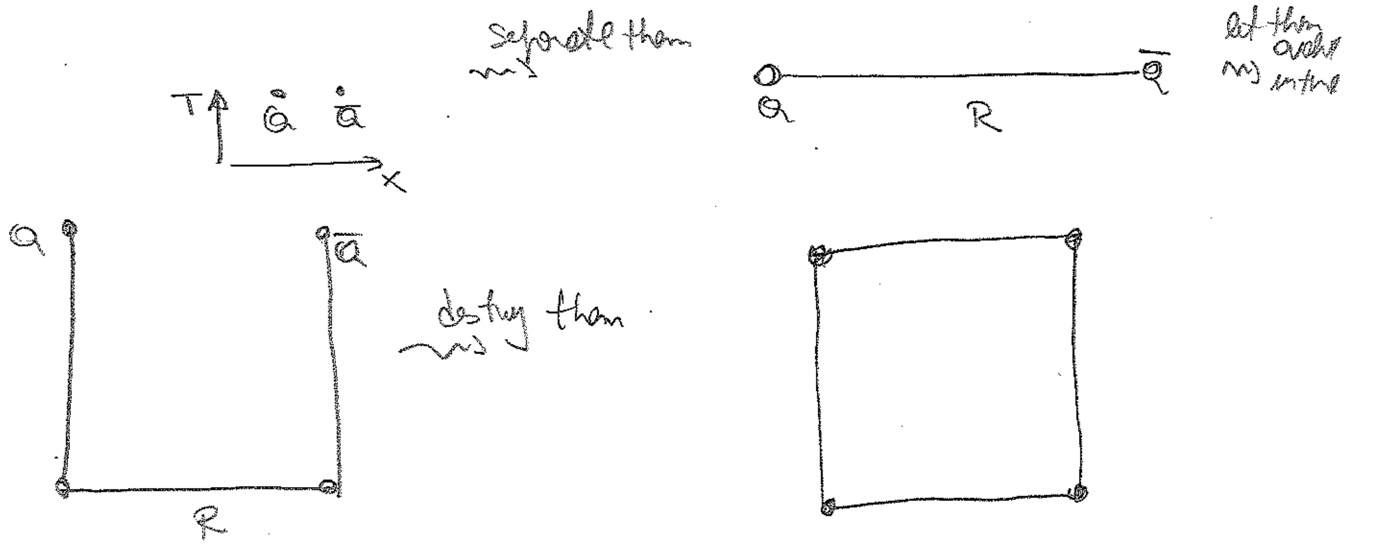
$\alpha' \to 0$ $\quad$ $J \to \infty$

Wilson loops

Wilson (1973) defined an object - gauge invariant - devised to measure the potential between a pair of external [non-dynamical] Quark - anti Quark

$$\langle W \rangle = \langle \mathrm{Tr}_P\, e^{\oint A_\mu dx^\mu} \rangle \cong e^{-V_{Q\bar{Q}}(R)\cdot T}$$

The idea: take a $Q\bar{Q}$ pair non dynamical (very heavy)



~~Calculate the interaction between the quarks and the gauge field~~

$$\cancel{\langle \bar{Q}(R,0)\, Q(0,0) \rangle = \int DA_\mu\, e^{-S[A]}\, \bar{Q}(R,0)\, e^{-\int_0^R A_x dx}\, Q(0,0)}$$

~~Similarly~~

The Action of the $Q\bar{Q}$ pair does not have a kinetic term and has a potential term

$$S_{E\,Q\bar{Q}} = -\int_0^T dt\, V_{Q\bar{Q}}(R) = -T\, V(R).$$

To compute the action we consider the interaction

$$S = \oint J_\mu A_\mu\, d^4x \quad \text{where} \quad J_\mu = \begin{cases} [\delta^3(x) - \delta^3(x-R)]\,\delta_{\mu t} \\ \delta_{\mu x}\, \theta\, \delta(t) \end{cases}$$

$$\int d^4x\, J_\mu A_\mu = \int_0^R A_x(x,0)\,dx + \int_0^T dt\,\left(A_t(0,t) - A_t(R,t)\right) + \int_R^0 A_x(x,T)\,dx = \oint A_\mu dx^\mu$$

$$-T \cdot V(R) = S_{Q\bar{Q}} = \oint A\,dx$$

$$e^{\oint A dx} = e^{-T\,V(R)}$$

So, the important thing is that

$$\underline{\left\langle e^{\oint A dx} \right\rangle = e^{-T\,V_{Q\bar{Q}}(R)}}$$

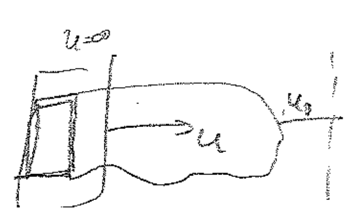
how to calculate this in AdS/CFT ?

The idea is the same as on GKP, W. We need to use a non-local object to compute the vev of a non-local operator

So we will use a string.

The proposal Maldacena, Rey-Yee 1998 is that $\left\langle e^{\oint A dx} \right\rangle = e^{-T \cdot V_{Q\bar{Q}}(R)}$

$$\simeq e^{-S_{NG}}\big|_b$$



Computation of Wilson Loop

Maldacena

Rey-Yee 1998

Proposal

conjecture

$$\langle W \rangle = \langle P\, e^{\oint_C A\,dx} \rangle \equiv Z_{string}[C] \approx e^{-S_{string}}\Big|_{C}$$

$\lambda \to \infty$

String is semiclassical

Let us then compute

$$S_{string} = \frac{1}{2\pi\alpha'}\int \sqrt{-\det G_{\alpha\beta}}\; d\sigma\, d\tau$$

$$ds^2 = \alpha'\left[\frac{u^2}{R^2}\,dx_{1,3}^2 + \frac{R^2}{u^2}\,du^2 + R^2\, d\Omega_5\right]$$

$\boxed{R^4 = g_s N_c 4\pi}$ $\quad [R] = L^0$

$[u] = \frac{R}{\alpha'} \to [u] = \frac{1}{L}$

Embedding

$t = \tau$

$x = \sigma$

$u = u(\sigma)$

Induced metric

$$G_{\alpha\beta} = \begin{array}{c|cc} & \sigma & \tau \\ \hline \sigma & g_{xx}x'^2 + g_{uu}u'^2 & 0 \\ \tau & 0 & g_{tt}\dot{t}^2 \end{array} = \begin{bmatrix} g_{xx} + g_{uu}u'^2 & 0 \\ 0 & g_{tt} \end{bmatrix}$$

$$G_{\alpha\beta} = g_{\mu\nu}\,\partial_\alpha X^\mu \partial_\beta X^\nu$$

$$\Rightarrow \boxed{\det G_{\alpha\beta} = g_{tt}\left(g_{xx} + g_{uu}u'^2\right)}$$

We compute this induced metric

$$
g_{\alpha\beta} = \begin{array}{c|cc} & \sigma & \tau \\ \hline \sigma & \frac{\alpha' u^2}{L^2} + \alpha' \frac{L^2}{u^2} u'^2 & 0 \\ \tau & 0 & -\frac{u^2}{L^2}\alpha' \end{array}
$$

$$
-\det g_{\alpha\beta} = \alpha'^2 \left[\left(\frac{u^2}{L^2}\right)\left(\frac{u^2}{L^2} + \frac{L^2}{u^2} u'^2\right)\right] \rightsquigarrow \sqrt{\det g_{\alpha\beta}} = \alpha' \cdot \frac{\cancel{u}}{\cancel{L}} \sqrt{\left(\frac{u^4}{L^4} + u'^2\right) \frac{\cancel{L^2}}{\cancel{u^2}}}
$$

$$
\int \sqrt{\det g_{\alpha\beta}}\, d^2\sigma = \frac{\cancel{\alpha'}}{2\pi\cancel{\alpha}} \int \sqrt{u'^2 + u^4/L^4}\; d\sigma\, d\tau =
$$

$$
\boxed{S_{NG} = \frac{\mathcal{T}}{2\pi} \cdot \int d\sigma \sqrt{u'^2 + u^4/L^4}}
$$

$\rightarrow$ Classical Mechanics System

$\rightarrow$ conserved Energy

Hamiltonian $\quad H = p_u u' - \mathcal{L}$

$$
p_u = \frac{\partial \mathcal{L}}{\partial u'} = \frac{u'}{\sqrt{u'^2 + u^4/L^4}}
$$

$$
\Rightarrow H = \frac{u^4/L^4}{\sqrt{u'^2 + u^4/L^4}} = \text{constant} = c \overset{u'=0}{=} \frac{u_0^2}{L^2}
$$

From here we compute $\frac{du}{dx}$ $\rightsquigarrow$ $\left(\frac{cL^4}{u^4}\right)^2 \left(u'^2 + u^4/L^4\right) = 1 \rightsquigarrow$

$$
u'^2 = \frac{u^8}{c^2 L^8} - \frac{u^4}{L^4} \Rightarrow \frac{du}{dx} = \sqrt{\frac{u^4}{L^4}\left[\frac{u^4}{L^4} - c^2\right]}
$$

$$\rightarrow \quad dx = \frac{cL^2\, du}{u^2\sqrt{\frac{u^4}{L^4} - c^2}}$$

$$L_{QQ} = \int_{u_0}^{\infty} \frac{c}{\left(u^2/L^2\right)} \frac{du}{\sqrt{u^4/L^4 - c^2}}$$

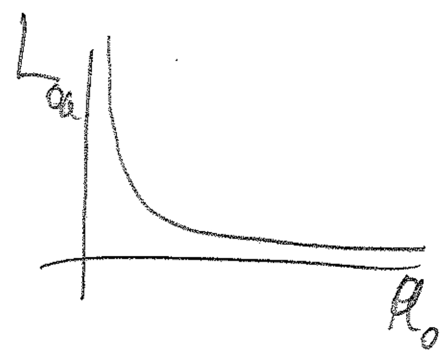
notice that $c = \frac{u_0^2}{L^2}$

$$\boxed{L_{QQ} = 2u_0^2 \int^{\infty} \frac{du}{u^2\sqrt{u^4/L^4 - c^2}}}$$

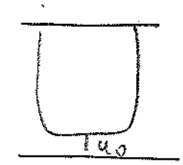
$\rightsquigarrow$ can be computed explicitly

Check that it gives

$$\boxed{L_{QQ} = \frac{(2\pi)^{3/2} L^2}{\Gamma(1/4)^2} \frac{1}{u_0}}$$



OK with naive expectation
large $u_0 \to$ small L
small $u_0 \to$ large L



The Energy of the $Q\bar{Q}$ pair

use $u'^2 = \frac{u^4}{c^2L^4}\left[\frac{u^4}{L^4} - c^2\right]$

$$E_{QQ} = \frac{S_{NG}}{T}\Big|_{\text{renormalized}} = \int d\sigma \sqrt{u'^2 + u^4/L^4} = \int d\sigma \sqrt{\frac{u^8}{c^2L^8} - \frac{u^4}{L^4} + \frac{u^4}{L^4}} = \int_{u_0}^{\infty} du\, \frac{u^4}{cL^4} \frac{d\sigma}{du}$$

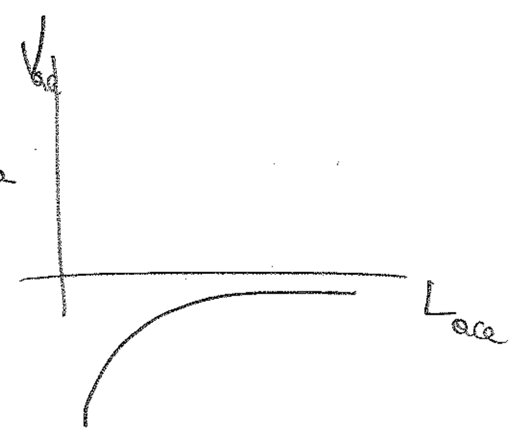
$$E_{Q\bar{Q}} = 2\int_{u_0}^{\infty} du \, \frac{u^4}{cL^4} \cdot \overbrace{\frac{cL^2}{u^2\sqrt{\frac{u^4}{L^4} - c^2}}}^{\frac{dx}{du}} \approx \int^{\infty} du \, \frac{u^4}{u^2 \cdot u^2} \longrightarrow \text{diverge linearly}$$

So we need to renormalize

$$E_{Q\bar{Q}} = \frac{S_{NG}}{T} - \underbrace{\frac{2}{2\pi}\int_0^{\infty} \sqrt{g_{tt}\, g_{uu}} \, du}_{\text{action of straight strings}} \qquad \sqrt{g_{tt}\, g_{uu}} = \alpha'$$

$$E_{Q\bar{Q}} = \frac{1}{2\pi}\left[2\int_{u_0}^{\infty} du \, \frac{u^2}{L^2\sqrt{\frac{u^4}{L^4} - c^2}} - 2\int_{u_0}^{\infty} 1 - 2\int_0^{u_0} 1 \right]$$

$$E_{Q\bar{Q}} = -\frac{(2\pi)^3 L^2}{[\Gamma(1/4)]^4} \frac{1}{L_{Q\bar{Q}}} \sim -\frac{\sqrt{\lambda}}{L_{Q\bar{Q}}}$$



Coulomb-law!

