Lecture 1: Clifford algebras: basic notions

Consider now a system of n units $\iota_1, \iota_2, \ldots, \iota_n$ such that the multiplication of any two of them is polar; that is, $\iota_r \iota_s = -\iota_s \iota_r$.

— William Kingdon Clifford, 1878

In this lecture we define the Clifford algebra of a quadratic vector space and view it from three different points of view: the contemporary categorical formulation, Clifford's original formulation and as a quantisation of the exterior algebra.

1.1 Quadratic vector spaces

Throughout $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let V be a finite-dimensional vector space over $\mathbb{K}$, let $B : V \times V \to \mathbb{K}$ be a (possibly degenerate) symmetric bilinear form and let $Q : V \to \mathbb{K}$ denote the corresponding quadratic form, defined by $Q(x) = B(x, x)$. One can recover B from Q by *polarisation*, namely

$$B(x, y) = \tfrac{1}{2}\big(Q(x+y) - Q(x) - Q(y)\big)\,. \tag{1}$$

The pair (V, Q) is called a **quadratic vector space** (over $\mathbb{K}$). They are the objects of a category **QVec** with morphisms $(V, Q_V) \to (W, Q_W)$ given by linear maps $f : V \to W$ such that $f^* Q_W = Q_V$, or explicitly that $Q_W(f(x)) = Q_V(x)$ for all $x \in V$. The zero vector space with the zero quadratic form is an initial object in **QVec**. The absence of terminal objects and (co)products is due to the fact that projections do not generally preserve norms.

We will see that the Clifford algebra $C\ell(V, Q)$ of a quadratic vector space (V, Q) is an associative, unital $\mathbb{K}$-algebra, with a natural filtration and a $\mathbb{Z}_2$-grading, and moreover that the assignment $(V, Q) \mapsto C\ell(V, Q)$ is functorial.

There are several ways to understand $C\ell(V, Q)$: from the very abstract to the very concrete. The latter is good for computations, whereas the former is good to prove theorems which may free us from computations. Therefore we will look at $C\ell(V, Q)$ in several ways, starting with the categorical definition.

All our associative algebras are unital, unless otherwise stated!

1.2 The Clifford algebra, categorically

Let (V, Q) be a quadratic vector space and let A be an associative $\mathbb{K}$-algebra. We say that a $\mathbb{K}$-linear map $\phi : V \to A$ is **Clifford** if for all $x \in V$,

$$\phi(x)^2 = -Q(x) 1_A\,, \tag{2}$$

where 1_A is the unit of A. Clifford maps from a fixed quadratic vector space (V, Q) are the objects of a category **Cliff**(V, Q), where a morphism from $V \to A$ to $V \to A'$ is given by a commuting triangle

$$\begin{tikzcd} & V \arrow[dl] \arrow[dr] & \\ A \arrow[rr, "f"] & & A' \end{tikzcd} \tag{3}$$

with $f : A \to A'$ a homomorphism of associative algebras.

1.2.1 Definition

Definition 1.1. The **Clifford algebra** — if it exists — is an initial object in **Cliff**(V, Q). In other words, it is given by an associative algebra $C\ell(V, Q)$ together with a Clifford map $i : V \to C\ell(V, Q)$ such that for every Clifford map $\phi : V \to A$ there is a *unique* algebra morphism $\Phi : C\ell(V, Q) \to A$ making the following triangle commute:

(4)
$$\begin{tikzcd} & V \arrow[dl, "i"'] \arrow[dr, "\phi"] & \\ C\ell(V,Q) \arrow[rr, "\Phi"] & & A \end{tikzcd}$$

Remark 1.2. There are several paraphrases of the defining property of the Clifford algebra. One can say that every Clifford map factors uniquely via the Clifford algebra, or that the Clifford algebra is *universal* for Clifford maps, or that every Clifford maps extends uniquely to a morphism of associative algebras from the Clifford algebra.

Remark 1.3. The mathematical literature is replete with such universal definitions. For example, if $\mathfrak{g}$ is a Lie algebra and A is an associative algebra (over the same ground field) then one can consider linear maps $\phi : \mathfrak{g} \to A$ such that, for all $X, Y \in \mathfrak{g}$,

(5)
$$\phi(X)\phi(Y) - \phi(Y)\phi(X) = \phi([X, Y])$$

Although it is not standard terminology, let us call such maps **Lie** within the confines of this remark. Then the *universal enveloping algebra* $U\mathfrak{g}$ of $\mathfrak{g}$ is universal for Lie maps; in other words, $U\mathfrak{g}$ is an associative algebra with a Lie map $i : \mathfrak{g} \to U\mathfrak{g}$ extending any Lie map $\phi : \mathfrak{g} \to A$ uniquely; i.e., there is a unique associative algebra morphism $\Phi : U\mathfrak{g} \to A$ such that the following triangle commutes:

(6)
$$\begin{tikzcd} & \mathfrak{g} \arrow[dl, "i"'] \arrow[dr, "\phi"] & \\ U\mathfrak{g} \arrow[rr, "\Phi"] & & A \end{tikzcd}$$

In other words, $U\mathfrak{g}$ is what allows us to "multiply" elements of $\mathfrak{g}$ as if they were matrices. One constructs the universal enveloping algebra as a quotient of the tensor algebra $T\mathfrak{g}$ of $\mathfrak{g}$ by the 2-sided ideal generated by $X \otimes Y - Y \otimes X - [X, Y]$ for all $X, Y \in \mathfrak{g}$. The construction of the Clifford algebra will proceed along similar lines.

