Spin Geometry

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http://empg.maths.ed.ac.uk/Activities/Spin

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These are the notes accompanying the lectures on **Spin Geometry**, a PG course taught in Edinburgh in the Spring of 2010.

The only requirement is a working familiarity with basic differential geometry and basic representation theory; although scholia on the necessary definitions will be scattered throughout the notes.

Any statement which is not proved to your satisfaction is to be thought of as an exercise, even if not explicitly labelled as such!

These notes are still in a state of flux and I am happy to receive comments and suggestions either by email or in person.

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Lecture 1: Clifford algebras: basic notions

Consider now a system of n units $\iota_1, \iota_2, \ldots, \iota_n$ such that the multiplication of any two of them is polar; that is, $\iota_r \iota_s = -\iota_s \iota_r$.

— William Kingdon Clifford, 1878

In this lecture we define the Clifford algebra of a quadratic vector space and view it from three different points of view: the contemporary categorical formulation, Clifford's original formulation and as a quantisation of the exterior algebra.

1.1 Quadratic vector spaces

Throughout $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let V be a finite-dimensional vector space over $\mathbb{K}$, let $B : V \times V \to \mathbb{K}$ be a (possibly degenerate) symmetric bilinear form and let $Q : V \to \mathbb{K}$ denote the corresponding quadratic form, defined by $Q(x) = B(x,x)$. One can recover B from Q by *polarisation*, namely

$$B(x,y) = \tfrac{1}{2}\big(Q(x+y) - Q(x) - Q(y)\big)\,. \tag{1}$$

The pair (V,Q) is called a **quadratic vector space** (over $\mathbb{K}$). They are the objects of a category **QVec** with morphisms $(V,Q_V) \to (W,Q_W)$ given by linear maps $f : V \to W$ such that $f^*Q_W = Q_V$, or explicitly that $Q_W(f(x)) = Q_V(x)$ for all $x \in V$. The zero vector space with the zero quadratic form is an initial object in **QVec**. The absence of terminal objects and (co)products is due to the fact that projections do not generally preserve norms.

We will see that the Clifford algebra $C\ell(V,Q)$ of a quadratic vector space (V,Q) is an associative, unital $\mathbb{K}$-algebra, with a natural filtration and a $\mathbb{Z}_2$-grading, and moreover that the assignment $(V,Q) \mapsto C\ell(V,Q)$ is functorial.

There are several ways to understand $C\ell(V,Q)$: from the very abstract to the very concrete. The latter is good for computations, whereas the former is good to prove theorems which may free us from computations. Therefore we will look at $C\ell(V,Q)$ in several ways, starting with the categorical definition.

All our associative algebras are unital, unless otherwise stated!

1.2 The Clifford algebra, categorically

Let (V,Q) be a quadratic vector space and let A be an associative $\mathbb{K}$-algebra. We say that a $\mathbb{K}$-linear map $\phi : V \to A$ is **Clifford** if for all $x \in V$,

$$\phi(x)^2 = -Q(x)1_A\,, \tag{2}$$

where 1_A is the unit of A. Clifford maps from a fixed quadratic vector space (V,Q) are the objects of a category **Cliff**(V,Q), where a morphism from $V \to A$ to $V \to A'$ is given by a commuting triangle

$$\begin{tikzcd} & V \arrow[dl] \arrow[dr] & \\ A \arrow[rr, "f"] & & A' \end{tikzcd} \tag{3}$$

with $f : A \to A'$ a homomorphism of associative algebras.

1.2.1 Definition

Definition 1.1. The **Clifford algebra** — if it exists — is an initial object in **Cliff**(V, Q). In other words, it is given by an associative algebra $C\ell(V, Q)$ together with a Clifford map $i : V \to C\ell(V, Q)$ such that for every Clifford map $\phi : V \to A$ there is a *unique* algebra morphism $\Phi : C\ell(V, Q) \to A$ making the following triangle commute:

(4)
$$\begin{tikzcd} & V \arrow[dl, "i"'] \arrow[dr, "\phi"] & \\ C\ell(V,Q) \arrow[rr, "\Phi"] & & A \end{tikzcd}$$

Remark 1.2. There are several paraphrases of the defining property of the Clifford algebra. One can say that every Clifford map factors uniquely via the Clifford algebra, or that the Clifford algebra is *universal* for Clifford maps, or that every Clifford maps extends uniquely to a morphism of associative algebras from the Clifford algebra.

Remark 1.3. The mathematical literature is replete with such universal definitions. For example, if $\mathfrak{g}$ is a Lie algebra and A is an associative algebra (over the same ground field) then one can consider linear maps $\phi : \mathfrak{g} \to A$ such that, for all $X, Y \in \mathfrak{g}$,

(5)
$$\phi(X)\phi(Y) - \phi(Y)\phi(X) = \phi([X, Y])$$

Although it is not standard terminology, let us call such maps **Lie** within the confines of this remark. Then the *universal enveloping algebra* $U\mathfrak{g}$ of $\mathfrak{g}$ is universal for Lie maps; in other words, $U\mathfrak{g}$ is an associative algebra with a Lie map $i : \mathfrak{g} \to U\mathfrak{g}$ extending any Lie map $\phi : \mathfrak{g} \to A$ uniquely; i.e., there is a unique associative algebra morphism $\Phi : U\mathfrak{g} \to A$ such that the following triangle commutes:

(6)
$$\begin{tikzcd} & \mathfrak{g} \arrow[dl, "i"'] \arrow[dr, "\phi"] & \\ U\mathfrak{g} \arrow[rr, "\Phi"] & & A \end{tikzcd}$$

In other words, $U\mathfrak{g}$ is what allows us to "multiply" elements of $\mathfrak{g}$ as if they were matrices. One constructs the universal enveloping algebra as a quotient of the tensor algebra $T\mathfrak{g}$ of $\mathfrak{g}$ by the 2-sided ideal generated by $X \otimes Y - Y \otimes X - [X, Y]$ for all $X, Y \in \mathfrak{g}$. The construction of the Clifford algebra will proceed along similar lines.

Initial objects in a category are unique up to unique isomorphism, hence the following should not be too surprising.

Proposition 1.4. *The Clifford algebra* $C\ell(V, Q)$*, if it exists, is unique up to a unique isomorphism.*

Proof. Let $i : V \to C$ and $i' : V \to C'$ be two Clifford algebras. Then since C is a Clifford algebra, there is a unique morphism $\Phi : C \to C'$ making the following triangle commute

(7)
$$\begin{tikzcd} & V \arrow[dl, "i"'] \arrow[dr, "i'"] & \\ C \arrow[rr, "\Phi"] & & C' \end{tikzcd}$$

whereas since C' is a Clifford algebra, there is a unique morphism $\Phi' : C' \to C$ making the following triangle commute

(8)
$$\begin{tikzcd} & V \arrow[dl, "i'"'] \arrow[dr, "i"] & \\ C' \arrow[rr, "\Phi'"] & & C \end{tikzcd}$$

Now the composition $\Phi' \circ \Phi : C \to C$ makes the following triangle commute