Let us now do this for a generic metric

$$ds^2 = g_{tt}\, dt^2 + g_{xx}\, dx^2 + g_{\rho\rho}\, d\rho^2 + \dots$$

The configuration for the string $\begin{cases} x = \sigma \\ t = \tau \\ \rho = \rho(\sigma) \end{cases} \rightsquigarrow g_{\alpha\beta} = \begin{array}{c|cc} & \tau & \sigma \\ \hline \tau & g_{tt} & 0 \\ \sigma & 0 & g_{xx} + g_{\rho\rho}\rho'^2 \end{array}$

$$\det g_{\alpha\beta} = g_{tt}\left[g_{xx} + g_{\rho\rho}\rho'^2\right]$$

$$S_{NG} = \frac{T}{2\pi\alpha'} \int d\sigma \sqrt{-[g_{tt}g_{xx} + g_{tt}g_{\rho\rho}\rho'^2]}$$

define $\begin{cases} f^2 = -g_{tt}g_{xx} \\ g^2 = -g_{tt}\, g_{\rho\rho} \end{cases}$

$$S = \frac{T}{2\pi\alpha'} \int d\sigma \sqrt{f^2 + g^2\rho'^2} \longrightarrow \text{conserved hamiltonian}$$

$$p_\rho = \frac{g^2\rho'}{\sqrt{f^2 + g^2\rho'^2}}$$

$$H = p_\rho \rho' - \mathcal{L} = \frac{g^2\rho'^2}{\sqrt{f^2+g^2\rho'^2}} - \sqrt{f^2+g^2\rho'^2}$$

$$H = \frac{-f^2}{\sqrt{f^2+g^2\rho'^2}} = \overset{\text{constant}}{-f(\rho_0)} \rightsquigarrow \frac{f^4}{f_0^2} = f^2 + g^2\rho'^2$$

$$\frac{f^2}{g^2}\cdot\left[\frac{f^2}{f_0^2} - 1\right] = \rho'^2 \longrightarrow \frac{d\rho}{dx} = \frac{f}{f_0\, g}\sqrt{f^2 - f_0^2} \rightsquigarrow$$

$$dx = f(\rho_0)\,\frac{g}{f\sqrt{f^2-f_0^2}}\,d\rho$$

$$\boxed{L_{QQ} = 2f(\rho_0)\cdot\int_{\rho_0}^{\infty}\frac{g}{f}\,\frac{d\rho}{\sqrt{f^2-f^2(\rho_0)}}}$$

$g_{tt} = \frac{u^2}{R^2} = g_{xx}$

$g_{uu} = \frac{R^2}{u^2}$

for AdS case

$$\left(\begin{array}{l} f^2 = \frac{u^4}{R^4} = -g_{tt}\,g_{xx} \\ g^2 = 1 = g_{tt}\,g_{uu} \end{array}\right) \rightarrow f = \frac{u^2}{R^2}$$

$$L = 2\,\frac{u_0^2}{R^2}\int_{\rho_0}^{\infty}\frac{R^4}{u^2}\,\frac{d\rho}{\sqrt{u^4-u_0^2}}$$

$$\boxed{L = 2R^2u_0^2\int_{u_0}^{\infty}\frac{d\rho}{u^2\sqrt{u^4-u_0^4}}}$$

Let us now

Compute the Energy

$$E_{QQ} = \frac{1}{T}S_{NG} = \int d\sigma\,\sqrt{f^2+g^2\rho'^2} - \underbrace{2\int_0^{\infty} g\,d\rho}_{\text{renormalization}}$$

$$d\sigma = f(\rho_0)\,\frac{g}{f}\,\frac{d\rho}{\sqrt{f^2-f^2(\rho_0)}}$$

$$E_{QQ} = 2f(\rho_0)\int_{\rho_0}^{\infty}d\rho\,\frac{g}{f\sqrt{f^2-f_0^2}} - 2\int_0^{\infty} g\,d\rho$$

$$\boxed{E_{QQ} = f(\rho_0)\cdot L_{QQ} + 2\int_{\rho_0}^{\infty}\frac{g}{f}\left[\sqrt{f^2-f_0^2}-f\right]d\rho - 2\int_0^{\rho_0} g\,d\rho}$$

$f(\rho_0) \neq 0 \rightsquigarrow E = \sigma L + \text{corrections} \rightarrow$ confinement

Correlation functions

Philosophy

AdS	CFT
field in AdS	operator in CFT
Spin	spin
mass	conformal dimension (scaling Δ)

As usual in any QFT, the object of interest is

$$\langle e^{\int \phi_0(x)\, \mathcal{O}(x)} \rangle$$

$\phi_0(x) = \text{Sources} = J(x)$

$\mathcal{O}(x) = \text{given operator}$

Since $\dfrac{\delta^n}{\delta J_0(x_1) \cdots \delta J_0(x_n)} \langle e^{\int J_0(x)\mathcal{O}(x)} \rangle = \langle \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle$

$$\frac{\delta^n}{\delta J_0(x_1) \cdots \delta J_0(x_n)} \log Z[J] \Big|_{J=0}$$

; where $\boxed{Z[J] = \frac{1}{Z(0)} \int \mathcal{D}\text{fields}\; e^{-S_{QFT} + \int \vec{J}\vec{\mathcal{O}}}}$

The proposal of AdS/CFT (GKP, Witten) 1998

$$\langle e^{-\int J_0(x)\mathcal{O}(x)} \rangle_{CFT} \equiv Z_{\substack{\text{IIB on} \\ AdS_5 \times S^5}} \Big[\overset{\text{fields of IIB}}{\{\phi\}_{IIB}} \xrightarrow[z \to 0]{} \overset{\text{Boundary conditions}}{\{J\}} \Big]$$

but the full partition function of IIB on $AdS_5 \times S^5$ we do not know how to compute as implies quantizing string theory in $AdS_5 \times S^5$.

So we better approximate it

$$Z_{\substack{\text{IIB on } AdS_5\times S^5}}\left[\{\phi_{IIB}\}\to\{\vec{J}_0\}\right] \underset{\text{Saddle point}}{\approx} e^{-S_{\text{IIB Sugra}}[\phi_{IIB}\to\{J_0\}]}\Big|$$

this is evaluated on the eqs of motion with boundary conditions $\{\phi\}\Rightarrow\{J\}$

when is this a good approximation?

Since the radius of AdS_5 $\quad R^4_{AdS_5} = (g_s N)\alpha'^2 \to \dfrac{R^4}{\alpha'^2} = g_s N = \lambda$

if $\underline{\lambda \to \infty}$

it is OK to approximate and neglect all stringy corrections.

So, what we will do is to study a 2-point correlator for a "field on AdS_5" that we will assume is one of the IIB fields fluctuating on the background.
we will do this in generic dimension "$d+1 \equiv AdS_{d+1}$"

$$S = \int d^{d+1}x \, \sqrt{g_{AdS_{d+1}}}\left[g^{AB}\partial_A\phi\partial_B\phi + m^2\phi^2 + \lambda\phi^3 + \cdots\right]$$

For the purpose of 2-point correlators we will ignore $\lambda\phi^3$ and higher terms

Aside: Let us compute the eq of motion

$$S = \int d^{d+1}x \sqrt{g}_{AdS_{d+1}} \left[g^{AB} \partial_A \phi \partial_B \phi + m^2 \phi^2 \right]$$

where the space time is

$$ds^2 = \left(\frac{L}{z}\right)^2 \left[dx^2_{1,d-1} + dz^2 \right] \quad \rightsquigarrow \underline{AdS_{d+1}}$$

$$\sqrt{g} = \left(\frac{L}{z}\right)^{d+1}$$

Let us see the eq of motion

$$\partial_K \frac{\partial \mathcal{L}}{\partial(\partial_K \phi)} = \partial_K \left[2 \sqrt{g}\, g^{KB} \partial_B \phi \right]$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = 2\sqrt{g}\, m^2 \phi$$

$$\boxed{\partial_K \left[\sqrt{g}\, g^{KB} \partial_B \phi \right] = \sqrt{g}\, m^2 \phi}$$

$$\downarrow$$

$$\underbrace{\frac{1}{\sqrt{g}} \partial_K \left[\sqrt{g}\, g^{KB} \partial_B \phi \right]} = m^2 \phi$$

$$\boxed{\Box \phi = m^2 \phi}$$

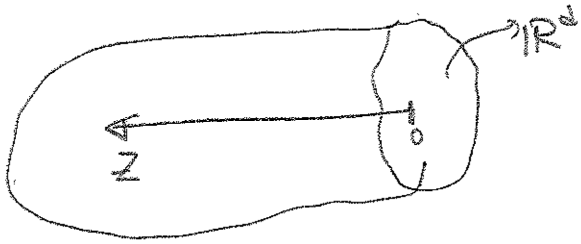
So, let us work back with the action

$$S = \int d^{d+1}x \sqrt{g} \left[g^{AB} \partial_A \phi \partial_B \phi + m^2 \phi^2 + \dots \right]$$

we will integrate by parts

$$S = \int d^{d+1}x \; \partial_A \left[\sqrt{g}\, g^{AB} \partial_B \phi \cdot \phi \right] - \partial_A \left[\sqrt{g}\, g^{AB} \partial_B \phi \right] \cdot \phi + m^2 \phi^2 \sqrt{g} + \dots \quad (*)$$

Now, we will use the fact that AdS_{d+1} has a boundary at $Z=0=\varepsilon$
regulator
and the interior is at $Z=\infty$



$$\int d^{d+1}x\ \partial_A\left[\sqrt{g}\, g^{AB}(\partial_B\phi)\cdot\phi\right] = \int d^dx\, dz\ \partial_Z\left[\sqrt{g}\, g^{ZB}(\partial_B\phi)\phi\right]$$

$$= \int d^dx\ \sqrt{g}\, g^{ZB}(\partial_B\phi)\,\phi\Big|_{Z=\varepsilon}$$

Notice that

$$\left.\begin{aligned} \sqrt{g}\Big|_{Z=\varepsilon} &= \left(\frac{L}{\varepsilon}\right)^{d+1} \\ g^{ZB}\Big|_{\varepsilon} = g^{ZZ}\Big|_{\varepsilon} &= \left(\frac{Z}{L}\right)^2\Big|_{\varepsilon} = \frac{\varepsilon^2}{L^2} \\ \partial_B\phi &= \partial_Z\phi\Big|_{Z=\varepsilon} \end{aligned}\right\} \int d^dx \left(\frac{L}{\varepsilon}\right)^{d-1}\left[(\partial_Z\phi)\phi\right]_{Z=\varepsilon}$$

Coming back to (+) in the previous page

$$S = \int d^dx\left(\frac{L}{\varepsilon}\right)^{d-1}\phi\,\partial_Z\phi\Big|_{Z=\varepsilon} + \int d^{d+1}_x\ \sqrt{g}\left\{-\frac{\phi}{\sqrt{g}}\partial_A\left[\sqrt{g}\, g^{AB}\partial_B\phi\right]+m^2\phi^2\right\}$$

or

$$S = \int d^dx \left(\frac{L}{\varepsilon}\right)^{d-1} \phi \partial_z \phi \Big|_{z=\varepsilon} + \int d^{d+1}x \sqrt{g}\, \phi \underbrace{\left\{ -\Box \phi + m^2 \phi \right\}}_{\text{eq of motion}}$$

So for a field satisfying the eq of motion $\boxed{\Box \phi = m^2 \phi}$

$$S = \int d^dx \left(\frac{L}{\varepsilon}\right)^{d-1} \phi \partial_z \phi \Big|_{z=\varepsilon}$$

the action just boils to a Boundary term evaluated on the solution to the eq of motion with a boundary condition $\phi(z=\varepsilon, x) = J_0$

So, in summary: the gravity action that we need to compute correlators

$$\left\langle e^{\int J \mathcal{O} d^dx} \right\rangle \simeq e^{-S_{grav}[\phi] \to [J]}$$

$$S_{grav} = \int d^dx \left(\frac{L}{\varepsilon}\right)^{d-1} \phi \partial_z \phi \Big|_{z=\varepsilon}$$

ϕ: satisfies eq of motion

So: we will

1. solve eq of motion on AdS_{d+1}
2. study its asymptotics
3. compute $\left(\frac{L}{\varepsilon}\right)^{d-1} \phi \partial \phi \Big|$ ...
4. study $\frac{\delta^2 \langle e^{\int d^dx J \mathcal{O}} \rangle}{\delta J(x) \delta J(y)} = \langle \mathcal{O}(x) \mathcal{O}(y) \rangle$

① Eq of motion

$$\Box\phi - m^2\phi = 0$$

$$\frac{1}{\sqrt{g}}\partial_A\left[\sqrt{g}\, g^{AB}\partial_B\phi\right] - m^2\phi$$

$$\sqrt{g} = \left(\frac{L}{z}\right)^{d+1} \; ; \; g^{zz} = g^{xx} = \frac{z^2}{L^2}$$

we will separate variables $\boxed{\phi(x,z) = \int \frac{d^dk}{(2\pi)^d}\, \phi_k(z)\, e^{ikx}}$

Note that $\partial_{x_\mu}\phi(x,z) = \int \frac{d^dx}{(2\pi)^d}\, \phi_k(z)\, i k_\mu\, e^{ikx}$

eq of motion

$$\boxed{\partial_z\phi = \int \frac{d^dk}{(2\pi)^d}\partial_z\phi_k(z)e^{ikx} \; ; \; \partial_x^2\phi = \partial_\mu\partial^\mu\phi = \int \frac{d^dk}{(2\pi)^d}(-k^2)\,\phi_k(z)e^{ikx}}$$

$$\frac{z^{d+1}}{L^{d+1}}\partial_z\left[\left(\frac{L}{z}\right)^{d+1} g^{zz}\partial_z\phi\right] + \frac{z^{d+1}}{L^{d+1}}\partial_x\left[\left(\frac{L}{z}\right)^{d+1} g^{xx}\partial_x\phi\right] - m^2\phi = 0$$

$$\left(\frac{z}{L}\right)^{d+1}\left\{\partial_z\left[\left(\frac{L}{z}\right)^{d-1}\partial_z\phi\right]\right\} + \frac{z^2}{L^2}\partial_x^2\phi - m^2\phi = 0$$

$$\cancel{\left(\frac{L}{z}\right)^{d-1}\partial_z^2\phi + (d-1)\left(\frac{L}{z}\right)^{d-2}\partial_z\phi + \frac{z^2}{L^2}\partial_x^2\phi - m^2\left(\frac{L}{z}\right)^{d+1}\phi = 0}$$

use Fourier decomposition

$$\cancel{\left(\frac{L}{z}\right)^{d-1}\left\{\partial_z^2\phi + (d-1)\frac{z}{L}\partial_z\phi + \left(\frac{z}{L}\right)^{d+1}\partial_x^2\phi - m^2\frac{L^2}{z^2}\phi\right\} = 0}$$

$$\boxed{\cancel{\partial_z^2\phi + (d-1)\frac{z}{L}\partial_z\phi - k^2\left(\frac{z}{L}\right)^{d+1}\phi - m^2\frac{L^2}{z^2}\phi = 0}}$$