Initial objects in a category are unique up to unique isomorphism, hence the following should not be too surprising.

Proposition 1.4. *The Clifford algebra* $C\ell(V, Q)$, *if it exists, is unique up to a unique isomorphism.*

Proof. Let $i : V \to C$ and $i' : V \to C'$ be two Clifford algebras. Then since C is a Clifford algebra, there is a unique morphism $\Phi : C \to C'$ making the following triangle commute

(7)
$$\begin{tikzcd} & V \arrow[dl, "i"'] \arrow[dr, "i'"] & \\ C \arrow[rr, "\Phi"] & & C' \end{tikzcd}$$

whereas since C' is a Clifford algebra, there is a unique morphism $\Phi' : C' \to C$ making the following triangle commute

(8)
$$\begin{tikzcd} & V \arrow[dl, "i'"'] \arrow[dr, "i"] & \\ C' \arrow[rr, "\Phi'"] & & C \end{tikzcd}$$

Now the composition $\Phi' \circ \Phi : C \to C$ makes the following triangle commute

(9)
$$\begin{tikzcd} & V \arrow[dl, "i"'] \arrow[dr, "i"] & \\ C \arrow[rr, "\Phi'\circ\Phi"] & & C \end{tikzcd}$$

and so does the identity $1_C : C \to C$, whence $\Phi' \circ \Phi = 1_C$. A similar argument shows that $\Phi \circ \Phi' = 1_{C'}$, whence $\Phi : C \to C'$ is an isomorphism. □

Assuming for a moment that Clifford algebras exist, we have the following

Proposition 1.5. *The Clifford algebra defines a functor* $C\ell$ *from* ***QVec*** *to the category of associative algebras.*

Proof. Indeed, let (V, Q_V) and (W, Q_W) be quadratic vector spaces and $i_V : V \to C\ell(V, Q_V)$ and $i_W : W \to C\ell(W, Q_W)$ the corresponding Clifford algebras. Now let $f : V \to W$ with $f^* Q_W = Q_V$ be a morphism in **QVec** and consider $i_W \circ f : V \to C\ell(W, Q_W)$. We observe that it is a Clifford map:

(10)
$$(i_W \circ f)(x)^2 = f(x)^2 = -Q_W(f(x))\mathbf{1}_W = -Q_V(x)\mathbf{1}_W\,,$$

where $\mathbf{1}_W$ is the identity in $C\ell(W, Q_W)$. Therefore by universality, there is a unique morphism $C\ell(f) : C\ell(V, Q_V) \to C\ell(W, Q_W)$. It is clear that if $1_V : V \to V$ is the identity transformation, then uniqueness forces $C\ell(1_V) = 1_{C\ell(V,Q)} : C\ell(V, Q) \to C\ell(V, Q)$ to be the identity morphism (not to be confused with the unit $\mathbf{1}$ in the Clifford algebra). Similarly, if (X, Q_X) is a third quadratic vector space and $g : W \to X$ with $g^* Q_X = Q_W$, then universality gives a morphism $C\ell(g) : C\ell(W, Q_W) \to C\ell(X, Q_X)$ and the composition $C\ell(g) \circ C\ell(f)$ has to agree (again by uniqueness) with $C\ell(g \circ f)$ where $g \circ f : V \to X$ is the composition Clifford map. □

Remark 1.6. The universal enveloping algebra also defines a functor from the category of Lie algebras to the category of associative algebras which is left adjoint to the functor which sends an associative algebra to the Lie algebra it becomes under the commutator. The functor defined by the Clifford algebra does not seem to be an adjoint functor in any interesting way.

1.2.2 Construction

Let $T^\bullet V = \bigoplus_{p \ge 0} V^{\otimes p}$ denote the tensor algebra of V, where $V^{\otimes 0} = \mathbb{K}$, $V^{\otimes 1} = V$ and $V^{\otimes p}$ is spanned by monomials $x_1 \otimes x_2 \otimes \cdots \otimes x_p$ with $x_i \in V$. The multiplication $V^{\otimes p} \times V^{\otimes q} \to V^{\otimes (p+q)}$, given by extending bilinearly the concatenation of monomials

(11)
$$(x_1 \otimes \cdots \otimes x_p)(y_1 \otimes \cdots \otimes y_q) = x_1 \otimes \cdots \otimes x_p \otimes y_1 \otimes \cdots \otimes y_q,$$

makes $T^\bullet V$ a graded algebra. The identity is given by $1 \in V^{\otimes 0}$. The tensor algebra is universal for linear maps $\phi : V \to A$, where A is an associative algebra. Indeed, any such map extends uniquely to an algebra morphism $\Phi : TV \to A$ defined by $\Phi(\lambda) = \lambda 1_A$ for $\lambda \in \mathbb{K}$, $\Phi(x) = \phi(x)$ for $x \in V$, and more generally

(12)
$$\Phi(x_1 \otimes \cdots \otimes x_p) = \phi(x_1) \cdots \phi(x_p).$$

In fact, the tensor algebra is the free associative algebra generated by V. The tensor algebra defines a functor T from the category of vector spaces to the category of associative algebras, which is left adjoint to the forgetful functor going in the opposite direction.