(9)
$$\begin{tikzcd} & V \arrow[dl, "i"'] \arrow[dr, "i"] & \\ C \arrow[rr, "\Phi' \circ \Phi"] & & C \end{tikzcd}$$

and so does the identity $1_C : C \to C$, whence $\Phi' \circ \Phi = 1_C$. A similar argument shows that $\Phi \circ \Phi' = 1_{C'}$, whence $\Phi : C \to C'$ is an isomorphism. ☐

Assuming for a moment that Clifford algebras exist, we have the following

Proposition 1.5. *The Clifford algebra defines a functor* $C\ell$ *from* ***QVec*** *to the category of associative algebras.*

Proof. Indeed, let (V, Q_V) and (W, Q_W) be quadratic vector spaces and $i_V : V \to C\ell(V, Q_V)$ and $i_W : W \to C\ell(W, Q_W)$ the corresponding Clifford algebras. Now let $f : V \to W$ with $f^* Q_W = Q_V$ be a morphism in **QVec** and consider $i_W \circ f : V \to C\ell(W, Q_W)$. We observe that it is a Clifford map:

(10)
$$(i_W \circ f)(x)^2 = -Q_W(f(x))\mathbf{1}_W = -Q_V(x)\mathbf{1}_W \ ,$$

where $\mathbf{1}_W$ is the identity in $C\ell(W, Q_W)$. Therefore by universality, there is a unique morphism $C\ell(f) : C\ell(V, Q_V) \to C\ell(W, Q_W)$. It is clear that if $1_V : V \to V$ is the identity transformation, then uniqueness forces $C\ell(1_V) = 1_{C\ell(V,Q)} : C\ell(V, Q) \to C\ell(V, Q)$ to be the identity morphism (not to be confused with the unit $\mathbf{1}$ in the Clifford algebra). Similarly, if (X, Q_X) is a third quadratic vector space and $g : W \to X$ with $g^* Q_X = Q_W$, then universality gives a morphism $C\ell(g) : C\ell(W, Q_W) \to C\ell(X, Q_X)$ and the composition $C\ell(g) \circ C\ell(f)$ has to agree (again by uniqueness) with $C\ell(g \circ f)$ where $g \circ f : V \to X$ is the composition Clifford map. ☐

Remark 1.6. The universal enveloping algebra also defines a functor from the category of Lie algebras to the category of associative algebras which is left adjoint to the functor which sends an associative algebra to the Lie algebra it becomes under the commutator. The functor defined by the Clifford algebra does not seem to be an adjoint functor in any interesting way.

1.2.2 Construction

Let $T^\bullet V = \bigoplus_{p \geq 0} V^{\otimes p}$ denote the tensor algebra of V, where $V^{\otimes 0} = \mathbb{K}$, $V^{\otimes 1} = V$ and $V^{\otimes p}$ is spanned by monomials $x_1 \otimes x_2 \otimes \cdots \otimes x_p$ with $x_i \in V$. The multiplication $V^{\otimes p} \times V^{\otimes q} \to V^{\otimes(p+q)}$, given by extending bilinearly the concatenation of monomials

(11)
$$(x_1 \otimes \cdots \otimes x_p)(y_1 \otimes \cdots \otimes y_q) = x_1 \otimes \cdots \otimes x_p \otimes y_1 \otimes \cdots \otimes y_q,$$

makes $T^\bullet V$ a graded algebra. The identity is given by $1 \in V^{\otimes 0}$. The tensor algebra is universal for linear maps $\phi : V \to A$, where A is an associative algebra. Indeed, any such map extends uniquely to an algebra morphism $\Phi : TV \to A$ defined by $\Phi(\lambda) = \lambda 1_A$ for $\lambda \in \mathbb{K}$, $\Phi(x) = \phi(x)$ for $x \in V$, and more generally

(12)
$$\Phi(x_1 \otimes \cdots \otimes x_p) = \phi(x_1) \cdots \phi(x_p).$$

In fact, the tensor algebra is the free associative algebra generated by V. The tensor algebra defines a functor T from the category of vector spaces to the category of associative algebras, which is left adjoint to the forgetful functor going in the opposite direction.

By definition, the Clifford algebra $C\ell(V, Q)$ is universal for Clifford maps to associative algebras. Since the tensor algebra is universal for linear maps to associative algebras, we expect $C\ell(V, Q)$ to be a quotient of TV by an ideal which imposes the condition that a linear map is Clifford. To this end, let us consider the 2-sided ideal I_Q of TV generated by elements of the form $x \otimes x + Q(x) \in V^{\otimes 2} \oplus V^{\otimes 0}$. Explicitly, I_Q is spanned (over $\mathbb{K}$) by elements of the form

(13)
$$x_1 \otimes \cdots \otimes x_p \otimes (z \otimes z + Q(z)) \otimes y_1 \otimes \cdots \otimes y_q$$

for some p, q and $x_i, y_j, z \in V$.

If $\phi : V \to A$ is a Clifford map and $\widetilde{\Phi} : TV \to A$ the unique extension of ϕ to the tensor algebra, then it is easy to see that $\widetilde{\Phi}$ annihilates I_Q precisely because ϕ is Clifford:

$$\widetilde{\Phi}(\Theta \otimes (z \otimes z + Q(z)) \otimes \Xi) = \widetilde{\Phi}(\Theta)\left(\phi(z)^2 + Q(z)1_A\right)\widetilde{\Phi}(\Xi) = 0, \tag{14}$$

for any $\Theta, \Xi \in TV$. Hence $\widetilde{\Phi}$ factors through a unique map $\Phi : TV/I_Q \to A$ from the quotient. We define $C\ell(V,Q) = TV/I_Q$ to be the quotient algebra, and the map $i : V \to C\ell(V,Q)$ is defined by the commutativity of the triangle

$$\begin{tikzcd} V \arrow[r] \arrow[dr, "i"'] & TV \arrow[d] \\ & C\ell(V,Q) \end{tikzcd} \tag{15}$$

We remark that i is injective because the ideal only "kicks in" at $V^{\otimes \geq 2}$, whence in many cases we will not write i explicitly and think of V as sitting inside $C\ell(V,Q)$.

Since the ideal I_Q is not homogeneous, $C\ell(V,Q)$ does not inherit a grading from TV, but since the ideal has even parity, $C\ell(V,Q)$ does inherit a $\mathbb{Z}_2$-grading. We will see this later from a different point of view, where we also show that it inherits a filtration from the canonical filtration of TV.

1.3 The Clifford algebra as Clifford would have written it

We now discuss $C\ell(V,Q)$ in a way more suitable to computations. This is the way that Clifford introduced the algebras and the way they are still taught in Physics courses, following Dirac.