Bessel eq

$$\frac{z^{d+1}}{\cancel{L^2}}\partial_z\left[z^{1-d}\partial_z\phi\right] - \frac{z^2}{\cancel{L^2}}k^2\phi - m^2\cancel{L^2}\phi = 0 \;\Rightarrow\; \boxed{z^{d+1}\partial_z\left[z^{1-d}\partial_z\phi\right] - (z^2k^2 + m^2L^2)}$$

So we obtain a Bessel eq for the scalar on AdS_{d+1}

$$z^{d+1}\partial_z\left[z^{1-d}\partial_z\phi\right]-(m^2L^2+z^2k^2)\phi=0$$

$\phi=\phi_k(z)$

Solution

$$\phi_k(z)=A_k\, z^{d/2}K_\nu(kz)+B_k\, z^{d/2}I_\nu(kz)$$

$$\boxed{\nu=\sqrt{d^2/4+m^2L^2}}$$

Let us see the behaviour near $z=0$, $z=\infty$ of K_ν, I_ν

$\boxed{x^2y''+xy'-(x^2+\alpha^2)y=0}$

$\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\frac{1}{\Gamma(\nu+\ell+1)}\left(\frac{kz}{2}\right)^{\nu+2\ell}$

$$K_\nu(kz)\xrightarrow[z=0]{}\frac{1}{2}\frac{(\nu-1)!}{(kz)^\nu}\qquad I_\nu(kz)\xrightarrow[z\to 0]{}\left(\frac{kz}{2}\right)^\nu$$

$$K_\nu(kz)\xrightarrow[z\to\infty]{}e^{-kz}\qquad I_\nu(kz)\xrightarrow[z\to\infty]{}e^{kz}$$

to impose regularity in the interior that is that the fluctuation is still small $\rightsquigarrow$ vanishing coefficient B to avoid I_ν to diverge

---

Aside

If we study the eq for $z\to 0$.

$$z^{d+1}\partial_z(z^{1-d}\partial_z\phi)-(m^2L^2+z^2k^2)=0$$

$$\boxed{z^2\partial_z^2\phi+(1-d)z\partial_z\phi-(m^2L^2+z^2k^2)\phi=0}\qquad \phi=z^\Delta$$

$z^2\phi''=z^{\Delta-2+2}\,\Delta(\Delta-1)$

$z\,\partial_z\phi=z^\Delta(1-d)$

$$\rightsquigarrow z^\Delta\left[\Delta(\Delta-d)-m^2L^2-k^2\cancel{z^2}^{\,0}\right]=0$$

$$\Delta(\Delta-d)=m^2L^2\rightarrow\boxed{\Delta_\pm=\frac{d}{2}\pm\sqrt{d^2/4+m^2L^2}}\qquad\boxed{\Delta_+-\Delta_-=\nu\ (\text{above})}$$

$$\phi_k(z) = a\, z^{d/2} K_\nu(kz) + b\, z^{d/2} I_\nu(kz)$$

$\underline{z \to 0}$

$$\boxed{\phi_{z\to 0} = a\, z^{d/2 - \nu} + b\, z^{d/2+\nu} = a\, z^{\Delta_-} + b\, z^{\Delta_+}}$$

So we see that if the field were going like z^{Δ_+} then it would grow a lot for $z \to \infty$ $\to$ dominate in the interior $\rightsquigarrow$ relevant operator

$$\Delta_+ = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2 L^2}$$

$$\Delta_- = \frac{d}{2} - \sqrt{\frac{d^2}{4} + m^2 L^2}$$

if $\underline{m^2 > 0}$ $\quad \Delta_+ > d \to$ irrelevant operator $\quad \int d^dx\, \mathcal{L} + \frac{1}{m^{\Delta_+ - d}} \mathcal{O}_{\Delta_+}$

$m^2 = 0 \quad \Delta = \Delta_+ = d \to$ marginal operator $\quad \int d^dx\, \mathcal{L} + \frac{1}{m^0} \mathcal{O}_{\Delta_+}$

$m^2 < 0 \quad \Delta < d \longrightarrow$ relevant operator

$\quad \hookrightarrow \left[m^2 \geq -\left(\frac{d}{2L}\right)^2 \right]$ is ok $\rightsquigarrow$ $\underline{\text{BF bound}}$

end of aside

Let us come back to the computation of $\langle \mathcal{O}\, \mathcal{O} \rangle$.

We have a solution to the eqs of motion

$$\phi_k(z) = a\, z^{d/2} K_\nu(kz) + \cancel{b}^{\,0}\, z^{d/2} I_\nu(kz)$$

Let us $\underline{\text{normalize}}$ ϕ_k $\quad$ choose $\underline{a}$

$$\boxed{\phi_k(z) = \frac{1}{\varepsilon^{d/2} K_\nu(k\varepsilon)}\, z^{d/2} K_\nu(kz)} \rightsquigarrow \boxed{\phi(x,z) = \frac{1}{(2\pi)^d} \int d^dk\, e^{ikx}\, \phi_k(z)\, \phi_0(k,\varepsilon)}$$

$\phi_0(k,\varepsilon)$

The on shell Action that was after all manipulations

$$S = \int d^d x \left(\frac{L}{z}\right)^{d-1} \phi\, \partial_z \phi \Big|_{z=\varepsilon}$$

$$\partial_z \phi = \int \frac{d^d k}{(2\pi)^d} \cdot e^{ikx} \left(\partial_z \phi_k(z)\right) \phi_0(k,\varepsilon)$$

$$\phi = \int \frac{d^d \ell}{(2\pi)^d}\, e^{i\ell x}\, \phi_\ell(z)\, \phi_0(\ell,\varepsilon)$$

$$S = \int d^d x \int \frac{d^d \ell\, d^d k}{(2\pi)^{2d}} \left(\frac{L}{\cancel{z}}\right)^{d-1} \phi_0(\ell,\varepsilon)\phi_0(k,\varepsilon)\, e^{i(k+\ell)x}\, \frac{1}{z^{d-1}} \phi_\ell(z)\, \partial_z \phi_k(z) \Big|_{z=\varepsilon}$$

make the x-integral $\to \delta(k+\ell)$

$$S = \int \frac{d^d k}{(2\pi)^d} L^{d-1} \phi_0(k,\varepsilon)\phi_0(-k,\varepsilon) \overbrace{\frac{1}{z^{d-1}} \phi_{-k}(z)\, \partial_z \phi_k(z) \Big|_{z=\varepsilon}}^{\mathcal{F}_\varepsilon(k)}$$

Since $\langle \mathcal{O}(x_1)\, \mathcal{O}(x_2)\rangle = \dfrac{\delta^2}{\delta J(x_1)\,\delta J(x_2)} \langle e^{\int J \mathcal{O}}\rangle = \dfrac{\delta^2}{\delta \phi_0(x_1)\,\delta\phi_0(x_2)} e^{-S_{grav}}$

In Fourier space

$$\langle \mathcal{O}(k_1)\, \mathcal{O}(k_2)\rangle = \frac{\delta^2}{\delta\phi_0(k_1)\,\delta\phi_0(k_2)} e^{-S_{grav}} = (2\pi)^d \delta^d(k_1+k_2)\, \mathcal{F}_\varepsilon(k)\Big|_{\varepsilon\to 0}$$

we need to know $\boxed{\mathcal{F}_k(\varepsilon) = \frac{1}{z^{d-1}} \phi_{-k}(z)\, \partial_z \phi_k(z)\Big|_{z=\varepsilon}}$ $\left(\phi_k(z) = z^{d/2} K_\nu(kz)\right)$

$$\mathcal{F}(k,\epsilon) = \frac{1}{z^{d-1}} \phi_{-k}(z)\, \partial_z \phi_k(z) \Big|_{z=\epsilon}$$

$$\boxed{\phi_k(z) = \frac{z^{d/2}}{\epsilon^{d/2}} \frac{K_\nu(kz)}{K_\nu(k\epsilon)}}$$

$$\partial_z \phi_k(z) = \frac{d}{2} z^{\frac{d}{2}-1} K_\nu(kz) + z^{d/2} \partial_z K_\nu(zk) = z^{d/2}\left[\frac{d}{2z} K_\nu(kz) + \partial_z K_\nu(kz)\right]$$

$$\phi_{-k} = z^{d/2} K_\nu(-kz)$$

$$\mathcal{F}(k,\epsilon) = \frac{1}{z^{d-1}} z^{d/2} \cdot z^{d/2} \cdot K_\nu(-kz) \cdot \left[\frac{d}{2z} K_\nu(kz) + K_\nu'(kz)\right]$$

$$= z\, K_\nu(-kz) \cdot \left[\frac{d}{2z} K_\nu(kz) + K_\nu'(kz)\right]\Big|_{z=\epsilon}$$

Now we expand this in series for $\underline{\epsilon \to 0}$

Let us discuss the formalism of Wilson loops in a generic metric

$$ds^2 = -g_{tt}\,dt^2 + g_{xx}\,dx^2 + g_{\rho\rho}\,d\rho^2 + \dots$$

$$\left(g_{\alpha\beta} = G_{\mu\nu}\,\partial_\alpha X^\mu \partial_\beta X^\nu \qquad \alpha,\beta = \sigma,\tau\right)$$

we propose a configuration (as before)

$$\left.\begin{array}{l} x = \sigma \\ \rho = \rho(\sigma) \\ t = \tau \end{array}\right\} \quad ds^2_{ind} = -g_{tt}\,d\tau^2 + \left(g_{xx} + g_{\rho\rho}\rho'^2\right)d\sigma^2$$

$$g_{\alpha\beta} = \begin{array}{c c} & \begin{array}{cc} \tau & \sigma \end{array} \\ \begin{array}{c} \tau \\ \sigma \end{array} & \begin{bmatrix} -g_{tt} & 0 \\ 0 & (g_{xx} + g_{\rho\rho}\rho'^2) \end{bmatrix} \end{array}$$

$$\longrightarrow -\det g_{\alpha\beta} = g_{tt}\left(g_{xx} + g_{\rho\rho}\rho'^2\right)$$

$$S_{NG} = \int \sqrt{g_{tt}g_{xx} + g_{tt}g_{\rho\rho}\,\rho'^2}\;\frac{d\sigma\,d\tau}{2\pi\alpha'} \rightsquigarrow \text{define} \quad \begin{array}{l} f^2 = g_{tt}\,g_{xx} \\ g^2 = g_{tt}\,g_{\rho\rho} \end{array}$$

$$S_{NG} = \frac{T}{2\pi\alpha'}\int d\sigma\,\sqrt{f^2_{(\rho)} + g^2_{(\rho)}\rho'^2} \longrightarrow$$ 1-d system in classical mechanics $\longrightarrow$ conserved Energy

with the Nambu-Goto Action

$$S_{NG} = \frac{T}{2\pi\alpha'} \int d\sigma \sqrt{f^2 + g^2 \rho'^2} \longrightarrow H = p_\rho \rho' - \mathcal{L}$$

$$H = -\frac{f^2}{\sqrt{f^2 + g^2\rho'^2}} \overset{\swarrow \text{constant}}{=} -f(\rho_0) = -f_0$$

$\rightsquigarrow$ from here

$$\left[\frac{d\rho}{dx} = \frac{1}{f_0} \frac{f}{g} \sqrt{f^2 - f_0^2} \right] \rightsquigarrow \text{then the separation between } Q\bar{Q} \text{ pair is}$$

$$L_{Q\bar{Q}} = 2 \int d\sigma = 2 \cdot \int_{\rho_0}^{\infty} d\rho \, \frac{f_0 \, g}{f \sqrt{f^2 - f_0^2}}$$

$$\boxed{L_{Q\bar{Q}}^{(\rho_0)} = 2 f_0 \int_{\rho_0}^{\infty} d\rho \, \frac{g}{f \sqrt{f^2 - f_0^2}}} \rightsquigarrow \text{Let us now compute the Energy}$$

T.O.

$$E_{Q\bar{Q}} = \underbrace{\int d\sigma \sqrt{f^2 + g^2 \rho'^2}}_{\text{original } \frac{\text{NG Action}}{T}} - 2 \underbrace{\int_0^\infty g(\rho)\, d\rho}_{\text{mass of 2 straight strings } |\ |}$$

using as derived above $\left[d\rho \frac{f_0 g}{f\sqrt{f^2 - f_0^2}} = d\sigma \right]$

$$E_{Q\bar{Q}} = 2 f(\rho_0) \cdot \int_{\rho_0}^\infty \frac{g/f}{\sqrt{f^2 - f_0^2}}\, d\rho - 2 \int_0^\infty g\, d\rho$$

or using the expression for $L_{Q\bar{Q}}(\rho_0)$

$$\boxed{E_{Q\bar{Q}}(\rho_0) = f(\rho_0) L_{Q\bar{Q}}(\rho_0) + 2 \int_{\rho_0}^\infty \frac{g}{f}\left[\sqrt{f^2 - f_0^2} - f\right] d\rho - 2 \int_0^{\rho_0} g\, d\rho}$$

Notice:
if $f(\rho_0) \neq 0$
$E \approx f(\rho_0) L_{Q\bar{Q}}$
⇒ confinement!