By definition, the Clifford algebra $C\ell(V, Q)$ is universal for Clifford maps to associative algebras. Since the tensor algebra is universal for linear maps to associative algebras, we expect $C\ell(V, Q)$ to be a quotient of TV by an ideal which imposes the condition that a linear map is Clifford. To this end, let us consider the 2-sided ideal I_Q of TV generated by elements of the form $x \otimes x + Q(x) \in V^{\otimes 2} \oplus V^{\otimes 0}$. Explicitly, I_Q is spanned (over $\mathbb{K}$) by elements of the form

(13)
$$x_1 \otimes \cdots \otimes x_p \otimes (z \otimes z + Q(z)) \otimes y_1 \otimes \cdots \otimes y_q$$

for some p, q and $x_i, y_j, z \in \mathrm{V}$.

If $\phi : \mathrm{V} \to \mathrm{A}$ is a Clifford map and $\widetilde{\Phi} : \mathrm{TV} \to \mathrm{A}$ the unique extension of ϕ to the tensor algebra, then it is easy to see that $\widetilde{\Phi}$ annihilates $\mathrm{I_Q}$ precisely because ϕ is Clifford:

$$\widetilde{\Phi}(\Theta \otimes (z \otimes z + \mathrm{Q}(z)) \otimes \Xi) = \widetilde{\Phi}(\Theta)\left(\phi(z)^2 + \mathrm{Q}(z)1_{\mathrm{A}}\right)\widetilde{\Phi}(\Xi) = 0, \tag{14}$$

for any $\Theta, \Xi \in \mathrm{TV}$. Hence $\widetilde{\Phi}$ factors through a unique map $\Phi : \mathrm{TV}/\mathrm{I_Q} \to \mathrm{A}$ from the quotient. We define $\mathrm{C}\ell(\mathrm{V},\mathrm{Q}) = \mathrm{TV}/\mathrm{I_Q}$ to be the quotient algebra, and the map $i : \mathrm{V} \to \mathrm{C}\ell(\mathrm{V},\mathrm{Q})$ is defined by the commutativity of the triangle

$$\begin{tikzcd} \mathrm{V} \arrow[r] \arrow[dr, "i"'] & \mathrm{TV} \arrow[d] \\ & \mathrm{C}\ell(\mathrm{V},\mathrm{Q}) \end{tikzcd} \tag{15}$$

We remark that i is injective because the ideal only "kicks in" at $\mathrm{V}^{\otimes\geq 2}$, whence in many cases we will not write i explicitly and think of V as sitting inside $\mathrm{C}\ell(\mathrm{V},\mathrm{Q})$.

Since the ideal $\mathrm{I_Q}$ is not homogeneous, $\mathrm{C}\ell(\mathrm{V},\mathrm{Q})$ does not inherit a grading from TV, but since the ideal has even parity, $\mathrm{C}\ell(\mathrm{V},\mathrm{Q})$ does inherit a $\mathbb{Z}_2$-grading. We will see this later from a different point of view, where we also show that it inherits a filtration from the canonical filtration of TV.

1.3 The Clifford algebra as Clifford would have written it

We now discuss $\mathrm{C}\ell(\mathrm{V},\mathrm{Q})$ in a way more suitable to computations. This is the way that Clifford introduced the algebras and the way they are still taught in Physics courses, following Dirac.

1.3.1 Clifford algebra in terms of generators and relations

We start by choosing a $\mathbb{K}$-basis (e_i) for V, where $i = 1, \ldots, n = \dim \mathrm{V}$, relative to which $\mathrm{B}(e_i, e_j) = \mathrm{B}_{ij} = \mathrm{B}_{ji}$. Let Γ_i denote the image of e_i under $i : \mathrm{V} \to \mathrm{C}\ell(\mathrm{V},\mathrm{Q})$. Then the Γ_i satisfy the relation

$$\Gamma_i\Gamma_j + \Gamma_j\Gamma_i = -2\mathrm{B}_{ij}\mathbf{1}, \tag{16}$$

where $\mathbf{1}$ is the unit in the Clifford algebra. The Clifford algebra is thus the associative algebra generated by the Γ_i subject to the above relation. This is enough to write down the product of any two generators:

$$\Gamma_i\Gamma_j = \tfrac{1}{2}(\Gamma_i\Gamma_j - \Gamma_j\Gamma_i) + \tfrac{1}{2}(\Gamma_i\Gamma_j + \Gamma_j\Gamma_i) = \Gamma_{ij} - \mathrm{B}_{ij}\mathbf{1}\,, \tag{17}$$