1.3.1 Clifford algebra in terms of generators and relations

We start by choosing a $\mathbb{K}$-basis (e_i) for V, where $i = 1, \ldots, n = \dim V$, relative to which $B(e_i, e_j) = B_{ij} = B_{ji}$. Let Γ_i denote the image of e_i under $i : V \to C\ell(V,Q)$. Then the Γ_i satisfy the relation

$$\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = -2B_{ij}\mathbf{1}, \tag{16}$$

where $\mathbf{1}$ is the unit in the Clifford algebra. The Clifford algebra is thus the associative algebra generated by the Γ_i subject to the above relation. This is enough to write down the product of any two generators:

$$\Gamma_i \Gamma_j = \tfrac{1}{2}(\Gamma_i \Gamma_j - \Gamma_j \Gamma_i) + \tfrac{1}{2}(\Gamma_i \Gamma_j + \Gamma_j \Gamma_i) = \Gamma_{ij} - B_{ij}\mathbf{1}\,, \tag{17}$$

where we have introduced the notation $\Gamma_{ij} = \frac{1}{2}(\Gamma_i \Gamma_j - \Gamma_j \Gamma_i)$. It seems to be a new object, since it cannot be reduced further using the relations. With a little bit more energy, one can compute the product

$$\Gamma_i \Gamma_{jk} = \Gamma_{ijk} - B_{ij}\Gamma_k + B_{ik}\Gamma_j\,, \tag{18}$$

where we have defined the alternating product of three generators

$$\Gamma_{ijk} = \tfrac{1}{6}\left(\Gamma_i\Gamma_j\Gamma_k - \Gamma_i\Gamma_k\Gamma_j + \Gamma_j\Gamma_k\Gamma_i - \Gamma_j\Gamma_i\Gamma_k + \Gamma_k\Gamma_i\Gamma_j - \Gamma_k\Gamma_j\Gamma_i\right). \tag{19}$$

More generally define

$$\Gamma_{i_1 \cdots i_p} = \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} (-1)^\sigma \Gamma_{i_{\sigma(1)}} \cdots \Gamma_{i_{\sigma(p)}}\,, \tag{20}$$

where $(-1)^\sigma$ is the sign of the permutation σ of $\{1, 2, \ldots, p\}$. Continuing in this way, and since $C\ell(V,Q)$ is generated by V and the identity, we see that $C\ell(V,Q)$ is the linear span of $\mathbf{1}$, Γ_i, Γ_{ij},... In total there are $1 + n + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$ such monomials, whence $\dim C\ell(V,Q) = 2^{\dim V}$. This is the same dimension of the exterior algebra ΛV and in fact we can establish a vector space isomorphism $\Lambda V \cong C\ell(V,Q)$ by sending $1 \mapsto \mathbf{1}$, $e_i \mapsto \Gamma_i$ and $e_{i_1} \wedge \cdots \wedge e_{i_p} \mapsto \Gamma_{i_1 \cdots i_p}$.

In the next section we will see this isomorphism from a different perspective. Namely we will show that $C\ell(V,Q)$ is a filtered algebra whose associated graded algebra is the exterior algebra. Of course, unless $Q = 0$, $C\ell(V,Q)$ and ΛV are not isomorphic as algebras; instead we will be able to interpret $C\ell(V,Q)$ as a quantisation of ΛV, much in the same way that the universal enveloping algebra $U\mathfrak{g}$ is a quantisation of the symmetric algebra $\operatorname{Sym}\mathfrak{g}$. But before doing that let us consider some low-dimensional examples.

1.3.2 Low-dimensional Clifford algebras

We now specialise to $\mathbb{K} = \mathbb{R}$. In a quadratic real vector space it is always possible to choose a basis (e_i), for $i = 1, \ldots, n$ for which the matrix of the bilinear form B has the form

$$[\mathrm{B}_{ij}] = \begin{pmatrix} \mathbf{0}_r & & \\ & +\mathbf{1}_s & \\ & & -\mathbf{1}_t \end{pmatrix} \tag{21}$$

where $n = r + s + t$ and $\mathbf{1}_k$ is the $k \times k$ identity matrix and $\mathbf{0}_k$ is the $k \times k$ zero matrix. Let us specialise to the case $r = 0$, whence B is nondegenerate. Then it defines an inner product of signature (s, t) and we call the corresponding Clifford algebra $\mathrm{C}\ell(s, t)$. We will now look at the first few cases.

The first "trivial" case (which is nondegenerate despite appearances!) is $\mathrm{C}\ell(0,0)$. This is an associative algebra without generators, so it is isomorphic to $\mathbb{R}$, the isomorphism being given by $x\mathbf{1} \longleftrightarrow x$.

$\mathrm{C}\ell(1,0)$ is generated by Γ obeying $\Gamma^2 = -\mathbf{1}$, whence it is isomorphic to $\mathbb{C}$ (as a real associative algebra), with isomorphism $x\mathbf{1} + y\Gamma \longleftrightarrow x + iy$.

$\mathrm{C}\ell(2,0)$ is generated by Γ_1, Γ_2 obeying $\Gamma_1^2 = -\mathbf{1} = \Gamma_2^2$ and $\Gamma_1\Gamma_2 = -\Gamma_2\Gamma_1$. Hence $\mathrm{C}\ell(2,0) \cong \mathbb{H}$, with explicit isomorphism

$$x_0\mathbf{1} + x_1\Gamma_1 + x_2\Gamma_2 + x_3\Gamma_1\Gamma_2 \longleftrightarrow x_0 + x_1 i + x_2 j + x_3 k\,. \tag{22}$$

You might be forgiven for thinking that $\mathrm{C}\ell(3,0)$ is related to the octonions, but only if you immediately discard this after realising that the octonions are not associative. In fact, we will see in the next lecture that $\mathrm{C}\ell(3,0) \cong \mathbb{H} \oplus \mathbb{H}$.

$\mathrm{C}\ell(0,1)$ is generated by Γ with $\Gamma^2 = \mathbf{1}$. We define complementary idempotents $p_\pm = \frac{1}{2}(\mathbf{1} \pm \Gamma)$, which obey $p_+ + p_- = \mathbf{1}$, $p_+p_- = 0$ and $p_\pm^2 = p_\pm$. This decomposes the Clifford algebra and indeed $\mathrm{C}\ell(0,1) \cong \mathbb{R} \oplus \mathbb{R}$, with explicit isomorphism $xp_+ + yp_- \longleftrightarrow (x, y)$.

$\mathrm{C}\ell(1,1)$ is generated by Γ_1, Γ_2 satisfying $\Gamma_1^2 = -\mathbf{1}$ and $\Gamma_2^2 = \mathbf{1}$ with $\Gamma_1\Gamma_2 = -\Gamma_2\Gamma_1$. The resulting algebra is isomorphic to the algebra of 2×2 real matrices, with the explicit isomorphism being given by

$$x\mathbf{1} + y\Gamma_1 + z\Gamma_2 + w\Gamma_1\Gamma_2 \longleftrightarrow \begin{pmatrix} x + z & y + w \\ -y + w & x - z \end{pmatrix}. \tag{23}$$

Finally, $\mathrm{C}\ell(0,2)$ is generated by Γ_1, Γ_2 satisfying $\Gamma_1^2 = \mathbf{1} = \Gamma_2^2$ with $\Gamma_1\Gamma_2 = -\Gamma_2\Gamma_1$. The resulting algebra is again isomorphic to the algebra of 2×2 real matrices, but with a different isomorphism:

$$x\mathbf{1} + y\Gamma_1 + z\Gamma_2 + w\Gamma_1\Gamma_2 \longleftrightarrow \begin{pmatrix} x + y & z + w \\ z - w & x - y \end{pmatrix}. \tag{24}$$

These results fill in a little corner of the tableau of Clifford algebras $\mathrm{C}\ell(s, t)$:

$\mathbb{R}(2)$			
$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$		
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	

Clifford's purpose in introducing the eponymous algebras in 1878 [Cli78] was the extension of the first row of the above tableau beyond the quaternions. In the next lecture, we will fill in the rest of the tableau!