Computing correlation functions

The quantity of interest to a quantum field theorist is

$$Z[J] = \frac{1 \int \mathcal{D}\text{fields}\; e^{-S[\phi_{\text{fields}}] - \int d^4x\, J(x)\, \phi_{\text{fields}}}}{\int \mathcal{D}\text{fields}\; e^{-S[\phi_{\text{fields}}]}}$$

because doing functional derivatives

$$\left. \frac{\delta^n Z[J]}{\delta J(x_1)\, \delta J(x_2) \cdots \delta J(x_n)} \right|_{J=0} = \langle 0 | \phi_{\text{field}}(x_1)\, \phi_{\text{field}}(x_2) \cdots \phi(x_n) | 0 \rangle \rightsquigarrow$$

this is what you measure in the laboratory

In AdS/CFT Gubser-Klebanov-Polyakov and Witten (1998) proposed that

$$\left\langle e^{-\int d^4x\, J_i(x)\, \Phi(x)} \right\rangle \equiv Z_{\text{string on } AdS_5 \times S^5}[\Phi_{\text{string}} \to J_i] \underset{\substack{g_s N_c \to \infty \\ \alpha' \to 0}}{\approx} \left. e^{-S_{\text{supergravity}}[\Phi \to J]} \right|$$

Then an n-point correlator.

$$\frac{\delta^n}{\delta J(x_1)\cdots\delta J(x_n)} \left\langle e^{-\int d^4x\, J(x)\, \Phi(x)} \right\rangle \cong \frac{\delta^n}{\delta J_1 \cdots \delta J_n}\, e^{-S_{\text{supergravity}}[\Phi \to J_i]}$$

In the following we will scketch the steps to compute a 2-point correlator.

We will consider a Scalar field in AdS_5.

this scalar will be associated with a Scalar (spin 0) operator in the CFT

the Scalar will have a mass "m" that as we will see will be associated with the dimension of the CFT operator.

We will proceed in various different steps

First step

Let us compute the eq. of motion for a massive scalar in AdS_5

[by the way, we will use coordinates / $ds^2 = \frac{L^2}{z^2}(-dt^2 + dx_1^2 + dx_2^2 + dz^2)$

basically $z = \frac{1}{u}$ of the previous radial coordinate $\rightsquigarrow$ the boundary will be $\underline{\underline{z=0}}$]

Let us do this in $\underline{AdS_{d+1}}$ $\rightsquigarrow$ it will give us a better understanding.

$$S = \int d^{d+1}x \sqrt{g_{AdS_{d+1}}} \left[g^{AB} \partial_A \phi \partial_B \phi + m^2 \phi^2 \right]$$

Check that

$$ds^2 = \left(\frac{L}{z}\right)^2 (dx_{1,3}^2 + dz^2)$$

$$\sqrt{g_{AdS_{d+1}}} = \left(\frac{L}{z}\right)^{d+1}$$

$\rightsquigarrow$ eq of motion $\left\{ \begin{aligned} \frac{\partial \mathcal{L}}{\partial \phi} &= 2\sqrt{g}\, m^2 \phi \\ \partial_M \frac{\partial \mathcal{L}}{\partial(\partial_M \phi)} &= 2\partial_M\left[\sqrt{g}\, g^{MB} \partial_B \phi\right] \end{aligned} \right.$ $\rightsquigarrow$ $\boxed{\frac{1}{\sqrt{g}} \partial_M \left[\sqrt{g}\, g^{MB} \partial_B \phi\right] = m^2 \phi}$

this is the eq of motion

Second step

We will work a bit with the Action

$$S = \int d^{d+1}x \, \sqrt{g} \left[\underbrace{g^{AB} \partial_A \phi \partial_B \phi}_{\text{I}} + m^2 \phi \right] \longrightarrow \text{integrate by parts in I}$$

$$\text{I} = \int d^{d+1}x \; \partial_A \left[\sqrt{g}\, g^{AB} \phi \partial_B \phi \right] - \Phi \, \partial_A \left[\sqrt{g}\, g^{AB} \partial_B \phi \right]$$

$$\Rightarrow S = \int d^{d+1}x \; \frac{1}{\sqrt{g}} \partial_A \left[\sqrt{g}\, g^{AB} \phi \partial_B \phi \right] + \sqrt{g}\, \phi \underbrace{\left[-\frac{1}{\sqrt{g}} \partial_A \left[\sqrt{g}\, g^{AB} \partial_B \phi \right] + m^2 \phi \right]}_{\text{Eq of motion}}$$

We will perform the integral in Z: $\int_0^\infty = \int_\varepsilon^\infty$ imposing $\phi(\infty) \to 0$

$$\Rightarrow S \Big|_{\text{eq of motion}} = \int d^{d+1}x \; \frac{1}{\sqrt{g}} \partial_A \left[\sqrt{g}\, g^{AB} \phi \partial_B \phi \right] = \int d^{d}x \; \sqrt{g}\, g^{ZB} \, \Phi(x,z) \, \partial_B \phi(z,B) \Big|_{Z=\varepsilon} =$$

evaluated in a regulated boundary $Z = \varepsilon$

So we get that the Action (once eqs of motion are used) is

$$S = \int d^dx \sqrt{g}\, g^{zB} \phi(x,z)\, \partial_z \phi(z,x)\Big|_{z=\epsilon} =$$

use $\sqrt{g}\Big|_{z=\epsilon} = \left(\frac{L}{\epsilon}\right)^{d+1}$

$g^{zB} = g^{zz}\delta_{Bz}\Big|_{z=\epsilon} = \frac{\epsilon^2}{L^2}$

$$\boxed{S = \int d^dx \left(\frac{L}{\epsilon}\right)^{d-1} \phi(x,z)\, \partial_z \phi(x,z)\Big|_{z=\epsilon}}$$

$\rightsquigarrow$ this is our "$S_{\text{Supergravity}}$"

in $\langle e^{\int J\mathcal{O}} \rangle \approx e^{-S_{\text{Supergrav}}}$

In this second step we have used that

$\phi(x,z) \rightsquigarrow$ • solves eqs of motion $\longrightarrow$ So next we will

• (1) solve the eq of motion ; (3) compute $\left(\frac{L}{\epsilon}\right)^{d-1} \phi\, \partial_z \phi\Big|_{z=\epsilon}$

(2) study the asymptotics

third step

solve the eq of motion $\frac{1}{\sqrt{g}}\partial_A\left[\sqrt{g}\, g^{AB}\partial_B\phi\right] = m^2\phi$. use $\sqrt{g} = \left(\frac{L}{z}\right)^{d+1}$

$g^{AB} = \frac{z^2}{L^2}\eta_{AB}$

→ we will "separate variables" with a Fourier transform.

$$\phi(x,z) = \int \frac{d^d k}{(2\pi)^d}\, \phi_k(z)\, e^{ikx}$$

The eq of motion is explicitly

$$\frac{z^{d+1}}{L^{d+1}}\left\{\partial_z\left[\frac{L^{d+1}}{z^{d+1}}\frac{z^2}{L^2}\partial_z\phi\right] + \frac{L^{d+1}}{z^{d+1}}\frac{z^2}{L^2}\partial_\mu\partial_\nu\phi\,\eta^{\mu\nu}\right\} = m^2\phi$$

using the decomposition [check that!]

$$\boxed{z^{d+1}\partial_z\left[z^{1-d}\partial_z\phi\right] - m^2L^2\phi - z^2k^2\phi = 0}$$ ⇝ this is a Bessel eq

Solution is $\phi_k(z) = A\, z^{d/2} K_\nu(kz) + B\, z^{d/2} I_\nu(kz)$ where $\nu = \sqrt{\frac{d^2}{4} + m^2L^2}$

I will ask you to check that the asymptotics of $K_\nu(kz)$; $I_\nu(kz)$ are

$z \to 0$ (UV) $\qquad\qquad z \to \infty$ (IR)

$K_\nu(kz) \longrightarrow \dfrac{(\nu-1)!}{2(kz)^\nu}$ $\qquad\qquad K_\nu(kz) \longrightarrow e^{-kz}$

$I_\nu(kz) \longrightarrow \left(\dfrac{kz}{2}\right)^\nu$ $\qquad\qquad I_\nu(kz) \longrightarrow e^{kz}$ [Regularity for $z\to\infty$ $\rightsquigarrow B=0$]

At this stage we make a <u>small</u> <u>detour</u> and study the eq of motion near <u>$z=0$</u>

$$z^{d+1}\partial_z\left[z^{1-d}\partial_z\phi\right]-m^2L^2\phi-\cancel{z^2k^2\phi}=0$$

<u>neglect this term</u>

$\rightsquigarrow$ solution [<u>check</u>] $\quad \phi = A z^{\Delta_+} + B z^{\Delta_-}$

$$\Delta_+ = \frac{d}{2} + \sqrt{\frac{d^2}{4}+m^2L^2}$$

$$\Delta_- = \frac{d}{2} - \sqrt{\frac{d^2}{4}+m^2L^2}$$

- $m^2>0$ $\quad \Delta_+ > d \longrightarrow$ irrelevant operator
- $m^2=0$ $\quad \Delta_+ = d \longrightarrow$ marginal operator
- $m^2<0$ $\quad \Delta_+ < d \longrightarrow$ relevant operator

<u>in $d=4$</u> $\quad \boxed{\Delta_\pm = 2 \pm \sqrt{1+m^2L^2}}$

So, let us come back to the correlator.

our Solution was

$$\phi_k(z) = a\, z^{d/2} K_\nu(kz)$$

[avoids divergent terms as $z \to \infty$]

with $\phi(x,z) = \int \frac{d^d k}{(2\pi)^d} e^{ikz} \phi_k(z)$

We "just" need to compute

$$S = \int d^d x \left(\frac{L}{\varepsilon}\right)^{d-1} \phi(x,z)\, \partial_z \phi(x,z) \Big|_{z=\varepsilon}$$

We compute this and then expand for $\underline{\varepsilon \to 0}$

Actually

$$\langle \mathcal{O}(x_1)\, \mathcal{O}(x_2) \rangle = \frac{\delta^2}{\delta J(x_1)\, \delta J(x_2)}\, e^{-S} \Big|_{J=0} \sim \frac{1}{|x_1 - x_2|^{2\Delta_+}}$$

Lecture III

How can we apply AdS/CFT to learn about QCD?
[or another interesting QFT from a phenomenological viewpoint].

Let us compare these theories

	QCD	$\mathcal{N}=4$ SYM
gauge group:	$SU(N_c)$	$SU(N_c)$
global group:	$SU(N_f)\times SU(N_f)\times U(1)_B$	$SO(6)_R$
SUSY:	0 susy.	32 susy
	$SO(1,3)$	$SO(2,4)$
field content	A_μ^a $\quad a:1,\ldots,N_c^2-1$ $\psi_i^\alpha, \bar{\psi}_i^\alpha$ $\quad \alpha:1,\ldots,N_c$; $i=1,\ldots,N_f$	A_μ^a, $6\times\phi^a$, $4\times\lambda^a$ $a:1,\ldots,N_c^2-1$

So, we see that the main differences are in

- global symmetries, SUSY, conformality
- particle content

one must conclude that is very little what we can learn about QCD using AdS/CFT. But it must be said that (strong coupling) at finite temperature, some of these differences are not relevant for the dynamics.

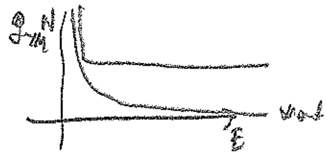
$\underline{\text{4 ideas}}$

How to extend the conjecture?

① take $\mathcal{N}=4 \longrightarrow AdS_5\times S^5$ at high Energy	SUSY
	add a black hole

$\mathcal{N}=4 \xrightarrow{\text{want } \sigma,\ \sigma\langle\sigma\rangle\ \} \text{SUSY}} \mathcal{N}=0,\ \mathcal{N}=1\ YM$

$AdS_5\times S^5 \xrightarrow{\text{deform the metric, turn on new fields}} \text{some background}$

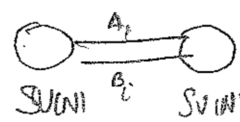


P.S set up

② change the $S^5 \longrightarrow$ another manifold   for example $T^{1,1}$, $T^{p,q}$

$$\boxed{ds^2 = \frac{1}{6}d\Omega_2 + \frac{1}{6}d\tilde{\Omega}_2 + \frac{1}{9}(d\psi + \cos\theta\, d\varphi + \cos\tilde{\theta}\, d\tilde{\varphi})}$$

hy me solucari $AdS_5\times T^{1,1} \longrightarrow$ SUSY $\mathcal{N}=1$ CFT

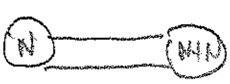


$$\mathcal{W} = (A_i B_i)^2 - A_i B_j A^i B^j$$

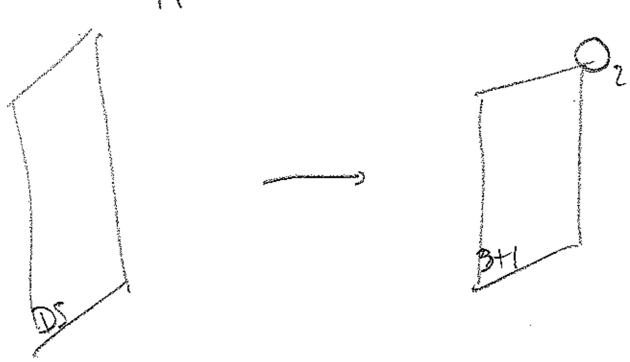
Add RR-fields so that conformality is broken $\longrightarrow (N = N+M)$

$$ds^2 = H^{-1/2}(dx_{1,3}^2) + H^{1/2}\underbrace{(dr^2 + r^2 dS_{T^{1,1}})}_{dM_5} \quad \text{KT}$$

$\underset{\text{deformed conifold}}{\sim}$ KS



③ use wrapped branes



Estos son formas de romper conformality, usando Dp branas.