where we have introduced the notation $\Gamma_{ij} = \frac{1}{2}(\Gamma_i\Gamma_j - \Gamma_j\Gamma_i)$. It seems to be a new object, since it cannot be reduced further using the relations. With a little bit more energy, one can compute the product

$$\Gamma_i\Gamma_{jk} = \Gamma_{ijk} - \mathrm{B}_{ij}\Gamma_k + \mathrm{B}_{ik}\Gamma_j\,, \tag{18}$$

where we have defined the alternating product of three generators

$$\Gamma_{ijk} = \tfrac{1}{6}\left(\Gamma_i\Gamma_j\Gamma_k - \Gamma_i\Gamma_k\Gamma_j + \Gamma_j\Gamma_k\Gamma_i - \Gamma_j\Gamma_i\Gamma_k + \Gamma_k\Gamma_i\Gamma_j - \Gamma_k\Gamma_j\Gamma_i\right). \tag{19}$$

More generally define

$$\Gamma_{i_1\cdots i_p} = \frac{1}{p!}\sum_{\sigma\in\mathfrak{S}_p}(-1)^\sigma\Gamma_{i_{\sigma(1)}}\cdots\Gamma_{i_{\sigma(p)}}\,, \tag{20}$$

where $(-1)^\sigma$ is the sign of the permutation σ of $\{1, 2, \ldots, p\}$. Continuing in this way, and since $\mathrm{C}\ell(\mathrm{V},\mathrm{Q})$ is generated by V and the identity, we see that $\mathrm{C}\ell(\mathrm{V},\mathrm{Q})$ is the linear span of $\mathbf{1}$, Γ_i, Γ_{ij},... In total there are $1 + n + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$ such monomials, whence $\dim \mathrm{C}\ell(\mathrm{V},\mathrm{Q}) = 2^{\dim \mathrm{V}}$. This is the same dimension of the exterior algebra $\Lambda\mathrm{V}$ and in fact we can establish a vector space isomorphism $\Lambda\mathrm{V} \cong \mathrm{C}\ell(\mathrm{V},\mathrm{Q})$ by sending $1 \mapsto \mathbf{1}$, $e_i \mapsto \Gamma_i$ and $e_{i_1} \wedge \cdots \wedge e_{i_p} \mapsto \Gamma_{i_1\cdots i_p}$.

In the next section we will see this isomorphism from a different perspective. Namely we will show that $\mathrm{C}\ell(\mathrm{V},\mathrm{Q})$ is a filtered algebra whose associated graded algebra is the exterior algebra. Of course, unless $\mathrm{Q} = 0$, $\mathrm{C}\ell(\mathrm{V},\mathrm{Q})$ and $\Lambda\mathrm{V}$ are not isomorphic as algebras; instead we will be able to interpret $\mathrm{C}\ell(\mathrm{V},\mathrm{Q})$ as a quantisation of $\Lambda\mathrm{V}$, much in the same way that the universal enveloping algebra $\mathrm{U}\mathfrak{g}$ is a quantisation of the symmetric algebra $\mathrm{Sym}\,\mathfrak{g}$. But before doing that let us consider some low-dimensional examples.

1.3.2 Low-dimensional Clifford algebras

We now specialise to $\mathbb{K} = \mathbb{R}$. In a quadratic real vector space it is always possible to choose a basis (e_i), for $i = 1, \dots, n$ for which the matrix of the bilinear form B has the form

$$[\mathrm{B}_{ij}] = \begin{pmatrix} \mathbf{0}_r & & \\ & +\mathbf{1}_s & \\ & & -\mathbf{1}_t \end{pmatrix} \tag{21}$$

where $n = r + s + t$ and $\mathbf{1}_k$ is the $k \times k$ identity matrix and $\mathbf{0}_k$ is the $k \times k$ zero matrix. Let us specialise to the case $r = 0$, whence B is nondegenerate. Then it defines an inner product of signature (s, t) and we call the corresponding Clifford algebra $\mathrm{C}\ell(s, t)$. We will now look at the first few cases.

The first "trivial" case (which is nondegenerate despite appearances!) is $\mathrm{C}\ell(0,0)$. This is an associative algebra without generators, so it is isomorphic to $\mathbb{R}$, the isomorphism being given by $x\mathbf{1} \longleftrightarrow x$.

$\mathrm{C}\ell(1,0)$ is generated by Γ obeying $\Gamma^2 = -\mathbf{1}$, whence it is isomorphic to $\mathbb{C}$ (as a real associative algebra), with isomorphism $x\mathbf{1} + y\Gamma \longleftrightarrow x + iy$.