1.4 The Clifford algebra and the exterior algebra

1.4.1 Filtered and associated graded algebras

Every graded algebra has a canonical filtration, which in the case of TV is given by $\mathrm{F}^p\mathrm{TV} = \bigoplus_{\ell \le p} \mathrm{V}^{\otimes \ell}$, so that $\mathrm{F}^0\mathrm{TV} = \mathbb{K}$, $\mathrm{F}^1\mathrm{TV} = \mathbb{K} \oplus \mathrm{V}$, $\mathrm{F}^2\mathrm{TV} = \mathbb{K} \oplus \mathrm{V} \oplus \mathrm{V}^{\otimes 2}$,... It is convenient to introduce $\mathrm{F}^{-1}\mathrm{TV} = 0$ and in this way arrive at a semi-infinite filtration

$$0 = \mathrm{F}^{-1}\mathrm{TV} \subset \mathrm{F}^0\mathrm{TV} \subset \mathrm{F}^1\mathrm{TV} \subset \mathrm{F}^2\mathrm{TV} \subset \cdots \tag{25}$$

The multiplication respects the filtration in that $F^p TV \times F^q TV \to F^{p+q} TV$, making it into a *filtered algebra*.

Every filtered algebra has an associated graded algebra. For the tensor algebra with the canonical filtration, the *associated graded algebra* $\operatorname{Gr}^\bullet FTV = \bigoplus_{p\geq 0} \operatorname{Gr}^p FTV$ is defined by

$$\operatorname{Gr}^p FTV = F^p TV / F^{p-1} TV. \tag{26}$$

It follows that $\operatorname{Gr}^\bullet FTV$ is indeed a graded algebra in that the product defines a bilinear map

$$\operatorname{Gr}^p FTV \times \operatorname{Gr}^q FTV \to \operatorname{Gr}^{p+q} FTV. \tag{27}$$

Of course, in this case $\operatorname{Gr}^p FTV = V^{\otimes p}$ and $\operatorname{Gr}^\bullet FTV \cong T^\bullet V$ as graded algebras. This only recapitulates the fact that TV is a graded algebra and FTV is the canonical filtration associated to that grading. In general, filtered algebras need not be graded and hence will not be isomorphic (as algebras) to their associated graded algebra; although they will be isomorphic as vector spaces.

For example, the universal enveloping algebra $U\mathfrak{g}$ inherits a filtration from the tensor algebra $T\mathfrak{g}$, whose associated graded algebra is the symmetric algebra $\operatorname{Sym}^\bullet \mathfrak{g}$. Filtered algebras whose associated graded algebras are commutative (or supercommutative) can be interpreted as quantisations of their associated graded algebra, which inherits a Poisson bracket from the (super)commutator in the filtered algebra. This is precisely what happens for the Clifford algebra as we will now see.

1.4.2 The $\mathbb{Z}_2$-grading revisited

The orthogonal group of a quadratic vector space acts on the Clifford algebra via automorphisms. Indeed, if $f : V \to V$ is an orthogonal transformation of V, so that $f^*Q = Q$, functoriality gives $C\ell(f) : C\ell(V,Q) \to C\ell(V,Q)$, which is an automorphism. In particular we can consider the simple orthogonal transformation $f(x) = -x$ for all $x \in V$. Since $f \circ f = 1_V$, it follows that $C\ell(f) \circ C\ell(f) = 1_{C\ell(V,Q)}$, and thus we can decompose $C\ell(V,Q) = C_0 \oplus C_1$ into eigenspaces of $C\ell(f)$:

$$C_0 = \left\{\alpha \in C\ell(V,Q) \middle| C\ell(f)\alpha = \alpha\right\} \qquad \text{and} \qquad C_1 = \left\{\alpha \in C\ell(V,Q) \middle| C\ell(f)\alpha = -\alpha\right\}. \tag{28}$$

Since $C\ell(f)$ is an automorphism, this makes $C\ell(V,Q)$ into a $\mathbb{Z}_2$-graded algebra, so that under the Clifford algebra multiplication

$$C_i \times C_j \to C_{i+j}, \tag{29}$$

where we add the subscripts modulo 2. The same is true for the tensor algebra TV and we have $TV = TV_0 \oplus TV_1$ where

$$TV_0 = \bigoplus_{k\geq 0} V^{\otimes 2k} \qquad \text{and} \qquad TV_1 = \bigoplus_{k\geq 0} V^{\otimes(2k+1)}. \tag{30}$$

In this case, the $\mathbb{Z}_2$-grading is the reduction mod 2 of the $\mathbb{Z}$-grading. Since the ideal I_Q is homogeneous, the projection $TV \to C\ell(V,Q)$ restricts to projections $TV_i \to C_i$ for $i = 0,1$. (Of course, for $i = 1$ this is only a projection of vector spaces, since neither TV_1 nor C_1 are algebras.)

1.4.3 The filtration of the Clifford algebra

The canonical filtration of TV defines a filtration on $C\ell(V,Q)$ as follows. First of all notice that we can filter TV_0 and TV_1 separately. We let

$$F^{2k} TV_0 = \bigoplus_{\ell\leq k} V^{\otimes 2\ell} \qquad \text{and} \qquad F^{2k+1} TV_1 = \bigoplus_{\ell\leq k} V^{\otimes(2\ell+1)}, \tag{31}$$

so that

$$\begin{gathered} 0 = F^{-2} TV_0 \subset F^0 TV_0 \subset F^2 TV_0 \subset \cdots \\ 0 = F^{-1} TV_1 \subset F^1 TV_1 \subset F^3 TV_1 \subset \cdots \end{gathered} \tag{32}$$

are filtrations of TV_0 and TV_1 respectively. We now define $F^{2k}C_0$ to be the image of $F^{2k}TV_0$ under the projection $TV_0 \to C_0$ and similarly $F^{2k+1}C_1$ to be the image of $F^{2k+1}TV_1$ under the projection $TV_1 \to C_1$. It follows that

$$\begin{aligned} 0 = F^{-2}C_0 \subset F^0C_0 \subset F^2C_0 \subset \cdots \\ 0 = F^{-1}C_1 \subset F^1C_1 \subset F^3C_1 \subset \cdots \end{aligned} \tag{33}$$

are filtrations of the Clifford algebra. We will use the shorthand

$$F^pC = \begin{cases} F^pC_0 & \text{if } p \text{ is even, and} \\ F^pC_1 & \text{if } p \text{ is odd.} \end{cases} \tag{34}$$

Since $TV \to C\ell(V,Q)$ is an algebra homomorphism, it follows that Clifford multiplication respects the filtration: $F^pC \times F^qC \to F^{p+q}C$. Notice now that $F^pC/F^{p-2}C \cong \Lambda^pV$, since the corrections involved in replacing, for $x, y \in V$, xy by $-yx$ in the Clifford algebra involves terms of degree 2 less. Of course, if $Q = 0$ then there are no corrections and $C\ell(V,0) \cong \Lambda V$ as graded associative algebras.