En realidad hay otra forma muy interesante y natural.
comenzar con $AdS_5 \times S^5$ e introducir un operador o dar un $\langle VEV \rangle$ a un operador de manera que se rompa conformality, parte de la SUSY y la teoría fluye a un IR interesante. El mejor ejemplo:
- Polchinski Strassler.
- Freedman-Bianchi-Skenderis
- GPPZ

E
- CFT ($\mathcal{N}=4$ SYM)
- insert $\mathcal{O}$ or $\langle \mathcal{O} \rangle$
- Flow
- $\mathcal{N}=1$, $\mathcal{N}=0$ YM like thing

$\xleftrightarrow{\text{duality}}$

Ed
- $AdS_5 \times X^5$
- some deformation breaks SUSY, $SO(2,4)$
- flow
- Some other background ($\mathcal{N}=1$, $\mathcal{N}=0$)

Un ejemplo muy explícito (Freedman, Nunez, Schnabl, Skenderis), (Bak, Gutperle, Hirano)

Prologo

$$AdS_5 = \left\{ \frac{dr^2}{1+\alpha e^{-2r}} + e^{2r} d\Sigma_4 \right.$$

$\alpha = 0 \quad d\Sigma_4 = Mink_4$

$\alpha = -1 \quad d\Sigma_4 = AdS_4$

$\alpha = +1 \quad d\Sigma_4 = dS_4$

~~La solución~~ hay una solución de IIB con dilaton, F_5 y $g_{\mu\nu}$

$$ds^2 = \frac{dr^2}{1+\alpha e^{-2r} + c\, e^{-8r}} + e^{2r} d\Sigma_4 + d\Omega_5$$

$\boxed{\text{esta solución rompe SUSY y es estable}}$

$$F_5 = \mathrm{Vol}\, S_5 + *$$

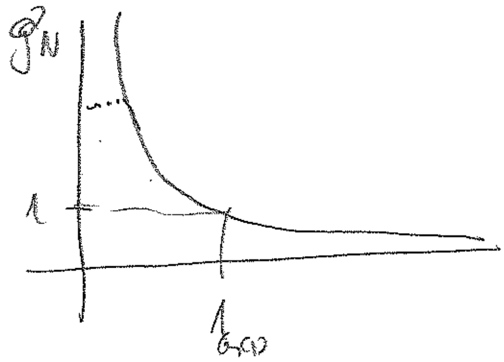
si $c=0 \to AdS_5 \times S^5$ (~~...~~)

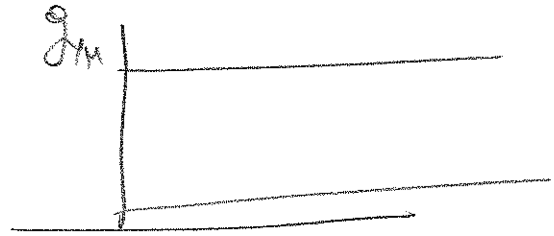
en $r \to \infty$ $e^{-8r} \to 0$ la métrica se parece a $\sim AdS_5$

(en general $\alpha=-1$ evita singularidades)

$$\phi(r) = c \int \frac{dr}{\sqrt{e^{8r} + \alpha e^{6r} + c^2/24}}$$

④ Play easy game

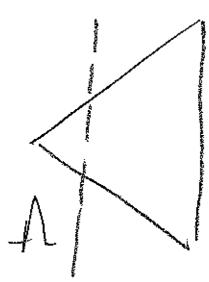




The idea is to replace the dotted line ⇝ by a strongly coupled CFT

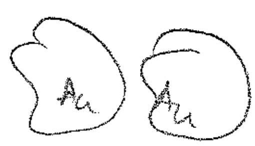
to simulate the scale

$$ds^2 = \frac{a^2}{\sqrt{g^2N}} dx_{1,3}^2 + \frac{a^2 dr^2}{g^2 N} + d\Omega_5$$

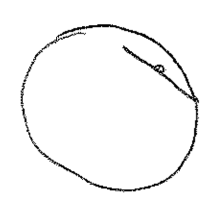


→ RHIC

. $\frac{\eta}{s} = \frac{1}{4\pi}$



. Jet quenching parameter



. Spectrum of mesons.

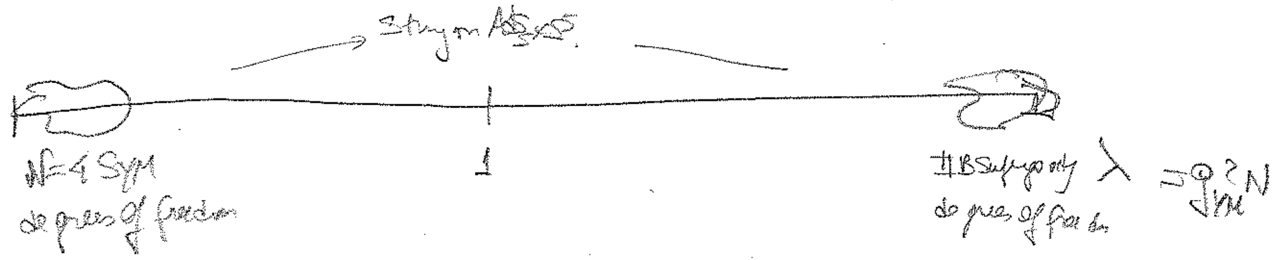
etc

Lecture: Models for phenomenologically interesting GFT's

Goal: compute observables [correlations of gauge invariant operators] in interesting GFT's. $\longrightarrow$ various methods $\begin{cases}\text{Lattice} \\ \vdots \\ \text{effective field theory}\end{cases}$

We will use AdS/CFT duality similar to Thirring / Sine gordon

$\mathcal{N}=4$ SYM $\equiv$ strings on $AdS_5 \times S^5$.



We will work with generalizations of this mostly on the supergravity side

$\alpha' R^2_{AdS} \sim \frac{1}{\sqrt{\lambda}}$; $e^{\phi} = $ const

$\longrightarrow$ Inadequacy of original conjecture for a ph-interesting GFT $\left\{\begin{array}{l}\text{wilson loop} \\ \exists \text{ mesons dynamical} \\ \text{running coupling} \\ \text{spectrum}\end{array}\right\}$ Sector theory

The first generalization $\longrightarrow$ study Dp-branes theories in (p+1) dim

[Maldacena, Itzhaki, Sonnenschein, Yankielowicz, '98] 16 SUSY

$SO(1,p) \times SO(8-p)_R$ symmetry

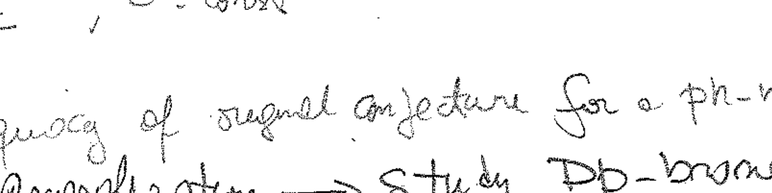
$$ds^2 = h^{-1/2} dx^2_{1,p} + h^{1/2}(d\rho^2 + \rho^2 d\Omega_{8-p})$$

$$\boxed{h(\rho) = \frac{c}{\rho^{7-p}}}$$

F_{p+2} ; $e^{\phi} \sim \frac{1}{\rho^{(7-p)(3-p)/4}}$

The idea there is that there is a RG-flow in the degrees of freedom

$p+1 < 4$
D2, D1, D0

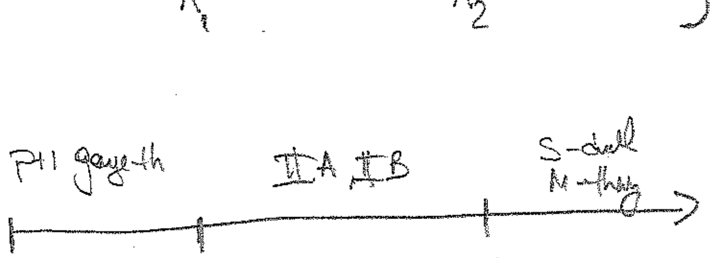


$\Lambda_1 \leftrightarrow e^{\phi} \sim 1$

$\Lambda_2 \leftrightarrow \alpha' R_{eff} \sim 1$

$$g^2_{eff} \sim \frac{1}{\alpha' R_{eff}}$$

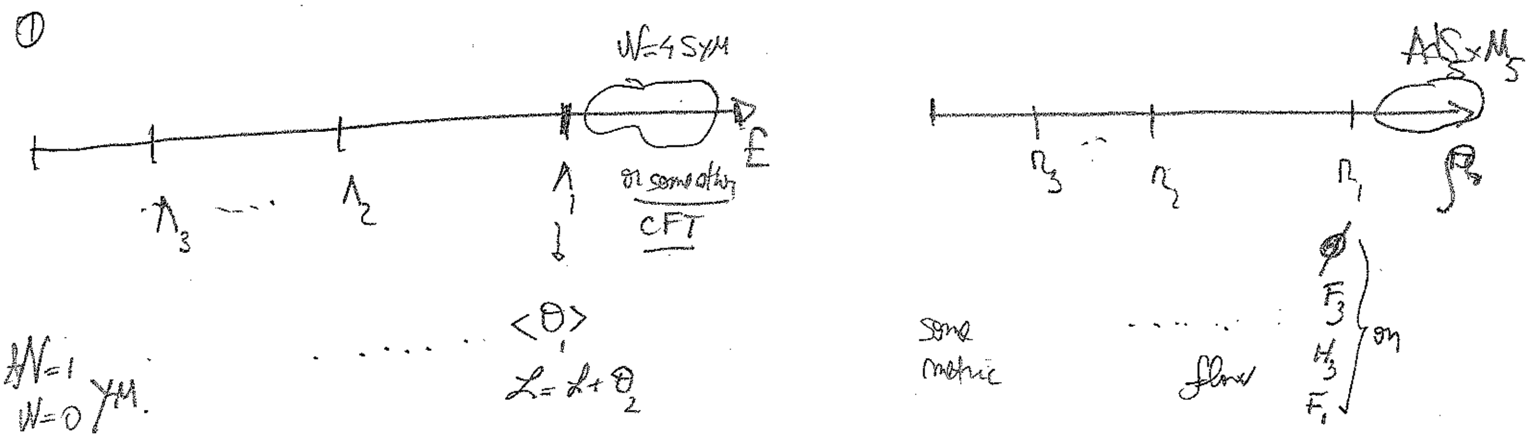
$p+1 > 4$
D4, D5, D6



$$\boxed{\alpha' R = \frac{1}{g^2_{eff}} = \sqrt{\frac{\rho^{3-p}}{g^2_{YM} N_c}} \qquad e^{\phi} \sim g_{YM}^{\frac{7-p}{2}} \ldots \rho^{(3-p)(p-3)/4} \ldots}$$

The Idea I would like to emphasize is the existence of a R-G flow in degrees of freedom to better describe the system

This kind of Idea was very used in developing different approaches to Phenomenologically interesting QFTs and theories in 3+1 dim with $\mathcal{N}=0$, $\mathcal{N}=1$ susy.



Propose a duality between the metric and the QFT UV completed
Polchinski–Strassler ; Porrati Girardello, Pasate, Zaffaroni, etc.

Example

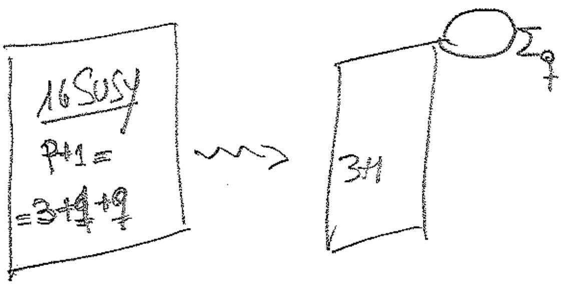
Janusian $\quad ds^2_{10} = e^{2r} AdS_4 + \dfrac{dr^2}{1 - e^{-2r} + b e^{-8r}} + d\Omega_5$

$e^{\phi} = b \displaystyle\int \frac{dr}{\sqrt{e^{8r} \mp e^{6r} + \frac{b^2}{24}}}$

- Klebanov

- Wrapped branes

Wrapped branes take a Dp brane $\to$ compactify q dimensions $D_{p+1-q} = D_{(3+1)}$



- Low Energy the theory is effectively $(3+1)$ dim.

- the wrapping is such that SUSY $\nearrow$ partially / $\searrow$ totally broken

Same for R-symmetries/other global

- Notice unconventional UV-completion
  not a CFT typically

Let us focus on two Examples.

① Witten's model for Yang-Mills
② wrapped D_5 branes.