$\mathrm{C}\ell(2,0)$ is generated by Γ_1, Γ_2 obeying $\Gamma_1^2 = -\mathbf{1} = \Gamma_2^2$ and $\Gamma_1\Gamma_2 = -\Gamma_2\Gamma_1$. Hence $\mathrm{C}\ell(2,0) \cong \mathbb{H}$, with explicit isomorphism

$$x_0\mathbf{1} + x_1\Gamma_1 + x_2\Gamma_2 + x_3\Gamma_1\Gamma_2 \longleftrightarrow x_0 + x_1 i + x_2 j + x_3 k\,. \tag{22}$$

You might be forgiven for thinking that $\mathrm{C}\ell(3,0)$ is related to the octonions, but only if you immediately discard this after realising that the octonions are not associative. In fact, we will see in the next lecture that $\mathrm{C}\ell(3,0) \cong \mathbb{H} \oplus \mathbb{H}$.

$\mathrm{C}\ell(0,1)$ is generated by Γ with $\Gamma^2 = \mathbf{1}$. We define complementary idempotents $p_\pm = \frac{1}{2}(\mathbf{1} + \Gamma)$, which obey $p_+ + p_- = \mathbf{1}$, $p_+p_- = 0$ and $p_\pm^2 = p_\pm$. This decomposes the Clifford algebra and indeed $\mathrm{C}\ell(0,1) \cong \mathbb{R} \oplus \mathbb{R}$, with explicit isomorphism $xp_+ + yp_- \longleftrightarrow (x, y)$.

$\mathrm{C}\ell(1,1)$ is generated by Γ_1, Γ_2 satisfying $\Gamma_1^2 = -\mathbf{1}$ and $\Gamma_2^2 = \mathbf{1}$ with $\Gamma_1\Gamma_2 = -\Gamma_2\Gamma_1$. The resulting algebra is isomorphic to the algebra of 2×2 real matrices, with the explicit isomorphism being given by

$$x\mathbf{1} + y\Gamma_1 + z\Gamma_2 + w\Gamma_1\Gamma_2 \longleftrightarrow \begin{pmatrix} x + z & y + w \\ -y + w & x - z \end{pmatrix}. \tag{23}$$

Finally, $\mathrm{C}\ell(0,2)$ is generated by Γ_1, Γ_2 satisfying $\Gamma_1^2 = \mathbf{1} = \Gamma_2^2$ with $\Gamma_1\Gamma_2 = -\Gamma_2\Gamma_1$. The resulting algebra is again isomorphic to the algebra of 2×2 real matrices, but with a different isomorphism:

$$x\mathbf{1} + y\Gamma_1 + z\Gamma_2 + w\Gamma_1\Gamma_2 \longleftrightarrow \begin{pmatrix} x + y & z + w \\ z - w & x - y \end{pmatrix}. \tag{24}$$

These results fill in a little corner of the tableau of Clifford algebras $\mathrm{C}\ell(s, t)$:

$\mathbb{R}(2)$			
$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$		
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	

Clifford's purpose in introducing the eponymous algebras in 1878 [Cli78] was the extension of the first row of the above tableau beyond the quaternions. In the next lecture, we will fill in the rest of the tableau!

1.4 The Clifford algebra and the exterior algebra

1.4.1 Filtered and associated graded algebras

Every graded algebra has a canonical filtration, which in the case of TV is given by $\mathrm{F}^p\mathrm{TV} = \bigoplus_{\ell \le p} \mathrm{V}^{\otimes \ell}$, so that $\mathrm{F}^0\mathrm{TV} = \mathbb{K}$, $\mathrm{F}^1\mathrm{TV} = \mathbb{K} \oplus \mathrm{V}$, $\mathrm{F}^2\mathrm{TV} = \mathbb{K} \oplus \mathrm{V} \oplus \mathrm{V}^{\otimes 2}$,... It is convenient to introduce $\mathrm{F}^{-1}\mathrm{TV} = 0$ and in this way arrive at a semi-infinite filtration

$$0 = \mathrm{F}^{-1}\mathrm{TV} \subset \mathrm{F}^0\mathrm{TV} \subset \mathrm{F}^1\mathrm{TV} \subset \mathrm{F}^2\mathrm{TV} \subset \cdots \tag{25}$$

The multiplication respects the filtration in that $\mathrm{F}^p\mathrm{TV} \times \mathrm{F}^q\mathrm{TV} \to \mathrm{F}^{p+q}\mathrm{TV}$, making it into a *filtered algebra*.

Every filtered algebra has an associated graded algebra. For the tensor algebra with the canonical filtration, the *associated graded algebra* $\mathrm{Gr}^\bullet \mathrm{FTV} = \bigoplus_{p\geq 0} \mathrm{Gr}^p \mathrm{FTV}$ is defined by

$$\mathrm{Gr}^p \mathrm{FTV} = \mathrm{F}^p\mathrm{TV}/\mathrm{F}^{p-1}\mathrm{TV}. \tag{26}$$

It follows that $\mathrm{Gr}^\bullet \mathrm{FTV}$ is indeed a graded algebra in that the product defines a bilinear map

$$\mathrm{Gr}^p \mathrm{FTV} \times \mathrm{Gr}^q \mathrm{FTV} \to \mathrm{Gr}^{p+q} \mathrm{FTV}. \tag{27}$$

Of course, in this case $\mathrm{Gr}^p \mathrm{FTV} = \mathrm{V}^{\otimes p}$ and $\mathrm{Gr}^\bullet \mathrm{FTV} \cong \mathrm{T}^\bullet\mathrm{V}$ as graded algebras. This only recapitulates the fact that TV is a graded algebra and FTV is the canonical filtration associated to that grading. In general, filtered algebras need not be graded and hence will not be isomorphic (as algebras) to their associated graded algebra; although they will be isomorphic as vector spaces.

