Lecture 6: The spin connection

On the tangent bundle of a riemannian manifold (M, g) there is a privileged connection called the Levi-Civita connection. Thinking of the tangent bundle as an associated vector bundle to the bundle $O(M)$ of orthonormal frames, we will see that this connection is induced from a connection on $O(M)$, which restricts to a connection on $SO(M)$ when (M, g) is orientable and lifts to a connection on any spin bundle $Spin(M)$ if (M, g) is spin. That being the case, it defines a connection on the spinor bundles which is usually called the spin connection.

6.1 The Levi-Civita connection

Let (M, g) be a riemannian manifold. We summarise here the basic definitions and results of the riemannian geometry of (M, g).

Theorem 6.1 (The fundamental theorem of riemannian geometry)**.** *There is a unique connection on the tangent bundle* TM *which is*

1. metric-compatible*:*

$$\nabla_X g = 0 \qquad \textit{equivalently} \qquad Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \ ,$$

2. *and* torsion-free*:*

$$\nabla_X Y - \nabla_Y X = [X, Y] \ ,$$

where X, Y, Z *are vector fields on* M *and* $[X, Y]$ *denotes the Lie bracket of vector fields.*

Proof. The proof consists in finding an explicit formula for the connection in terms of the metric. Let $X, Y, Z \in \mathscr{X}(M)$. The metric compatibility condition says that

$$\begin{aligned} Xg(Y,Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ Yg(Z,X) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\ Zg(X,Y) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \ , \end{aligned}$$

whereas the vanishing of the torsion allows to rewrite the middle equation as

$$Yg(Z,X) = g(\nabla_Y Z, X) + g(Z, \nabla_X Y) + g(Z, [X, Y]) \ .$$

We now compute

$$Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) = 2g(\nabla_X Y, Z) + g(Y, \nabla_X Z - \nabla_Z X) + g(\nabla_Y Z - \nabla_Z Y, X) + g(Z, [X, Y])$$

and use the torsionless condition once again to arrive at the **Koszul formula**

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(Y, [X,Z]) - g([Y,Z], X) - g(Z, [X,Y]) \tag{79}$$

which determines $\nabla_X Y$ uniquely. □

The connection so defined is called the **Levi-Civita connection**. Its curvature, defined by

$$R(X,Y)Z = \nabla_{[X,Y]} Z - \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z \ , \tag{80}$$

gives rise to the **Riemann curvature tensor**

$$R(X,Y,Z,W) := g(R(X,Y)Z, W) \ .$$

Proposition 6.2. *The curvature satisfies the following identities*

1. symmetry conditions:

$$R(X,Y)Z = -R(Y,X)Z \qquad \textit{and} \qquad R(X,Y,Z,W) = -R(X,Y,W,Z)\ ,$$

2. algebraic Bianchi identity:

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0\ ,$$

3. differential Bianchi identity:

$$\nabla_X R(Y,Z) + \nabla_Y R(Z,X) + \nabla_Z R(X,Y) = 0\ .$$

A tensor satisfying the symmetry conditions and the algebraic Bianchi identity is called an **algebraic curvature tensor**.

If we fix $X, Y \in \mathscr{X}(M)$, the curvature defines a linear map $Z \mapsto R(X,Z)Y$, whose trace is the **Ricci (curvature) tensor** $r(X,Y)$.

Proposition 6.3. *The Ricci tensor is symmetric:* $r(X,Y) = r(Y,X)$.

The trace (relative to the metric g) of the Ricci tensor is called the **scalar curvature** of (M,g) and denoted s.

Definition 6.4. A riemannian manifold (M,g) is said to be **Einstein** if $r(X,Y) = \lambda g(X,Y)$ for some $\lambda \in \mathbb{R}$. Clearly $\lambda = s/n$ where n is the dimension of M. It is said to be **Ricci-flat** if $r = 0$ and **flat** if $R = 0$.

If $h, k \in C^\infty(M, S^2T^*M)$ are two symmetric tensors, their **Kulkarni–Nomizu product** $h \odot k$ is the algebraic curvature tensor defined by

(81) $$(h \odot k)(X,Y,Z,W) = h(X,Z)k(Y,W) + h(Y,W)k(X,Z) - h(X,W)k(Y,Z) - h(Y,Z)k(X,W)\ ,$$

for all $X, Y, Z, W \in \mathscr{X}(M)$.

Proposition 6.5. *The Riemann curvature tensor can be decomposed as*

$$R = \frac{s}{2n(n-1)} g \odot g + \frac{1}{n-2}\left(r - \frac{s}{n} g\right) \odot g + W$$

where W *is the* **Weyl (curvature) tensor**.