Witten's model for YM_{3+1} (1998)

D_4 branes on S^1 with $\cancel{SUSY}$ boundary conditions on S^1

$$\underset{16\,SUSY}{(4+1)\,SYM} \longrightarrow (3+1)\,YM + UV\ \text{completion}$$

$$(A^a_\mu,\ 5\times\phi^a,\ 4\,\lambda^a) \longrightarrow (A^a_\mu,\ \rho^a) + \text{massive modes}$$

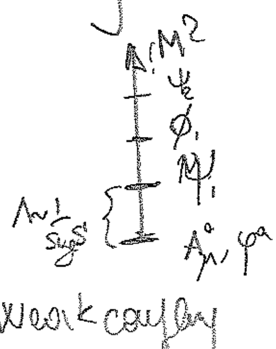
type IIA D4 branes.

$$ds^2_{D4_{on}S^1} = h_1(\rho)\left[-dt^2+dx_1^2+dx_2^2+dx_3^2+f(\rho)\,dy^2\right] + \frac{1}{h_1(\rho)}\left[\frac{d\rho^2}{f(\rho)} + \rho^2 d\Omega_4^2\right]$$

$$\phi \sim \log h_1\ ;\quad F_4 \sim \mathrm{vol}\, S^4$$

$$h_1 = \left(\frac{\rho}{R}\right)^{3/2} .\quad f = 1 - \frac{\Lambda^3}{\rho^3}$$

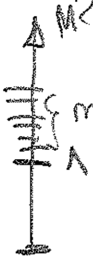
The field theory



$$\mathcal{L} = \mathrm{Tr}\left[\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}|D_\mu\varphi|^2 + \text{UV completion}\right]$$

Weak coupling

at strong coupling



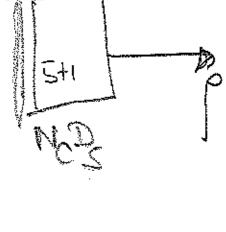
mix between glueballs coming from $F_{\mu\nu}^a$ coming from UV completion

The background describes the strong dynamics of this QFT [just like if there were a chiral Lagrangian with a UV completion]

Another example

D_5 branes and a dual to $\underline{\mathcal{N}=1 \text{ SYM}}$ + UV completion $\underline{\text{in } (3+1) \dim}$

The flat D_5 branes



$$ds^2 = e^{\phi}\left[-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2 + d\rho^2 + \frac{\rho^2}{4}(\omega_1^2 + \omega_2^2 + \omega_3^2)\right]$$

$\phi(\rho)$

$$F_3 = \frac{N_c}{4}\tilde{\omega}_1 \wedge \tilde{\omega}_2 \wedge \tilde{\omega}_3$$

$$\begin{cases} \tilde{\omega}_1 = \cos\psi\, d\tilde{\theta} + \sin\psi \sin\tilde{\theta}\, d\tilde{\varphi} \\ \tilde{\omega}_2 = -\sin\psi\, d\tilde{\theta} + \cos\psi \sin\tilde{\theta}\, d\tilde{\varphi} \\ \tilde{\omega}_3 = d\psi + \cos\tilde{\theta}\, d\tilde{\varphi}. \end{cases}$$

Now on these branes there is an $SU(N_c)$ 16 SUSY ; $SO(4)_R$ (5+1) field theory.

We want to compactify 2 directions $(dx_4, dx_5) \equiv d\theta^2 + \sin^2\theta\, d\varphi^2$

Such that

- part of the SUSY is preserved
- some global symmetry is broken

A general metric satisfying this is

$$ds^2_5 = e^{\phi}\left[dx^2_{1,3} + e^{2h}(d\theta^2 + \sin^2\theta\, d\varphi^2) + \frac{e^{2g}}{4}\left((\tilde\omega_1 - a\,d\theta)^2 + (\omega_2 + a\sin\theta\, d\varphi)^2\right) + \frac{e^{2k}}{4}(\omega_3 + \cos\theta\, d\varphi)^2 + e^{2k}\, d\rho^2\right]$$

$$F_3 = \left\{\frac{N_c}{4}(\tilde\omega_1 + a\,d\theta) \wedge (\tilde\omega_2 + a\sin\theta\, d\varphi) + \frac{N_c}{4}(a^2 - 2ab + 1)\sin\theta\, d\theta \wedge d\varphi + \frac{N_c}{4}(b-a)\left[\sin\theta(\tilde\omega_1 - a\,d\theta)\wedge d\varphi + (\omega_2 + a\sin\theta\, d\varphi)\wedge d\theta\right]\right\}$$

$$\wedge\, (d\psi + \cos\theta\, d\varphi + \cos\tilde\theta\, d\tilde\varphi)$$

$$+ \frac{N_c}{2} b' \, d\rho \wedge (\omega_1 \wedge d\theta + \sin\theta\, d\varphi \wedge \omega_2) = \tilde\omega_1 \wedge \tilde\omega_2 \wedge \tilde\omega_3 + \cdots$$

---

the functions $\phi, h, g, k, a, b \rightarrow$ BPS eqs $\rightarrow$ non linear, coupled, ordinary $\}$ difficult

+ constraint

a good trick is to find another "base" of functions where one can decouple the BPS eqs, solve them. This basis almost exists.

$$[P, Q, \Phi, Y; \sigma]_{\tau} \qquad e^{2h} = \frac{(P^2 - Q^2)}{P\cosh\tau - Q}, \quad e^{2g} = P\cosh\tau - Q, \quad e^{2k} = 4Y$$

$$a = P\sinh\tau \qquad b = \sigma$$

So one can solve

Int constants

$[\rho_0,\ Q_0,\ \phi_0,\ P_1, P_2] \to$ 5 int 6 eqs constant

because of constraint

$$\mathrm{ch}\,\tau = \coth 2\rho$$

$$Q = N_c\,(2\rho\coth 2\rho - 1)$$

$$\sigma = N_c\,\frac{2\rho}{\mathrm{sh}\,2\rho}$$

$$e^{4\phi} = \frac{e^{4\phi_0}}{(P^2 - Q^2)}\,\mathrm{sh}^2(2\rho) \quad ; \quad Y = \frac{P'}{8}$$

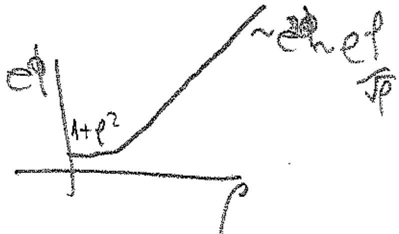
where

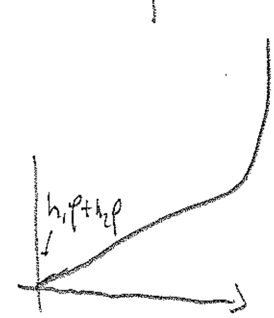
$$\boxed{P'' + P'\left[\frac{P' + Q'}{P - Q} + \frac{P' - Q'}{P + Q} - 4\coth 2\rho\right] = 0}$$

equivalent to all the BPS eqs

There are different solutions to this eq $\to$ exact; $\to$ series + numerics.

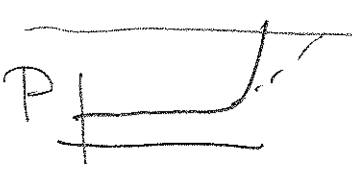
$P = 2N_c\,\rho \;\to$ exact



$P =$ 

$$\rho\to 0 \quad P = h_1\rho + \frac{4h_1}{15}\left(1 - \frac{4N_c^2}{h_1^2}\right)\rho^2 + \frac{165}{525}h_1\left(1 - \frac{4N_c^2}{3h_1^2} - \frac{32N_c^4}{3h_1^4}\right)\rho^5$$

$$\rho\to\infty \quad P = e^{\frac{4}{3}\rho}\left[c + \frac{1}{64c}\left[64.4.N_c^2\rho^2 + 128.2.N_c^2\rho\right]e^{-\frac{8}{3}\rho}\right]$$

P 

numerically (Sonnenschein)

$$P = P_0 + c_1 P_0\rho^3 \quad (\rho\to 0)$$

what is the dual QFT?

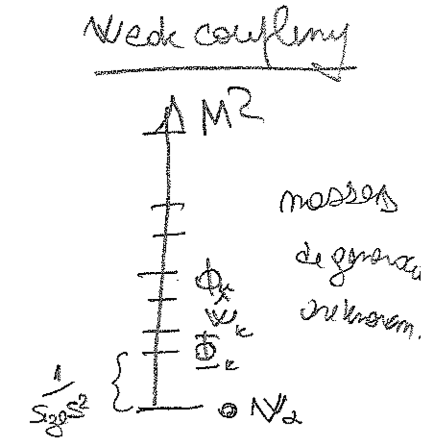
Dorey + Andrews 2007 ×2

careful study of the twisted compactification

$W_\alpha = (A_\mu, \lambda)$ vector multiplet

$\Phi_k = (\varphi_k, \psi_k)$ chiral multiplet

$\hat{W}_\alpha = (\xi_k, a^{(k)}_\mu, \psi_k, \Lambda_k)$ massive vector multiplet



Let me comment in a bit more detail about one "exact" solution.

Start with master eq

$$P'' + P'\left[\frac{P'+Q'}{P-Q} + \frac{P'-Q'}{P+Q} - 4\coth 2\rho\right] = 0$$

{Same as

$$\partial_\rho\left[\frac{P^2-Q^2}{\sinh^2(2\rho)}P'\right] + 4\,\frac{P'Q'Q}{\sinh^2(2\rho)} = 0$$

$\to$ Integrate twice

$$P^3 - 3Q^2P + 6\int^\rho d\rho'\, PQ'Q + 12\int^\rho d\rho' \sinh^2(2\rho)\cdot\int^{\rho'} d\rho''\,\frac{P'QQ'}{\sinh^2(2\rho)}$$

$$= c\left[\cos^3\alpha + \sin^3\alpha\,(\sinh 4\rho - 4\rho)\right]$$

(other version of master eq.)

One can propose a solution

$$P = cP_1 + P_0 + \frac{P_{-1}}{c} + \frac{P_{-2}}{c^2} + \dots = \sum_{k=0}^{\infty} c^{1-k} P_{1-k}(\rho)$$

one consider large "c" $\to$ few terms are good approximation

So one can solve recursively

$$P_1 = \left[\cos^3\alpha + \sin^3\alpha\,(\sinh 4\rho - 4\rho)\right]^{1/3}$$

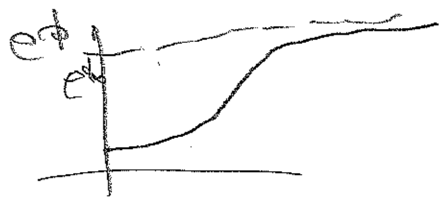
$$P_0 = P_{-2} = P_{-4} = \dots P_{-2k} \equiv 0$$

$$P_{-1} =$$

$$P_{-2k-1} = \text{Recursive relation}$$

In principle → exact solution but the integrals cannot be exactly computed

<u>the dilaton</u>



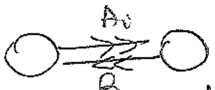
Notice, this is the same dilaton as in Butti et al solution

if one compares F_3 → they are also equal

Is there any relation between these solutions?

Another idea

Start with another QFT in the UV and deform it.



	local		global			
	$SU(N)$	$\times SU(N)$	$\times SU(2)_L$	$\times SU(2)_R$	$\times U(1)_B$	$\times U(1)_R$
A_i^{ab}	N	$\bar{N}$	2	1	1	$\frac{1}{2}$
B_i^{ab}	$\bar{N}$	N	1	2	-1	$\frac{1}{2}$
$W_\alpha^{(1)}$	1	adj	—	—	—	—
$W_\alpha^{(2)}$	adj	1	—	—	—	—

$i = 1,2$

$a: 1, \ldots N$

$b: 1, \ldots N$

with a superpotential $W = \eta\, \varepsilon_{\alpha\beta}\, \varepsilon_{ij}\, A_\alpha B_i A_\beta B_j$

$\rightsquigarrow$ this is a CFT

at a conformal point

$\dim \theta = \frac{3}{2} R[\theta]$

$(\dim \theta = \Delta + m\gamma/2)$

$\beta = 3N_c - N_f(1-\gamma_{\ldots})$

$$\beta_{g_1} = \bar{\theta}\left[3N_1 - 2N_2(1-\gamma)\right] = \bar{\theta}\left[3N - 2N\left(\tfrac{3}{2}\right)\right] = 0$$

$$\beta_{g_2} = -\left[3N_2 - 2N_1(1-\gamma)\right] = -\left[3N - 2N\left(\tfrac{3}{2}\right)\right] = 0$$

$$\Delta\theta_1 = 2 . N . 1 - 2 . 2 . N . (-\tfrac{1}{2}) = 0$$

Same for $\Delta\theta_2$.

So the idea is to deform it by an imbalance in the gauge groups

	$SU(N)$	$\times SU(N+M)$	$\times SU(2)_L$	$\times SU(2)_R$	$\times U(1)_B$	$\times U(1)_R$
A_i	N	$N+M$	2	1	1	$\frac{1}{2}$
B_i	$\bar{N}$	$\overline{N+M}$	1	2	-1	$\frac{1}{2}$
$W_\alpha^{(1)}$						
$W_\alpha^{(2)}$						

$$W = \eta\, \varepsilon_{\alpha\beta}\, \varepsilon_{ij}\, A_\alpha B_i A_\beta B_j$$

$(\gamma \approx -\frac{1}{2})$

$$\beta_{g_1} = -\left[3(N) - 2.(N+M)(1-(-\tfrac{1}{2}))\right] = 3M$$

weakens to the IR

$$\beta_{g_2} = -3M \rightarrow \text{grows to the IR}$$

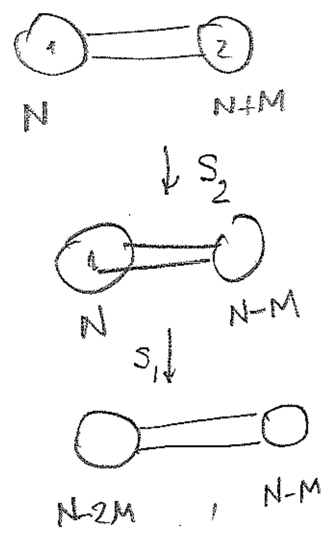
$$\Delta\theta_1 = -\Delta\theta_2 = 3M$$

R

So, while one of the groups becomes strong, the other weak

→ Seiberg duality.