For example, the universal enveloping algebra $\mathrm{U}\mathfrak{g}$ inherits a filtration from the tensor algebra $\mathrm{T}\mathfrak{g}$, whose associated graded algebra is the symmetric algebra $\mathrm{Sym}^\bullet \mathfrak{g}$. Filtered algebras whose associated graded algebras are commutative (or supercommutative) can be interpreted as quantisations of their associated graded algebra, which inherits a Poisson bracket from the (super)commutator in the filtered algebra. This is precisely what happens for the Clifford algebra as we will now see.

1.4.2 The $\mathbb{Z}_2$-grading revisited

The orthogonal group of a quadratic vector space acts on the Clifford algebra via automorphisms. Indeed, if $f : \mathrm{V} \to \mathrm{V}$ is an orthogonal transformation of V, so that $f^*\mathrm{Q} = \mathrm{Q}$, functoriality gives $\mathrm{C}\ell(f) : \mathrm{C}\ell(\mathrm{V},\mathrm{Q}) \to \mathrm{C}\ell(\mathrm{V},\mathrm{Q})$, which is an automorphism. In particular we can consider the simple orthogonal transformation $f(x) = -x$ for all $x \in \mathrm{V}$. Since $f \circ f = 1_\mathrm{V}$, it follows that $\mathrm{C}\ell(f) \circ \mathrm{C}\ell(f) = 1_{\mathrm{C}\ell(\mathrm{V},\mathrm{Q})}$, and thus we can decompose $\mathrm{C}\ell(\mathrm{V},\mathrm{Q}) = \mathrm{C}_0 \oplus \mathrm{C}_1$ into eigenspaces of $\mathrm{C}\ell(f)$:

$$\mathrm{C}_0 = \left\{\alpha \in \mathrm{C}\ell(\mathrm{V},\mathrm{Q}) \middle| \mathrm{C}\ell(f)\alpha = \alpha\right\} \qquad \text{and} \qquad \mathrm{C}_1 = \left\{\alpha \in \mathrm{C}\ell(\mathrm{V},\mathrm{Q}) \middle| \mathrm{C}\ell(f)\alpha = -\alpha\right\}. \tag{28}$$

Since $\mathrm{C}\ell(f)$ is an automorphism, this makes $\mathrm{C}\ell(\mathrm{V},\mathrm{Q})$ into a $\mathbb{Z}_2$-graded algebra, so that under the Clifford algebra multiplication

$$\mathrm{C}_i \times \mathrm{C}_j \to \mathrm{C}_{i+j}, \tag{29}$$

where we add the subscripts modulo 2. The same is true for the tensor algebra TV and we have $\mathrm{TV} = \mathrm{TV}_0 \oplus \mathrm{TV}_1$ where

$$\mathrm{TV}_0 = \bigoplus_{k\geq 0} \mathrm{V}^{\otimes 2k} \qquad \text{and} \qquad \mathrm{TV}_1 = \bigoplus_{k\geq 0} \mathrm{V}^{\otimes (2k+1)}. \tag{30}$$

In this case, the $\mathbb{Z}_2$-grading is the reduction mod 2 of the $\mathbb{Z}$-grading. Since the ideal I_Q is homogeneous, the projection $\mathrm{TV} \to \mathrm{C}\ell(\mathrm{V},\mathrm{Q})$ restricts to projections $\mathrm{TV}_i \to \mathrm{C}_i$ for $i = 0,1$. (Of course, for $i = 1$ this is only a projection of vector spaces, since neither TV_1 nor C_1 are algebras.)

1.4.3 The filtration of the Clifford algebra

The canonical filtration of TV defines a filtration on $\mathrm{C}\ell(\mathrm{V},\mathrm{Q})$ as follows. First of all notice that we can filter TV_0 and TV_1 separately. We let

$$\mathrm{F}^{2k}\mathrm{TV}_0 = \bigoplus_{\ell\leq k} \mathrm{V}^{\otimes 2\ell} \qquad \text{and} \qquad \mathrm{F}^{2k+1}\mathrm{TV}_1 = \bigoplus_{\ell\leq k} \mathrm{V}^{\otimes (2\ell+1)}, \tag{31}$$

so that

$$\begin{aligned} 0 = \mathrm{F}^{-2}\mathrm{TV}_0 \subset \mathrm{F}^0\mathrm{TV}_0 \subset \mathrm{F}^2\mathrm{TV}_0 \subset \cdots \\ 0 = \mathrm{F}^{-1}\mathrm{TV}_1 \subset \mathrm{F}^1\mathrm{TV}_1 \subset \mathrm{F}^3\mathrm{TV}_1 \subset \cdots \end{aligned} \tag{32}$$