The Weyl tensor is the "traceless" part of the Riemann tensor. It is conformally invariant and if it vanishes, (M,g) is said to be *conformally flat.* If (M,g) is Einstein, then the middle term in R is absent. If only the first term is present then (M,g) is said to have *constant sectional curvature.*

6.2 The connection one-forms on O(M), SO(M) and Spin(M)

The Levi-Civita connection of a riemannian manifold induces a connection one-form ω on the orthonormal frame bundle and, if orientable, also on the oriented orthonormal frame bundle. Indeed, let us assume that M is orientable and let $\mathscr{E} : U \subset M \to SO(M)$ be local orthonormal frame, i.e., a local section of SO(M). Then we may pull ω back to a gauge field $\mathscr{E}^*\omega$ on U with values in $\mathfrak{so}(s,t)$, for (M,g) of signature (s,t). We can describe the gauge field explicitly as follows. Let (e_i) denote the elements in the frame $\mathscr{E}$. Being orthonormal, their inner products are given by $g(e_i, e_j) = \varepsilon_i \delta_{ij}$, where $\varepsilon_i = \pm 1$. Then we have

$$\mathscr{E}^*\omega = \tfrac{1}{2}\sum_{i,j} \omega_{ij} \varepsilon_i \varepsilon_j e_i \curlywedge e_j\ ,$$

where $\omega_{ij} \in \Omega^1(U)$ is defined by

(82) $$\omega_{ij}(X) = g(\nabla_X e_i, e_j)$$

for all $X \in \mathscr{X}(M)$ and $e_i \curlywedge e_j \in \mathfrak{so}(s,t)$ are the skewsymmetric endomorphisms defined by (53). It is convenient in calculations to introduce the **dual frame** $e^i = \varepsilon_i e_i$, where now $g(e_i, e^j) = \delta_{ij}$, and in terms of which

$$\mathscr{E}^*\omega = \tfrac{1}{2}\sum_{i,j} \omega_{ij} e^i \curlywedge e^j .$$

If $\mathscr{E}'$ is another local frame $\mathscr{E}' : U' \to SO(M)$, so that on $U \cap U'$, $\mathscr{E}' = \mathscr{E} h$ for some $h : U \cap U' \to SO(s,t)$, then on $U \cap U'$,

$$\mathscr{E}'^*\omega = h\mathscr{E}^*\omega h^{-1} - dh h^{-1} ,$$

whence it does indeed give rise to a gauge field.

Now let

$$\begin{array}{ccc} \mathrm{Spin}(M) & \xrightarrow{\ \varphi\ } & SO(M) \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

denote a spin bundle. The connection 1-form ω on $SO(M)$ pulls back to a connection 1-form $\varphi^*\omega$ on $\mathrm{Spin}(M)$, called the **spin connection**. Now given a local section $\mathscr{E}$ of $SO(M)$, let $\widetilde{\mathscr{E}}$ denote a local section of $\mathrm{Spin}(M)$ such that $\varphi \circ \widetilde{\mathscr{E}} = \mathscr{E}$. Then the gauge field associated to $\varphi^*\omega$ via $\widetilde{\mathscr{E}}$ coincides with the one associated to ω via $\mathscr{E}$:

(83) $$\widetilde{\mathscr{E}}^*\varphi^*\omega = (\varphi \circ \widetilde{\mathscr{E}})^*\omega = \mathscr{E}^*\omega .$$

If $\varrho : \mathrm{Spin}(s,t) \to GL(F)$ is any representation, then on sections of the associated vector bundle $\mathrm{Spin}(M) \times_{\mathrm{Spin}(s,t)} F$ we have a covariant derivative

(84) $$d^\nabla = d + \tfrac{1}{2}\sum_{i,j} \omega_{ij} \varrho(e^i \curlywedge e^j) ,$$

where we also denote by $\varrho : \mathfrak{so}(s,t) \to \mathfrak{gl}(F)$ the representation of the Lie algebra.

We shall be interested primarily in the spinor representations of $\mathrm{Spin}(s,t)$, which are induced by restriction from pinor representations of $C\ell(s,t)$. This means that the associated bundle $\mathrm{Spin}(M) \times_{\mathrm{Spin}(s,t)} F$ is (perhaps a subbundle of) a bundle $C\ell(TM) \times_{C\ell(s,t)} P$ of Clifford modules. In this case, it is convenient to think of the gauge field as taking values in the Clifford algebra. If we let $\rho : \mathfrak{so}(s,t) \to C\ell(s,t)$ denote the embedding defined in (55), then

(85) $$\rho(\mathscr{E}^*\omega) = \tfrac{1}{4}\sum_{i,j} \omega_{ij} e^i e^j ,$$

where $e^i e^j \in C\ell(s,t)$. The curvature two-form of this connection is given by

(86) $$\rho(\mathscr{E}^*\Omega) = \tfrac{1}{4}\sum_{i,j} \Omega_{ij} e^i e^j ,$$

where $\Omega_{ij}(X,Y) = g(R(X,Y)e_i, e_j)$ for all $X, Y \in \mathscr{X}(M)$, with $R(X,Y)$ defined by (80).