Since ΛV is supercommutative, the supercommutator of two elements $\alpha \in F^pC$ and $\beta \in F^qC$ belongs to $F^{p+q-2}C$. If we let $\overline{\alpha} \in \Lambda^pV$ and $\overline{\beta} \in \Lambda^qV$ be such that $\alpha = \overline{\alpha} \mod F^{p-2}C$ and $\beta = \overline{\beta} \mod F^{q-2}C$, then we define a bracket $[-,-] : \Lambda^pV \times \Lambda^qV \to \Lambda^{p+q-2}V$ by

$$[\overline{\alpha}, \overline{\beta}] := \alpha\beta - (-1)^{|\alpha||\beta|}\beta\alpha \mod F^{p+q-4}C. \tag{35}$$

It is an exercise to show that this is a Poisson bracket making ΛV into a Poisson superalgebra. It is in this sense that $C\ell(V,Q)$ is a quantisation of ΛV. We can think of ΛV as the functions on the "phase space" for a finite number of fermionic degrees of freedom and $C\ell(V,Q)$ as the corresponding quantum operator algebra. The Hilbert space of the quantum theory is then an irreducible representation of $C\ell(V,Q)$. We will see later than for V finite-dimensional and Q nondegenerate there are (up to equivalence) either one or two irreducible representations of $C\ell(V,Q)$. For V infinite-dimensional the situation is drastically different. A reasonable account of this can be found in [KS87].

1.4.4 The action of $C\ell(V,Q)$ on ΛV

We can understand the relation between the Clifford and the exterior algebras in a different way which does not involve filtrations. The bilinear form B defines a linear map $\flat : V \to V^*$ where V^* is the dual vector space by $x \mapsto x^\flat$, where $x^\flat(y) = B(x,y)$. If (and only if) B is nondegenerate, is $\flat$ an isomorphism. In that case its inverse is denoted $\sharp : V^* \to V$ and they are referred to together as the *musical isomorphisms* induced from the inner product B. We define a linear map $\phi : V \to \operatorname{End}\Lambda V$ by

$$\phi(x)\alpha = x \wedge \alpha - \imath_{x^\flat}\alpha\,, \tag{36}$$

where $\imath_{x^\flat}$ is the unique odd derivation defined by $\imath_{x^\flat}1 = 0$ and $\imath_{x^\flat}y = B(x,y)$ for $y \in V$. In other words, on a monomial it acts like

$$\imath_{x^\flat}(y_1 \wedge y_2 \wedge \cdots \wedge y_p) = \sum_{i=1}^{p} (-1)^{i-1} B(x,y_i)\, y_1 \wedge \cdots \wedge \widehat{y_i} \wedge \cdots y_p\,, \tag{37}$$

where the hat denotes omission, and we extend linearly to all of ΛV.

Lemma 1.7. *The map $\phi : V \to \operatorname{End}\Lambda V$ defined in (36) is Clifford.*

Proof. For every $x \in V$ and $\alpha \in \Lambda V$, we have

$$\begin{aligned} \phi(x)^2\alpha &= \phi(x)\left(x \wedge \alpha - \imath_{x^\flat}\alpha\right) \\ &= x \wedge (x \wedge \alpha - \imath_{x^\flat}\alpha) - \imath_{x^\flat}(x \wedge \alpha - \imath_{x^\flat}\alpha) \\ &= x \wedge x \wedge \alpha - x \wedge \imath_{x^\flat}\alpha - Q(x)\alpha + x \wedge \imath_{x^\flat} + \imath_{x^\flat}\imath_{x^\flat}\alpha \\ &= -Q(x)\alpha\,, \end{aligned}$$

where we have used that $x \wedge x = 0$, $\imath_{x^\flat}\imath_{x^\flat} = 0$ and that $\imath_{x^\flat}(x \wedge \alpha) = Q(x)\alpha - x \wedge \imath_{x^\flat}\alpha$. ☐

By universality of the Clifford algebra this extends to a unique algebra homomorphism

$$\Phi : C\ell(V,Q) \to \operatorname{End} \Lambda V ,$$

which composing with evaluation at $1 \in \Lambda V$ gives a linear map $\Phi_1 : C\ell(V,Q) \to \Lambda V$. This maps obeys $\Phi_1(\mathbf{1}) = 1$, and if $x \in V$, then $\Phi_1(i(x)) = x$, where $i : V \to C\ell(V,Q)$. Notice that this shows that $\Phi_1 \circ i$ is injective, whence it follows that i is injective without appealing to the construction of $C\ell(V,Q)$ from the tensor algebra. We can similarly calculate

$$\Phi_1(i(x)i(y)) = \Phi(i(x)i(y))1 = \Phi(i(x))\Phi(i(y))1 = \phi(x)\phi(y)1 = \phi(x)y = x \wedge y - B(x,y) \tag{38}$$

and

$$\Phi_1(xyz) = x \wedge y \wedge z - B(x,y)z + B(x,z)y - B(y,z)x , \tag{39}$$

et cetera. It is clear that Φ_1 surjects onto ΛV and counting dimensions we see that it is a vector space isomorphism, with inverse the map $\Lambda V \to C\ell(V,Q)$ defined by the complete skew-symmetrisation:

$$y_1 \wedge \cdots \wedge y_p \mapsto \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} (-1)^\sigma y_{i_{\sigma(1)}} \cdots y_{i_{\sigma(p)}} . \tag{40}$$

This map is an explicit *quantisation* of the exterior algebra.

1.4.5 The Clifford inner product

The exterior algebra ΛV inherits an inner product from V. Explicitly it is defined as follows: if $\Xi := x_1 \wedge \cdots \wedge x_p, \Upsilon := y_1 \wedge \cdots \wedge y_p \in \Lambda^p V$, then

$$\langle \Xi, \Upsilon \rangle = \det B(x_i, y_j) , \tag{41}$$

and we extend it bilinearly to all of $\Lambda^p V$, while declaring $\Lambda^p V$ and $\Lambda^q V$ perpendicular for $p \neq q$. The **Clifford inner product** is the unique inner product on $C\ell(V,Q)$ making the isomorphism $C\ell(V,Q) \to \Lambda V$ into an isometry.

Proposition 1.8. *Let $\alpha, \beta \in C\ell(V,Q)$. Then their Clifford inner product is given in terms of Clifford multiplication by*

$$\langle \alpha, \beta \rangle = \langle \mathbf{1}, \hat{\alpha}\beta \rangle$$

where $\hat{\alpha}$ is the image of α under the involutive antiautomorphism induced by multiplication by -1 on V. In other words, if $\alpha = x_1 \cdots x_p$, with $x_i \in V$, then $\hat{\alpha} = (-x_p)\cdots(-x_1) = (-1)^p x_p \cdots x_1$.