are filtrations of TV_0 and TV_1 respectively. We now define $F^{2k}C_0$ to be the image of $F^{2k}TV_0$ under the projection $TV_0 \to C_0$ and similarly $F^{2k+1}C_1$ to be the image of $F^{2k+1}TV_1$ under the projection $TV_1 \to C_1$. It follows that

$$\begin{aligned} 0 = F^{-2}C_0 \subset F^0C_0 \subset F^2C_0 \subset \cdots \\ 0 = F^{-1}C_1 \subset F^1C_1 \subset F^3C_1 \subset \cdots \end{aligned} \tag{33}$$

are filtrations of the Clifford algebra. We will use the shorthand

$$F^pC = \begin{cases} F^pC_0 & \text{if } p \text{ is even, and} \\ F^pC_1 & \text{if } p \text{ is odd.} \end{cases} \tag{34}$$

Since $TV \to C\ell(V,Q)$ is an algebra homomorphism, it follows that Clifford multiplication respects the filtration: $F^pC \times F^qC \to F^{p+q}C$. Notice now that $F^pC/F^{p+2}C \cong \Lambda^pV$, since the corrections involved in replacing, for $x, y \in V$, xy by $-yx$ in the Clifford algebra involves terms of degree 2 less. Of course, if $Q = 0$ then there are no corrections and $C\ell(V,0) \cong \Lambda V$ as graded associative algebras.

Since ΛV is supercommutative, the supercommutator of two elements $\alpha \in F^pC$ and $\beta \in F^qC$ belongs to $F^{p+q-2}C$. If we let $\overline{\alpha} \in \Lambda^pV$ and $\overline{\beta} \in \Lambda^qV$ be such that $\alpha = \overline{\alpha} \mod F^{p-2}C$ and $\beta = \overline{\beta} \mod F^{q-2}C$, then we define a bracket $[-,-] : \Lambda^pV \times \Lambda^qV \to \Lambda^{p+q-2}V$ by

$$[\overline{\alpha}, \overline{\beta}] := \alpha\beta - (-1)^{|\alpha||\beta|}\beta\alpha \mod F^{p+q-4}C. \tag{35}$$

It is an exercise to show that this is a Poisson bracket making ΛV into a Poisson superalgebra. It is in this sense that $C\ell(V,Q)$ is a quantisation of ΛV. We can think of ΛV as the functions on the "phase space" for a finite number of fermionic degrees of freedom and $C\ell(V,Q)$ as the corresponding quantum operator algebra. The Hilbert space of the quantum theory is then an irreducible representation of $C\ell(V,Q)$. We will see later than for V finite-dimensional and Q nondegenerate there are (up to equivalence) either one or two irreducible representations of $C\ell(V,Q)$. For V infinite-dimensional the situation is drastically different. A reasonable account of this can be found in [KS87].

1.4.4 The action of $C\ell(V,Q)$ on ΛV

We can understand the relation between the Clifford and the exterior algebras in a different way which does not involve filtrations. The bilinear form B defines a linear map $\flat : V \to V^*$ where V^* is the dual vector space by $x \mapsto x^\flat$, where $x^\flat(y) = B(x,y)$. If (and only if) B is nondegenerate, is $\flat$ an isomorphism. In that case its inverse is denoted $\sharp : V^* \to V$ and they are referred to together as the *musical isomorphisms* induced from the inner product B. We define a linear map $\phi : V \to \operatorname{End}\Lambda V$ by

$$\phi(x)\alpha = x \wedge \alpha - \imath_{x^\flat}\alpha\,, \tag{36}$$

where $\imath_{x^\flat}$ is the unique odd derivation defined by $\imath_{x^\flat}1 = 0$ and $\imath_{x^\flat}y = B(x,y)$ for $y \in V$. In other words, on a monomial it acts like

$$\imath_{x^\flat}(y_1 \wedge y_2 \wedge \cdots \wedge y_p) = \sum_{i=1}^{p} (-1)^{i-1} B(x,y_i)\, y_1 \wedge \cdots \wedge \widehat{y_i} \wedge \cdots y_p\,, \tag{37}$$

where the hat denotes omission, and we extend linearly to all of ΛV.