Seiberg on ②

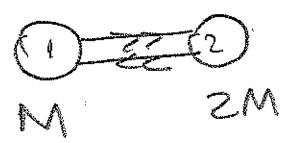


$\beta_1 = -3M$
$\beta_2 = 3M$

$\beta_1 = +3M$
$\beta_2 = -3M$

if we fine tune things so that $N = kM$

we arrive at a step in the cascade



$$\beta_1 = -\left[3M - 2M.2.\tfrac{3}{2}\right] = +3M$$
$$\beta_2 = -\left[3.2M - 2.M.\tfrac{3}{2}\right] = -3M$$

SQCD with $\underbrace{N_f}_{(2M)} = N_c \rightsquigarrow 2M$ → Seiberg studied this case

$$W = \xi\cdot\left[\det M - B\tilde{B} - \Lambda^{2M}\right] + \underbrace{W_{tree}}_{ABAB\,=\,MM}$$

one solution to $\dfrac{\partial W}{\partial \text{fields}} = 0$ $\begin{cases} \dfrac{\partial W}{\partial M_{ij}} = 0 \\ \dfrac{\partial W}{\partial B} = \dfrac{\partial W}{\partial \tilde{B}} = 0 \end{cases}$ ⟶ $M = 0$, $B \neq 0$, $\tilde{B} \neq 0$, $B\tilde{B} = \Lambda^{2M}$

$B = e^{i\alpha}\Lambda^M$
$\tilde{B} = e^{-i\alpha}\Lambda^M$

So, in principle one could give different vev's to $B, \tilde{B}$ → baryonic branch

Now, what is the string background describing this dynamics?
Klebanov – Witten – Tseytlin – Strassler

$$ds^2 = \hat{h}^{-1/2}\, dx_{1,3}^2 + \hat{h}^{1/2}\Big[e^{-6p-x}\left[d\rho^2 + (d\psi + A)^2\right] + e^{x+g}\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) + e^{x-g}\left((\omega_1 - a\, d\theta)^2 + (\omega_2 + a \sin\theta\, d\varphi)^2\right)\Big]$$

Not the notation of Klebanov

F_3

H_3

F_5

$\phi = \text{const}$

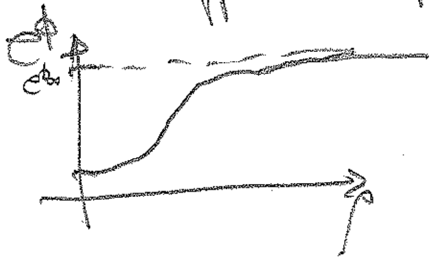
$\hat{h}$, e^{2x}, e^{2g}, e^{2p}, a are known exactly.

$$\hat{h} = \frac{8}{2^{1/3}} \int_\rho^{\infty} \frac{(2\rho \coth 2\rho - 1)\cdot(\mathrm{sh}\, 4\rho - 4\rho)^{1/3}}{\mathrm{sh}^2(2\rho)}\, d\rho \quad ; \quad a = \frac{-1}{\mathrm{ch}\, 2\rho}$$

$$e^{g} = \mathrm{th}\, 2\rho \; ; \; e^{6p+2x} = \frac{3}{2}\left(\coth 2\rho - \frac{2\rho}{\mathrm{sh}^2(2\rho)}\right) ; \; e^{2x} = (\mathrm{sh}\, 2\rho\, \mathrm{ch}\, 2\rho - 2\rho)^{2/3}\, \frac{\hat{h}(\rho)}{16}$$

$h \sim 1 + \kappa\rho^2$ $\qquad\longrightarrow\qquad$ $h \sim \frac{\log \rho}{\rho^4}$ $\quad \rho$

The solution that explores the Baryonic branch (Petrini, Grana, Zaffaroni, Mussio, Butti) has some differences respect to KS most notably the dilaton is not constant



the solution depends on a parameter "c"

related to the VEV of

$$\mathcal{U} = \mathrm{tr}\left(AA^{+} - BB^{+}\right)$$

$$\left[\langle \mathcal{U} \rangle \sim \frac{1}{c}\right]$$

Lecture Relation between wrapped branes and cascading theories [Maldacena, Martelli 2009; Gaillard, Martelli, Nunez, Papadimitriou]

In the last lecture we proposed there maybe a relation between the solution by Butti et al describing the baryonic branch of KS field theory with the solution based on $D5$ branes wrapping S^2 with asymptotically stable dilaton. We will make this precise in what follows.

Consider the wrapped brane metric

$$ds_E^2 = e^{\phi/2}\left[dx_{1,3}^2 + ds_6^2\right]$$

$$ds_6^2 = e^{2k}\left(d\rho^2 + \frac{1}{4}(\omega_3+\cos\theta\, d\varphi)^2\right) + e^{2h}(d\theta^2+\sin^2\theta\, d\varphi^2) + \frac{e^{2g}}{4}\left[(\omega_1 - a\,d\theta)^2 + (\omega_2 + a\sin\theta\, d\varphi)^2\right]$$

We introduce the obvious vielbein.

$$e^\rho = e^{\frac{\phi}{4}+k}\, d\rho \; ; \quad e^\theta = e^{\frac{\phi}{4}+h}\, d\theta \; ; \quad e^1 = \frac{e^{\frac{\phi}{4}+g}}{4}(\tilde{\omega}_1 - a\,d\theta)$$

$$e^3 = \frac{e^{\frac{\phi}{4}+k}}{2}(\omega_3 + \cos\theta\, d\varphi) \; ; \quad e^\varphi = e^{\frac{\phi}{4}+h}\sin\theta\, d\varphi \; ; \quad e^2 = \frac{e^{\frac{\phi}{4}+g}}{4}(\tilde{\omega}_2 + a\sin\theta\, d\varphi)$$

and define a 2-form $J_2 \equiv J$

3 form $\Omega_3 = \Omega$

$$J = e^\rho\wedge e^3 + e^\theta\wedge\left[-\cos\mu\, e^\varphi + \sin\mu\, e^2\right] + e^1\wedge\left[-\sin\mu\, e^\varphi - \cos\mu\, e^2\right]$$

$$\Omega = (e^\rho + i e^3)\wedge\left[e^\theta + i(-\cos\mu\, e^\varphi + \sin\mu\, e^2)\right]\wedge\left[e^1 + i(-\sin\mu\, e^\varphi - \cos\mu\, e^2)\right]$$

There is the following result by Martucci + Smyth 2005

any type IIB background of the form

$$ds_E^2 = e^{2\Delta}\left(dx_{1,3}^2 + ds_6^2\right)$$

$$F_5 = e^{4\Delta+\phi}\,(1+*_{10})\,\mathrm{Vol}\,M_4 \wedge f_1$$

$$H_3 = dB_2$$

has BPS eqs and expression for the fields F_3, H_3, $F_5 \sim f$, F_1, ϕ given by

---

$$d\left(e^{6\Delta+\phi/2}\,\Omega\right)=0 \quad ; \quad d\left(e^{8\Delta}\,J\wedge J\right)=0 \quad ; \quad d\left(e^{2\Delta-\phi/2}\cos\xi\right)=0$$

these are the BPS eqs

where the fluxes are given by

$$F_5 \sim f_1 = -\,e^{-4\Delta-\phi}\,d\left(e^{4\Delta}\sin\xi\right)$$

$$*_6 F_3 = -\frac{e^{-\phi}}{\cos\xi}\left[e^{-2\Delta-\phi/2}\,d\left(e^{4\Delta+\phi}J\right) + e^{2\Delta+3/2\,\phi}\sin\xi\; d\left(e^{-\phi}\sin\xi\right)\right]$$

$$H_3 = -\sin\xi\, e^{\phi} *_6 F_3 + \cos\xi\, e^{2\Delta+3/2\,\phi}\, d\left(e^{-\phi}\sin\xi\right)$$

$$*_6 F_1 = -\tfrac{1}{2}\, d\left(e^{-\phi}\sin\xi\right)\wedge J \wedge J$$

Now, let us assume that we want the RR 1-form $=0$ [not confuse this with f_1]

$$F_1 = 0 \rightsquigarrow d(e^{-\phi}\sin\xi) = 0 \rightsquigarrow \boxed{\sin\xi = \eta_2 e^{\phi}}$$

the BPS eqs turn into
and fluxes

$$d\left(e^{6\Delta+\phi/2}\Omega\right)=0 \; ; \; d\left(e^{8\Delta}J\wedge J\right)=0 \; ; \; d\left(e^{2\Delta-\phi/2}\cos\xi\right)=0$$

$$F_5 \to f_1 = -e^{-4\Delta-\phi}d(e^{4\Delta}\sin\xi) = \eta_2 e^{-4\Delta-\phi}\, d(e^{4\Delta+\phi})$$

$$*_6 F_3 = -\frac{e^{-\phi}}{\cos\xi}\left[e^{-2\Delta-\phi/2}\, d\left(e^{4\Delta+\phi}J\right)\right]$$

$$H_3 = -\sin\xi\, e^{\phi} *_6 F_3$$

Ok, now, let us analyze this for $\underline{\xi = 0}$

$$F_5 = H_3 = 0$$

$$d\left(e^{6\Delta+\phi/2}\Omega\right)=0 \; ; \; d\left(e^{8\Delta}J\wedge J\right)=0$$

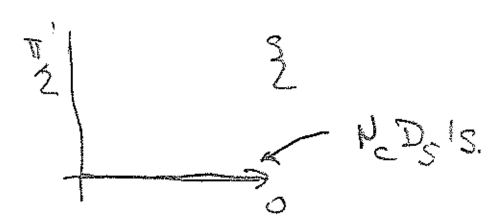
$$d\left(e^{2\Delta-\phi/2}\right)=0 \rightsquigarrow \boxed{e^{2\Delta} = \eta_1 e^{\phi/2}}$$

$$*_6 F_3 = e^{-2\Delta-3/2\phi}\, d\left(e^{4\Delta+\phi}J\right) = -e^{-2\phi}\, d\left(e^{2\phi}J\right)$$

this is equivalent to Master eq

F_3 we wrote D_5 on S^2

$\xi = 0$ corresponds to $N_c D_5$ on $S^2 \subset$ resolved conifold



Can we <u>rotate</u> in ξ?

Can we construct a solution with $\xi \neq 0$? certainly but this would imply to solve BPS all over again.!

But if we impose.

- Dilaton before / after is the same
- F_1 before / after is the same
- to preserve the <u>same BPS</u> $\longrightarrow$ no need to solve again

$$\phi_{New} = \phi_{old} = \phi$$

$$F_{1\,New} = d\left(e^{-\phi} \sin\xi\right) = F_{1\,old} = 0 \ \rightarrow \ \sin\xi = k_2 e^{\phi}$$

$$d\left(e^{6\Delta_{New} + \phi/2}\, \Omega_{New}\right) = d\left(e^{6\Delta_{old} + \phi/2}\, \Omega_{old}\right) = 0$$

$$d\left(e^{8\Delta_{New}} J_{New} \wedge J_{New}\right) = d\left(e^{8\Delta_{old}} J_{old} \wedge J_{old}\right) = 0$$

$$d\left(e^{2\Delta_{New} - \phi/2} \cos\xi\right) = d\left(e^{2\Delta_{old} - \phi/2}\right) = 0$$

we can satisfy all this by a <u>simple scaling</u> !

Lecture flavor in AdS/CFT

All the theories we have studied up to here are theories with only adjoint fields [all the theories were basically UV-completed versions of $\mathcal{N}=1$ SYM]

Now, let us compare $\mathcal{N}=1$ SYM and QCD in the same critical spirit as when we compared $\mathcal{N}=4$ SYM and QCD

$\mathcal{N}=1$ SYM	($\mathcal{N}=0$) QCD (massless)
$\mathcal{L}=-\frac{1}{4}\vec{F}_{\mu\nu}\vec{F}^{\mu\nu}+i\bar{\lambda}^{\prime}\not{D}\lambda$	$\mathcal{L}=-\frac{1}{2}\vec{F}_{\mu\nu}\vec{F}^{\mu\nu}+i\bar{\Psi}\not{D}\Psi$
$SU(N_c)$	$SU(N_c)$
A_μ^a, λ^a adjoint	A_μ^a adjoint
	ψ^α: fundamental
$U(1)_R$: global symmetry $\hookrightarrow Z_{2N_c}$ (anomaly) $\to Z_2$ (spont)	$SU(N_f)\times SU(N_f)\times U(1)_B$
confines	confines $\to$ screens
glueballs	glueballs, mesons $\to$ mixing
χ chiral symmetry	χ symmetry $\to$ broken $\to$ spont / $\to$ explicitly

So, the fact of having the fields in the fundamental (quarks) introduces very important differences [dynamical differences]

- more particles
- more symmetries
- confinement $\longrightarrow$ screening
- coupling runs differently and anomalies modified
- Seiberg-like dualities

So, how do we encode all this in a String background

We should first <u>go back 1970's</u>

1974 't Hooft proposed a scaling

$$\left.\begin{array}{l} g_{YM}^2 \to 0 \\ N_c \to \infty \end{array}\right\} g_{YM}^2 N_c = \text{fixed}$$

<u>quenched</u>

$$N_f \to \text{const} \quad / \quad \frac{N_f}{N_c} \to 0$$

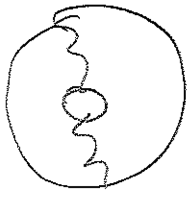
1977 Veneziano proposed

$$\left.\begin{array}{l} g_{YM}^2 \to 0 \\ N_c \to \infty \end{array}\right\} g_{YM}^2 N_c = \text{fixed}$$

<u>unquenched</u>

$$N_f \to \infty \quad / \quad \frac{N_f}{N_c} = x \ \text{fixed}$$

~~Different~~ Same diagrams behave differently under both scalings [Capella et al Phys Rept 1994]

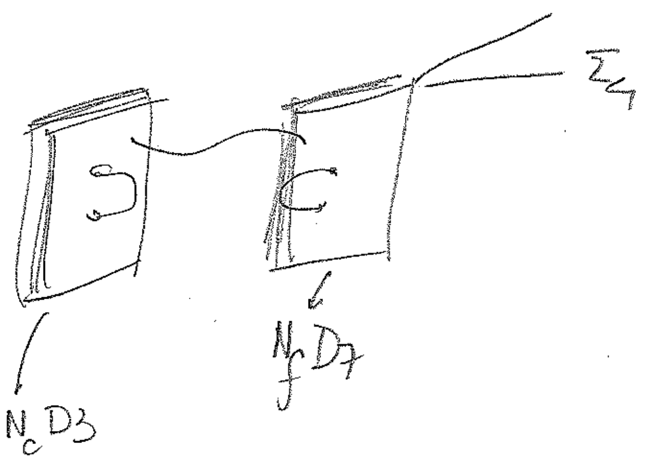
	't Hooft	Veneziano
	1	1
	$\left(\frac{N_f}{N_c}\right)^w \sim \frac{1}{N_c^w}$	$\left(\frac{N_f}{N_c}\right)^w \sim 1$
	$\sim \frac{1}{N_c^2}$	$\frac{1}{N_c^2}$

The interesting thing is that **both** scaling can be realized in string th.