Proof. Let (e_i) be an orthonormal basis for V; that is, $Q(e_i) = \pm 1$ and $B(e_i, e_j) = 0$ for $i \neq j$. If $I = (i_1, \ldots, i_p)$ is an increasing sequence, then let $e_I = e_{i_1} \wedge \cdots \wedge e_{i_p} \in \Lambda^p V$. It is clear that if I and J are distinct increasing sequences, then $\langle e_I, e_J \rangle = 0$, and otherwise

$$\langle e_I, e_I \rangle = Q(e_{i_1}) \cdots Q(e_{i_p}) .$$

On the other hand, the element in $C\ell(V,Q)$ corresponding to $e_I \in \Lambda^p V$ is $e_{i_1} \cdots e_{i_p}$ and

$$\langle e_{i_1} \cdots e_{i_p}, e_{i_1} \cdots e_{i_p} \rangle = \langle \mathbf{1}, (-e_{i_p}) \cdots (-e_{i_1}) e_{i_1} \cdots e_{i_p} \rangle = Q(e_{i_1}) \cdots Q(e_{i_p}) \langle \mathbf{1}, \mathbf{1} \rangle = Q(e_{i_1}) \cdots Q(e_{i_p}) ,$$

where we have used that $-e_i e_i = Q(e_i)$. Finally, if $I \neq J$ are increasing sequences,

$$\langle e_{i_1} \cdots e_{i_p}, e_{j_1} \cdots e_{j_p} \rangle = \langle \mathbf{1}, (-1)^p e_{i_p} \cdots e_{i_1} e_{j_1} \cdots e_{j_p} \rangle = 0 ,$$

since $e_{i_p} \cdots e_{i_1} e_{j_1} \cdots e_{j_p}$ will not be proportional to $\mathbf{1}$. $\square$

Lecture 2: Clifford algebras: the classification

The small part of ignorance that we arrange and classify we give the name of knowledge.

—Ambrose Bierce

In this section we will classify finite-dimensional real and complex Clifford algebras. Useful references are [ABS64] (which treats only the positive- and negative-definite cases, albeit lucidly) and [LM89, Har90].

2.1 A less-than-useful classification

We start with a result which is interesting in its own right, but perhaps not as useful as it may appear at first.

Given two quadratic vector spaces (V, Q_V) and (W, Q_W), we can form their orthogonal direct sum $(V \oplus W, Q_V \oplus Q_W)$. (Notice that although the direct sum is the coproduct in the category of vector spaces, it is not the coproduct in **QVec**.) One natural question is whether the Clifford algebra $C\ell(V \oplus W, Q_V \oplus Q_W)$ of the orthogonal direct sum is related to the Clifford algebras $C\ell(V, Q_V)$ and $C\ell(W, Q_W)$ of its summands. The answer is very simple, which shows why one should take the $\mathbb{Z}_2$-grading of the Clifford algebra very seriously!

Proposition 2.1. *Let (V, Q_V) and (W, Q_W) be quadratic vector spaces and let $(V \oplus W, Q_V \oplus Q_W)$ be their orthogonal direct sum. Then there is an isomorphism of $\mathbb{Z}_2$-graded associative algebras:*

$$C\ell(V \oplus W, Q_V \oplus Q_W) \cong C\ell(V, Q_V) \hat{\otimes} C\ell(W, Q_W) ,$$

where $\hat{\otimes}$ denotes the $\mathbb{Z}_2$-graded tensor product.

Tensor product of algebras

If A and B are associative algebras over the same ground field, their vector space tensor product $A \otimes B$ becomes an algebra by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$$

and extending bilinearly. However when $A = A_0 \oplus A_1$ and $B = B_0 \oplus B_1$ are themselves $\mathbb{Z}_2$-graded, we can define on the $\mathbb{Z}_2$-graded tensor product $A \hat{\otimes} B$, with $(A \hat{\otimes} B)_0 = (A_0 \otimes B_0) \oplus (A_1 \otimes B_1)$ and $(A \hat{\otimes} B)_1 = (A_0 \otimes B_1) \oplus (A_1 \otimes B_0)$, the following associative multiplication

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|a_2||b_1|} a_1 a_2 \otimes b_1 b_2 , \tag{42}$$

where a_i and b_i are homogeneous elements of parity $|a_i|$, $|b_i|$, respectively, and extending it bilinearly to all of $A \hat{\otimes} B$.

Proof. Define a linear map

$$\phi : V \oplus W \longrightarrow C\ell(V, Q_V) \hat{\otimes} C\ell(W, Q_W) \tag{43}$$

by $\phi(v + w) = v \otimes \mathbf{1} + \mathbf{1} \otimes w$, where $v \in V$ and $w \in W$ and we are identifying V with its image in $C\ell(V, Q_V)$ and similarly for W. One checks that this map is Clifford *precisely* because of the sign in the definition (42) of the multiplication in the $\mathbb{Z}_2$-graded tensor product. Indeed,

$$\begin{aligned} \phi(v + w)^2 &= (v \otimes \mathbf{1} + \mathbf{1} \otimes w)^2 \\ &= v^2 \otimes \mathbf{1} + v \otimes w + (-1)^{|v||w|} v \otimes w + \mathbf{1} \otimes w^2 \\ &= -(Q(v) + Q(w))\, \mathbf{1} \otimes \mathbf{1} . \end{aligned}$$

By universality it extends to a homomorphism of $\mathbb{Z}_2$-graded associative algebras

$$\Phi : C\ell(V \oplus W, Q_V \oplus Q_W) \longrightarrow C\ell(V, Q_V) \hat{\otimes} C\ell(W, Q_W) ,$$

which is surjective because the image contains $C\ell(V, Q_V)\hat{\otimes} 1$ and $1 \hat{\otimes} C\ell(W, Q_W)$, which together generate $C\ell(V, Q_V)\hat{\otimes} C\ell(W, Q_W)$ and hence, counting dimension, an isomorphism. □

Now every (finite-dimensional) real quadratic vector space (V, Q) is isomorphic to an orthogonal direct sum

$$\underbrace{\mathbb{R}(0) \oplus \cdots \oplus \mathbb{R}(0)}_{r} \oplus \underbrace{\mathbb{R}(+) \oplus \cdots \oplus \mathbb{R}(+)}_{s} \oplus \underbrace{\mathbb{R}(-) \oplus \cdots \oplus \mathbb{R}(-)}_{t} \tag{44}$$

where $\mathbb{R}(0)$ is the one-dimensional vector space with zero quadratic form and $\mathbb{R}(\pm)$ is the one dimensional vector space with quadratic form $Q(1) = \pm 1$. It then follows from the above proposition that

$$C\ell(V, Q) \cong \Lambda \mathbb{R}^r \hat{\otimes} \underbrace{C\ell(1,0) \hat{\otimes} \cdots \hat{\otimes} C\ell(1,0)}_{s} \hat{\otimes} \underbrace{C\ell(0,1) \hat{\otimes} \cdots \hat{\otimes} C\ell(0,1)}_{t} , \tag{45}$$

where we have used that the Clifford algebra associated to the zero quadratic form is the exterior algebra. Using that $C\ell(1,0) \cong \mathbb{C}$ and $C\ell(0,1) \cong \mathbb{R} \oplus \mathbb{R}$, the above result determines in principle all the finite-dimensional real Clifford algebras as $\mathbb{Z}_2$-graded associative algebras. This is nice, but one can do much better and actually identify the Clifford algebras in terms of the matrix algebras $\mathbb{K}(n)$, for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ as we started doing in the first lecture.