Lemma 1.7. *The map $\phi : V \to \operatorname{End}\Lambda V$ defined in (36) is Clifford.*

Proof. For every $x \in V$ and $\alpha \in \Lambda V$, we have

$$\begin{aligned} \phi(x)^2\alpha &= \phi(x)\left(x \wedge \alpha - \imath_{x^\flat}\alpha\right) \\ &= x \wedge (x \wedge \alpha - \imath_{x^\flat}\alpha) - \imath_{x^\flat}(x \wedge \alpha - \imath_{x^\flat}\alpha) \\ &= x \wedge w \wedge \alpha - x \wedge \imath_{x^\flat}\alpha - Q(x)\alpha + x \wedge \imath_{x^\flat} + \imath_{x^\flat}\imath_{x^\flat}\alpha \\ &= -Q(x)\alpha\,, \end{aligned}$$

where we have used that $x \wedge x = 0$, $\imath_{x^\flat}\imath_{x^\flat} = 0$ and that $\imath_{x^\flat}(x \wedge \alpha) = Q(x)\alpha - x \wedge \imath_{x^\flat}\alpha$. ∎

By universality of the Clifford algebra this extends to a unique algebra homomorphism

$$\Phi : C\ell(V,Q) \to \operatorname{End} \Lambda V ,$$

which composing with evaluation at $1 \in \Lambda V$ gives a linear map $\Phi_1 : C\ell(V,Q) \to \Lambda V$. This maps obeys $\Phi_1(\mathbf{1}) = 1$, and if $x \in V$, then $\Phi_1(i(x)) = x$, where $i : V \to C\ell(V,Q)$. Notice that this shows that $\Phi_1 \circ i$ is injective, whence it follows that i is injective without appealing to the construction of $C\ell(V,Q)$ from the tensor algebra. We can similarly calculate

$$\Phi_1(xy) = \Phi(xy)1 = \Phi(x)\Phi(y)1 = \phi(x)\phi(y)1 = \phi(x)y = x \wedge y - B(x,y) \tag{38}$$

and

$$\Phi_1(xyz) = x \wedge y \wedge z - B(x,y)z + B(x,z)y - B(y,z)x , \tag{39}$$

et cetera. It is clear that Φ_1 surjects onto ΛV and counting dimensions we see that it is a vector space isomorphism, with inverse the map $\Lambda V \to C\ell(V,Q)$ defined by the complete skew-symmetrisation:

$$y_1 \wedge \cdots \wedge y_p \mapsto \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} (-1)^\sigma y_{i_{\sigma(1)}} \cdots y_{i_{\sigma(p)}} . \tag{40}$$

This map is an explicit *quantisation* of the exterior algebra.

1.4.5 The Clifford inner product

The exterior algebra ΛV inherits an inner product from V. Explicitly it is defined as follows: if $\Xi := x_1 \wedge \cdots \wedge x_p, \Upsilon := y_1 \wedge \cdots \wedge y_p \in \Lambda^p V$, then

$$\langle \Xi, \Upsilon \rangle = \det B(x_i, y_j) , \tag{41}$$

and we extend it bilinearly to all of $\Lambda^p V$, while declaring $\Lambda^p V$ and $\Lambda^q V$ perpendicular for $p \neq q$. The **Clifford inner product** is the unique inner product on $C\ell(V,Q)$ making the isomorphism $C\ell(V,Q) \to \Lambda V$ into an isometry.

Proposition 1.8. *Let $\alpha, \beta \in C\ell(V,Q)$. Then their Clifford inner product is given in terms of Clifford multiplication by*

$$\langle \alpha, \beta \rangle = \langle \mathbf{1}, \hat\alpha\beta \rangle$$

where $\hat\alpha$ is the image of α under the involutive antiautomorphism induced by multiplication by -1 on V. In other words, if $\alpha = x_1 \cdots x_p$, with $x_i \in V$, then $\hat\alpha = (-x_p)\cdots(-x_1) = (-1)^p x_p \cdots x_1$.

Proof. Let (e_i) be an orthonormal basis for V; that is, $Q(e_i) = \pm 1$ and $B(e_i, e_j) = 0$ for $i \neq j$. If $I = (i_1, \ldots, i_p)$ is an increasing sequence, then let $e_I = e_{i_1} \wedge \cdots \wedge e_{i_p} \in \Lambda^p V$. It is clear that if I and J are distinct increasing sequences, then $\langle e_I, e_J \rangle = 0$, and otherwise

$$\langle e_I, e_I \rangle = Q(e_{i_1}) \cdots Q(e_{i_p}) .$$

On the other hand, the element in $C\ell(V,Q)$ corresponding to $e_I \in \Lambda^p V$ is $e_{i_1} \cdots e_{i_p}$ and

$$\langle e_{i_1} \cdots e_{i_p}, e_{i_1} \cdots e_{i_p} \rangle = \langle \mathbf{1}, (-e_{i_p}) \cdots (-e_{i_1}) e_{i_1} \cdots e_{i_p} \rangle = Q(e_{i_1}) \cdots Q(e_{i_p}) \langle \mathbf{1}, \mathbf{1} \rangle = Q(e_{i_1}) \cdots Q(e_{i_p}) ,$$

where we have used that $-e_i e_i = Q(e_i)$. Finally, if $I \neq J$ are increasing sequences,

$$\langle e_{i_1} \cdots e_{i_p}, e_{j_1} \cdots e_{j_p} \rangle = \langle \mathbf{1}, (-1)^p e_{i_p} \cdots e_{i_1} e_{j_1} \cdots e_{j_p} \rangle = 0 ,$$

since $e_{i_p} \cdots e_{i_1} e_{j_1} \cdots e_{j_p}$ will not be proportional to $\mathbf{1}$. □