How to add flavor ? $\longrightarrow$ add $\rightarrow$ symmetries $\longrightarrow$ gauge field in bulk
$\rightarrow$ particles $\longrightarrow$ meson fluctuations

$\longrightarrow$ add D-branes

Let us think the D3/D7 example [at weak coupling]



3-3 strings $\longrightarrow$ $SU(N_c)$ $\mathcal{N}=4$ SYM 3+1, $g^2_{YM}=g_s$

7-7 strings $\longrightarrow$ $SU(N_f)$ 16 SUSY SYM in 7+1, $g^2_{YM}=g_s\alpha'^2$

3-7, 7-3 string: 3+1 fundamentals

$Q^{a,b}$ $\quad a: 1,\dots,N_c$

$\tilde{Q}^{a,b}$ $\quad b: 1,\dots,N_f$

$g^2_{YM}=g_s$

in the limit $\alpha' \to 0$ the theory on 7-7 string decouple

what happens at strong coupling?

N_c D3 branes $\longrightarrow$ background

N_f D7 brane $\longrightarrow$?

here is where we realize both scalings

't Hooft Scaling

- probe (like an EM with a probe charge) the background with N_f D-flavor branes

$$S_{Brane} = S_{DBI} + S_{WZ} = T_{Dp} \int e^{-\phi} \sqrt{\det(g_{ab} + F_{ab})}\, d^{p+1}x - T_{Dp} \int e^{F} \wedge C_p$$

background fields! — quenched

The brane feels the effect of the background, but not viceversa

dynamics of the brane is dictated by background

Veneziano Scaling

new action → new modified dynamics

$$S = S_{IIB/A} + S_{Brane}$$ → new eqs of motion — influence on one another

unquenched

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}[\text{IIA/B fields}] + T^{brane}_{\mu\nu}$$ → new solutions

Notice that

- we are adding new degrees of freedom [D-branes, $T^{brane}_{\mu\nu}$]
- we are adding new symmetries [gauge fields on brane]
- color branes / flavor branes are not equally treated < closed string / open strings → reflects difference between local / global symmetry

I will focus on applying this to N_c D_5 branes + flavor.

We start all over again from $N_c D_5$ branes metric and F_3

$$ds^2_E = e^{\phi/2}\Big[dx^2_{1,3} + e^{2k}\big(d\rho^2 + \tfrac{1}{4}(d\psi+A)^2\big) + e^{2h}\big(d\theta^2 + \sin^2\theta\, d\varphi^2\big) + \frac{e^{2g}}{4}\big[(\omega_1 - a\,d\theta)^2 + (\omega_2 + a\sin\theta \ldots$$

$$F_3 = -2N_c\, e^{-2g-k}\, e^1\wedge e^2\wedge e^3 + \frac{N_c}{2}(a^2-2ab+1)\, e^{-2h-k}\, e^\theta\wedge e^\varphi\wedge e^3$$
$$+ N_c(b-a)\, e^{-g-h-k}\,(e^1\wedge e^\varphi + e^2\wedge e^\theta)\, e^3 + \frac{N_c}{2}\, b'\, e^{-g-h-k}\, e^\rho\wedge(e^1\wedge e^\theta + e^\varphi \ldots$$

$\phi(\rho)$.

Solution to

$$S_{IIB} = \frac{1}{2\kappa^2_{10}}\int d^{10}x\,\sqrt{g}\,\Big[R - \frac{1}{2}(\partial_\mu\phi)^2 - \frac{e^\phi}{12}F_3^2\Big].$$

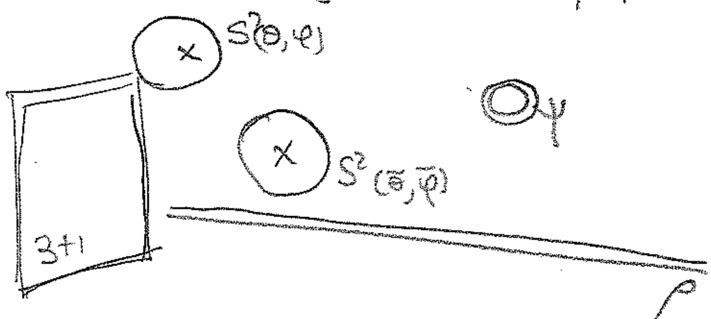
Remember that we can write a master eq here

$$P'' + P'\Big[\frac{P'+Q'}{P-Q} + \frac{P'-Q'}{P+Q} - 4\coth 2\rho\Big] = 0$$

$$e^{2h} = \frac{(P^2-Q^2)e^{-2g}}{4}$$
$$e^{2g} = P\coth\tau - Q$$
$$e^{2k} = 4Y$$
$$a = P\sinh\tau\, e^{-2g}\,, \quad b = \sigma/N_c$$

- $\cosh\tau = \coth 2\rho$
- $\sigma = \dfrac{N_c\, 2\rho}{\sinh 2\rho}$
- $Y = \dfrac{P'}{8}$

Now we add flavor $\to$ SUSY "probes"

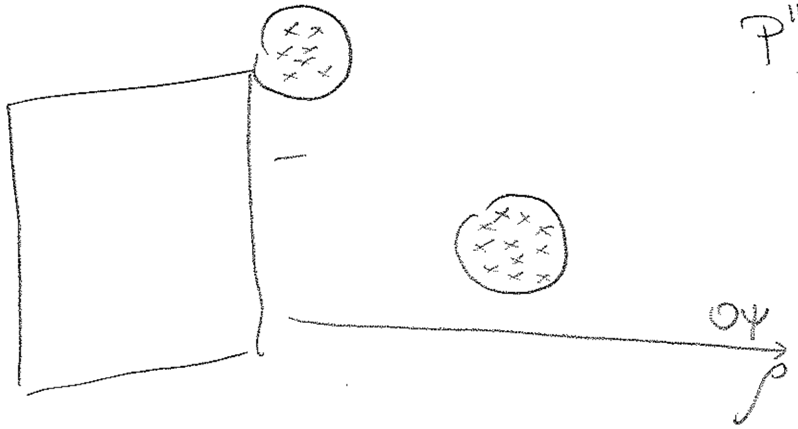


$N_f\ D_5 : [\vec{x}_{1,3}, \rho, \psi]$

$$S = S_{IIB} + \Big[-N_f T_{D_5}\int e^{\phi/2}\sqrt{\det g_{ind}} + N_f T_{D_5}\int C_6\Big]\delta[S^2\times\tilde S^2]$$

$\to$ Partial diff eq. (BPS) $\longrightarrow$ to make this easier <u>smear</u>

$$S = S_{IIB} + S_{Brane}\Big/\ S_{Brane} \sim \int e^{\phi/2}\sqrt{\det g_{ind}}\,|\Omega_4|\,d^{10}x - \int C_6\wedge\Omega_4 \to$$ BPS eqs ordinary non linear coupled



$$P'' + (P' + N_f)\left[\frac{P' + Q' + 2N_f}{P - Q} + \frac{P' - Q' + 2N_f}{P + Q} - 4\coth 2\rho\right] = 0.$$

$\longrightarrow$ new solutions containing N_f, N_c explicit.

Example

$$e^{4\phi}\Big|_{\rho\to 0} \sim e^{4\phi_0}\left[1 + \frac{64N_c^2}{h_1^2}\rho^2 + \mathcal{O}(\rho^4)\right]. \qquad N_f = 0$$

$$e^{4\phi}\Big|_{\rho\to 0} \sim e^{4\phi_0}\left[1 + \frac{4N_f}{P_0}\rho + \mathcal{O}(\rho^2)\right] \qquad N_f \neq 0$$

In these changes is hidden interesting gauge theory dynamics.

UV

$\cancel{e^{2h}\big|_{\rho\to\infty} =}$

$\cancel{e^{2h}\big|_{\rho\to\infty} =}$

$$P \sim 2N_c\rho$$
$$e^{2h} \sim N_c\rho \; ; \; e^{2g} \sim N_c, \; Y = \frac{N_c}{4} \; ; \; e^{4\phi} \sim \frac{e^{4\rho}}{\rho}$$

$$P \sim \left(\frac{2N_c - N_f}{2}\right)\rho \; ; \; e^{2h} \sim \frac{(2N_c - N_f)}{2}\rho \; ; \; e^{2g} \sim N_c \; ; \; Y \sim \frac{N_c}{4}$$

In these changes field theory aspects are hidden.

~~what can we learn?~~

Let us think a bit about the dual QFT

We start with a QFT ~~$\mathcal{N}=$~~

$W_\alpha = (A_\mu, \lambda)$, $\Phi_K = (\varphi_K, \psi_K)$

$$\mathcal{L} = \int d^2\theta\, W_\alpha W^\alpha + \int d^4\theta\, \Phi_K^\dagger e^V \Phi_K + \int d^2\theta\, \mu_K \Phi_K^2 + \int d^2\theta\, W(\Phi_K)$$

and to this one we are adding flavor multiplets

$Q = (q, \psi_q)$
$\tilde{Q} = (\tilde{q}, \tilde{\psi}_q)$

A nice way of thinking about this is to start from $\mathcal{N}=2$ SQCD

$$\mathcal{L} = \int d^4\theta\, \Phi^\dagger e^V \Phi + \tilde{Q}^\dagger e^{-V} Q + Q^\dagger e^V Q + \int d^2\theta\, W_\alpha W^\alpha + \underbrace{W(\Phi, Q, \tilde{Q})}_{\kappa \tilde{Q}\Phi Q}$$

and instead of having 1 scalar having many of them

$$\phi \longrightarrow \phi_K \quad K = 1, \ldots, \infty$$

this breaks $\underline{\mathcal{N}=2 \longrightarrow \mathcal{N}=1}$

and giving them masses. μ_K

$$\mathcal{L} = \int d^4\theta\, \tilde{Q}^\dagger e^{-V} Q + \tilde{Q}^\dagger e^V Q + \sum_K \Phi_K e^V \Phi_K + \int d^2\theta\, W_\alpha W^\alpha + \sum_K \kappa_K \tilde{Q} \phi_K Q + \mu_K \Phi_K^2$$

Now we look at this theory at Low Energy $\longrightarrow$ integrate out Φ_K

$$\mathcal{L} = \int d^4\theta\, \tilde{Q}^\dagger e^{-V} \tilde{Q} + Q^\dagger e^V Q + \int d^2\theta\, W_\alpha W^\alpha + \int d^2\theta\, \frac{\kappa}{\mu} (\tilde{Q} Q)^2 .$$

- Namely irrelevant deformation in $\mathcal{N}=1$ SQCD
- Various changes in QFT that can be learn from pure field theory methods

TX

Let me focus on two things

$$\frac{1}{g^2_{YM}} \sim \mathrm{Vol}\, S^2 = e^{2h} + \frac{e^{2g}}{4}(a-1)^2$$

$$\beta = \frac{d\, \frac{1}{g^2_{YM}}}{d\rho} \cdot \frac{d\rho}{d\log\mu_{/\Lambda}} = \frac{d}{d\rho}\left(e^{2h} + \frac{e^{2g}}{4}(a-1)^2\right) \cdot \left[\frac{d\rho}{d\log\mu_{/\Lambda}}\right]$$

$$e^{2h} \sim (2N_c - N_f)\,\rho$$

$$\rho \sim (2N_c - N_f)\left[\frac{d\rho}{d\log\mu_{/\Lambda}}\right]$$

anomalies

$$\Delta\theta = \int_{S^2} C_2 = \frac{(2N_c - N_f)}{2}\,\psi$$

Seiberg duality

BPS eqs [for illustrative purposes when turn off fibration $a=b=0$]

$$\left.\begin{aligned} H' &= \frac{N_c - N_f}{2} + 2Y \\ G' &= -\frac{N_c}{2} + 2Y \\ Y' &= -\frac{(N_f - N_c)}{2}\frac{Y}{H} - \frac{N_c}{2}\frac{Y}{G} - 2Y^2\left(\frac{1}{H} + \frac{1}{G}\right) + 4Y \\ \phi' &= -\frac{(N_c - N_f)}{4H} + \frac{N_c}{4G} \end{aligned}\right\}$$

invariant under

$H \longleftrightarrow G$

$Y \longleftrightarrow Y$

$\phi \longleftrightarrow \phi$

$N_c \longrightarrow N_f - N_c$

$N_f \longrightarrow N_f$

from the most generic view of the master eq

$$P'' + (P' + N_f)\left[\frac{P' + Q' + 2N_f}{P - Q} + \frac{P' - Q' + 2N_f}{P + Q} - 4\coth 2\rho\right] = 0$$

$P \to P \quad N_c \to 2N_f - N_c$

$Q \to -Q \quad N_f \to N_f$