The Clifford algebra-valued covariant derivative is compatible with Clifford action in the following sense. Suppose that $\theta \in C\ell(TM)$ and ψ is a section of a bundle of Clifford modules associated to $C\ell(TM)$. Then for all vector fields $X \in \mathscr{X}(M)$, we have that

(87) $$\nabla_X(\theta \cdot \psi) = \nabla_X\theta \cdot \psi + \theta \cdot \nabla_X \psi ,$$

where $\nabla_X\theta$ agrees with the action of the Levi-Civita connection on θ viewed as a section of ΛTM.

6.3 Parallel spinor fields

We can now define the notion of a parallel spinor field as a (nonzero) section of a spinor bundle which is covariantly constant. On a trivialising neighbourhood U of M, where Spin(M) is trivialised by a local section $\widetilde{\mathscr{E}}$ lifing a local orthonormal frame $\mathscr{E}$, a spinor field is given by a function $\psi : \mathrm{U} \to \mathrm{S}(s,t)$ taking values in the spinor representation, which we think of as the restriction to $\mathrm{Spin}(s,t)$ of an irreducible $\mathrm{C}\ell(s,t)$-module. Depending on (s,t), it may very well be the case that the $\mathrm{S}(s,t)$ so defined is not irreducible, in which case $\mathrm{S}(s,t) = \mathrm{S}(s,t)_+ \oplus \mathrm{S}(s,t)_-$ decomposes into two half-spinor irreducible representations of $\mathrm{Spin}(s,t)$. The covariant derivative of ψ is given by

(88) $$d^\nabla \psi = d\psi + \tfrac{1}{4}\sum_{i,j} \omega_{ij} e^i e^j \psi \,,$$

and we say that ψ is **covariantly constant** (or **parallel**) if $d^\nabla\psi = 0$. The fact (78) that d^∇ is covariant means that this equation is well-defined on global section of the spinor bundle.

Differentiating $d^\nabla\psi$ again we obtain an integrability condition for the existence of parallel spinor fields, namely

(89) $$d^\nabla d^\nabla \psi = \tfrac{1}{4}\sum_{i,j} \Omega_{ij} e^i e^j \psi = 0 \,.$$

This equation is equivalent to

(90) $$\mathrm{R}(\mathrm{X},\mathrm{Y})\psi = 0 \,,$$

where $\mathrm{R}(\mathrm{X},\mathrm{Y}) \in \mathrm{C}\ell(\mathrm{TM})$ acts on ψ via Clifford multiplication. Relative to the local orthonormal frame $\mathscr{E} = (e_i)$, we have

(91) $$\mathrm{R}(e_i,e_j)\cdot\psi = 0 \implies \sum_{k,\ell} \mathrm{R}_{ijk\ell} e^k e^\ell \psi = 0 \,.$$

If we multiply the above equation with e^j and sum over j, we obtain the following:

$$\begin{aligned} 0 &= \sum_{j,k,\ell} \mathrm{R}_{ijk\ell} e^j e^k e^\ell \psi \\ &= \sum_{j,k,\ell} \mathrm{R}_{ijk\ell}\left(e^{jk\ell} - g^{jk} e^\ell + g^{j\ell} e^k\right)\psi \\ &= \sum_{j,k,\ell} \mathrm{R}_{ijk\ell}\left(e^{jk\ell} + 2 g^{j\ell} e^k\right)\psi \,. \end{aligned}$$

The first term vanishes by the algebraic Bianchi identity and the second term yields the Ricci tensor, whence the integrability condition becomes

(92) $$\sum_j \mathrm{R}_{ij} e^j \psi = 0 \,.$$

More invariantly, this says the following. The Ricci tensor defines an endomorphism R of the tangent bundle called the **Ricci operator**, by $g(\mathrm{R}(\mathrm{X}),\mathrm{Y}) = r(\mathrm{X},\mathrm{Y})$. Then the above integrability condition says that $\mathrm{R}(\mathrm{X})\psi = 0$ for all $\mathrm{X} \in \mathscr{X}(\mathrm{M})$. Hitting this equation again with $\mathrm{R}(\mathrm{X})$, we see that $g(\mathrm{R}(\mathrm{X}),\mathrm{R}(\mathrm{X})) = 0$ for all X. If g is positive-definite, then $\mathrm{R}(\mathrm{X}) = 0$ and (M, g) is Ricci-flat. In indefinite signature, the image of the Ricci operator consists of null vectors, whence we could call such manifolds *Ricci-null.*

In the next lecture we will reformulate the question of which spin manifolds admit parallel spinor fields in terms of holonomy.