2.2 Complex Clifford algebras

Before presenting the classification, let us consider the complex Clifford algebras. Since a complex quadratic form has no signature, every (finite-dimensional) complex quadratic vector space is isomorphic to an orthogonal direct sum

$$\underbrace{\mathbb{C}(0) \oplus \cdots \oplus \mathbb{C}(0)}_{r} \oplus \underbrace{\mathbb{C}(+) \oplus \cdots \oplus \mathbb{C}(+)}_{n} , \tag{46}$$

where $\mathbb{C}(0)$ is the one-dimensional complex vector spaces with zero quadratic form and $\mathbb{C}(+)$ the one-dimensional complex vector space with $Q(1) = 1$. Proposition 2.1 then says that the corresponding complex Clifford algebra is isomorphic to

$$\Lambda \mathbb{C}^r \hat{\otimes} \underbrace{\mathbb{C}\ell(1) \hat{\otimes} \cdots \hat{\otimes} \mathbb{C}\ell(1)}_{n} , \tag{47}$$

where $\mathbb{C}\ell(1)$ denotes the Clifford algebra $C\ell(\mathbb{C}(+))$. To identify it, notice that $\mathbb{C}\ell(1)$ is the complex associative algebra generated by e obeying $e^2 = -\mathbf{1}$. This means that $(ie)^2 = \mathbf{1}$ and we define complementary projectors $p_\pm = \frac{1}{2}(\mathbf{1} \pm ie)$, which induce an isomorphism $\mathbb{C}\ell(1) \cong \mathbb{C} \oplus \mathbb{C}$ with $zp_+ + wp_- \leftrightarrow (z, w)$.

One can complexify real quadratic vector spaces as follows. Let (V, Q) be a real quadratic vector space and let $V_\mathbb{C} = V \otimes_\mathbb{R} \mathbb{C}$ be its complexification. We extend Q complex linearly to a quadratic form $Q_\mathbb{C}$ defined by $Q_\mathbb{C}(v \otimes z) = z^2 Q(v)$. This turns the pair $(V_\mathbb{C}, Q_\mathbb{C})$ into a complex quadratic vector space. It is natural to ask whether $C\ell(V, Q)$ and $C\ell(V_\mathbb{C}, Q_\mathbb{C})$ are related and the answer could not be nicer.

Proposition 2.2. *The Clifford functor* $C\ell$ *commutes with complexification; that is,*

$$C\ell(V_\mathbb{C}, Q_\mathbb{C}) \cong C\ell(V, Q) \otimes_\mathbb{R} \mathbb{C} .$$

Proof. The map $V \times \mathbb{C} \to C\ell(V, Q) \otimes_\mathbb{R} \mathbb{C}$ defined by $\phi(v, z) = v \otimes z$ is real bilinear, whence it defines an $\mathbb{R}$-linear map

$$V \otimes_\mathbb{R} \mathbb{C} \xrightarrow{\phi} C\ell(V, Q) \otimes_\mathbb{R} \mathbb{C} .$$

Since $\phi(v \otimes z) = v \otimes z = (v \otimes 1)z = \phi(v \otimes 1)z$, we see that it is also $\mathbb{C}$-linear. Also since $\phi(v \otimes 1)^2 = (v \otimes 1)^2 = -Q(v)(\mathbf{1} \otimes 1)$, we see that ϕ is Clifford, whence it extends uniquely to a homomorphism of complex algebras

$$\Phi : C\ell(V_{\mathbb{C}}, Q_{\mathbb{C}}) \longrightarrow C\ell(V, Q) \otimes_{\mathbb{R}} \mathbb{C} .$$

It is shown to be surjective by observing that the image of Φ contains $C\ell(V, Q) \otimes 1$ and $\mathbf{1} \otimes \mathbb{C}$ and these generate the right-hand side. Counting dimension we conclude that Φ is an isomorphism. □

As a corollary we have that upon complexification we lose the information about the signature:

$$C\ell(s,t) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}\ell(s+t) . \tag{48}$$

The absence of signature for a complex quadratic vector space means that the complex Clifford algebras have a much simpler structure than the real Clifford algebras.

2.3 Filling in the Clifford chessboard

From now on we will restrict attention to the case of nondegenerate quadratic forms, whence any real quadratic vector space is isomorphic to $\mathbb{R}^{s,t}$ for some s, t. We would like to identify the Clifford algebras $C\ell(s,t)$ for all $s, t \geq 0$. In the last lecture we already filled in a corner of the table of Clifford algebras:

$\mathbb{R}(2)$							
$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$						
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$					

$\xrightarrow{\quad s \quad}$

This corner of the table is enough to fill in the rest of the table, thanks to the following isomorphisms.

Theorem 2.3. *For all $n, s, t \geq 0$ we have the following isomorphisms*

$$\begin{aligned} C\ell(n,0) \otimes C\ell(0,2) &\cong C\ell(0,n+2) \\ C\ell(0,n) \otimes C\ell(2,0) &\cong C\ell(n+2,0) \\ C\ell(s,t) \otimes C\ell(1,1) &\cong C\ell(s+1,t+1) \end{aligned} \tag{49}$$

with $\otimes$ the ungraded tensor product.

Proof. As the three cases are very similar, we shall prove the second equation in (49) and leave the other two as exercises for the reader.

Let us write $\mathbb{R}^{n+2,0} = \mathbb{R}^{n,0} \oplus \mathbb{R}^{2,0}$. Let e_1, e_2 be an orthonormal basis for $\mathbb{R}^{2,0}$ and let us denote by the same symbols their image in $C\ell(2,0)$. This means that $e_1^2 = e_2^2 = -\mathbf{1}$ and $e_1 e_2 = -e_2 e_1$. The element $e_1 e_2 \in C\ell(2,0)$ satisfies the following easily verifiable identities: $(e_1 e_2)^2 = -\mathbf{1}$, $e_1 e_2 e_i = -e_i e_1 e_2$ for $i = 1,2$. Let us define a linear map

$$\phi : \mathbb{R}^{n+2,0} \longrightarrow C\ell(0,n) \otimes C\ell(2,0)$$

by

$$\phi(x) = x \otimes e_1 e_2 \qquad \text{and} \qquad \phi(e_i) = \mathbf{1} \otimes e_i ,$$

for $x \in \mathbb{R}^{n,0}$. This map is Clifford by virtue of the identities satisfied by $e_1 e_2$; indeed,

$$\begin{aligned}
\phi(x + \lambda e_1 + \mu e_2)^2 &= (x \otimes e_1 e_2 + \lambda \mathbf{1} \otimes e_1 + \mu \mathbf{1} \otimes e_2)^2 \\
&= -x^2 \otimes \mathbf{1} - \lambda^2 \mathbf{1} \otimes \mathbf{1} - \mu^2 \mathbf{1} \otimes \mathbf{1} \\
&\quad + \lambda x \otimes (e_1 e_1 e_2 + e_1 e_2 e_1) + \mu x \otimes (e_2 e_1 e_2 + e_1 e_2 e_2) + \lambda\mu \mathbf{1} \otimes (e_1 e_2 + e_2 e_1) \\
&= -(x^2 + \lambda^2 + \mu^2) \mathbf{1} \otimes \mathbf{1} \\
&= -Q(x + \lambda e_1 + \mu e_2) \mathbf{1} \otimes \mathbf{1} \,.
\end{aligned}$$

Hence ϕ extends uniquely to an algebra homomorphism $\Phi : C\ell(n+2,0) \to C\ell(0,n) \otimes C\ell(2,0)$ which is surjective (the image contains a generating set) and by dimension must be an isomorphism. □

Notice that the first two isomorphisms in (49) allows us to fill the left column and the bottom row in the table, whereas the last isomorphism allows us to move diagonally. Since any square in the table lies on some diagonal, it can in principle be determined by using the isomorphisms. Moving diagonally (one step) is the same as tensoring with $C\ell(1,1) \cong \mathbb{R}(2)$. To apply this we need to make use of the following standard isomorphism of matrix algebras.

Lemma 2.4. *Let $\mathbb{K}$ stand for any of $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ and let $\mathbb{K}(n)$ denote the* real *algebra of $n \times n$ matrices with entries in $\mathbb{K}$. Then we have the following isomorphisms of real associative algebras:*

(50) $$\mathbb{K}(m) \otimes_{\mathbb{R}} \mathbb{R}(n) \cong \mathbb{K}(mn) \,.$$

This already allows us to fill in five of the diagonals in the table:

					$\mathbb{R}(64)$	$\mathbb{R}(64) \oplus \mathbb{R}(64)$	$\mathbb{R}(128)$
				$\mathbb{R}(32)$	$\mathbb{R}(32) \oplus \mathbb{R}(32)$	$\mathbb{R}(64)$	$\mathbb{C}(64)$
			$\mathbb{R}(16)$	$\mathbb{R}(16) \oplus \mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$
		$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	
	$\mathbb{R}(4)$	$\mathbb{R}(4) \oplus \mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$		
$\mathbb{R}(2)$	$\mathbb{R}(2) \oplus \mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$			
$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$				
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$					

$\xrightarrow{s}$

To continue it is necessary to extend the bottom row and the left column. For example, let us continue with the bottom row. From the second of the isomorphisms in (49), we have

$$C\ell(3,0) \cong C\ell(0,1) \otimes C\ell(2,0) \cong (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{H} = \mathbb{H} \oplus \mathbb{H} \,,$$

where we have used the distributivity of $\otimes$ over $\oplus$. In the same way we obtain

$$C\ell(4,0) \cong C\ell(0,2) \otimes C\ell(2,0) \cong \mathbb{R}(2) \otimes \mathbb{H} = \mathbb{H}(2) \,.$$

We cannot continue along the bottom row without first extending the left column. Using the first of the isomorphisms in (49), we find

$$C\ell(0,3) \cong C\ell(1,0) \otimes C\ell(0,2) \cong \mathbb{C} \otimes \mathbb{R}(2) \cong \mathbb{C}(2),$$

whereas

$$C\ell(0,4) \cong C\ell(2,0) \otimes C\ell(0,2) \cong \mathbb{H} \otimes \mathbb{R}(2) \cong \mathbb{H}(2),$$

$$C\ell(0,5) \cong C\ell(3,0) \otimes C\ell(0,2) \cong (\mathbb{H} \oplus \mathbb{H}) \otimes \mathbb{R}(2) \cong \mathbb{H}(2) \oplus \mathbb{H}(2),$$

and

$$C\ell(0,6) \cong C\ell(4,0) \otimes C\ell(0,2) \cong \mathbb{H}(2) \otimes \mathbb{R}(2) \cong \mathbb{H}(4) \,.$$

This allows us to fill in six more diagonals in the table!

	$\mathbb{H}(8)$	$\mathbb{H}(8)\oplus\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$	$\mathbb{R}(64)$	$\mathbb{R}(64)\oplus\mathbb{R}(64)$	$\mathbb{R}(128)$
$\mathbb{H}(4)$	$\mathbb{H}(4)\oplus\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	$\mathbb{R}(32)\oplus\mathbb{R}(32)$	$\mathbb{R}(64)$	$\mathbb{C}(64)$
$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16)\oplus\mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$
$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16)\oplus\mathbb{H}(16)$
$\mathbb{C}(2)$	$\mathbb{R}(4)$	$\mathbb{R}(4)\oplus\mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8)\oplus\mathbb{H}(8)$	$\mathbb{H}(16)$
$\mathbb{R}(2)$	$\mathbb{R}(2)\oplus\mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$\mathbb{H}(4)\oplus\mathbb{H}(4)$	$\mathbb{H}(8)$	
$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$		
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$			

$\xrightarrow[s]{}$

To continue we have to determine $C\ell(5,0)$. From the second of the isomorphisms in (49), we find

$$C\ell(5,0) \cong C\ell(0,3)\otimes C\ell(2,0) \cong \mathbb{C}(2)\otimes\mathbb{H} \cong ?$$

To answer the question we need the following result.

Lemma 2.5. *The following are isomorphisms of real associative algebras:*

1. $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{H} \cong \mathbb{C}(2)$
2. $\mathbb{H}\otimes_{\mathbb{R}}\mathbb{H} \cong \mathbb{R}(4)$
3. $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C} \cong \mathbb{C}\oplus\mathbb{C}$

Proof.

1. To prove the first isomorphism, let us give $\mathbb{H}$ the structure of a complex vector space by left multiplication by the complex subalgebra of $\mathbb{H}$ generated by i, say. Then we construct a real bilinear map
$$\phi : \mathbb{C}\times\mathbb{H} \longrightarrow \operatorname{End}_{\mathbb{C}}(\mathbb{H}) \quad \text{by} \quad \phi(z,q)x = zx\overline{q},$$
for all $z \in \mathbb{C}$ and $x, q \in \mathbb{H}$. By universality of the tensor product, it defines a real linear map
$$\Phi : \mathbb{C}\otimes_{\mathbb{R}}\mathbb{H} \longrightarrow \operatorname{End}_{\mathbb{C}}(\mathbb{H}) \quad \text{by} \quad \Phi(z\otimes q) = \phi(z,q).$$
We check that Φ is a homomorphism of real algebras:
$$\Phi(z_1\otimes q_1)\Phi(z_2\otimes q_1)x = z_1(z_2x\overline{q_2})\overline{q_1} = (z_1z_2)x\overline{q_1q_2} = \Phi(z_1z_2\otimes q_1q_2)x\,.$$
It is clearly injective because $\mathbb{C}$ and $\mathbb{H}$ are division algebras and counting dimension ($\dim_{\mathbb{R}} = 8$) we see that Φ must be an isomorphism. Now $\mathbb{H}\cong\mathbb{C}^2$ as a complex vector space, whence $\operatorname{End}_{\mathbb{C}}(\mathbb{H}) \cong \mathbb{C}(2)$.

2. This is proved in a very similar manner to the first isomorphism. Namely we define a real bilinear map
$$\phi : \mathbb{H}\times\mathbb{H} \longrightarrow \operatorname{End}_{\mathbb{R}}(\mathbb{H}) \quad \text{by} \quad \phi(q_1,q_2)x = q_1x\overline{q_2},$$
for all $q_i, x \in \mathbb{H}$, which by universality of the tensor product induces a real linear map
$$\Phi : \mathbb{H}\otimes_{\mathbb{R}}\mathbb{H} \longrightarrow \operatorname{End}_{\mathbb{R}}(\mathbb{H}) \quad \text{by} \quad \Phi(q_1\otimes q_2) = \phi(q_1,q_2).$$
It is clear that it is injective and counting dimension ($\dim_{\mathbb{R}} = 16$), it is an isomorphism of real vector spaces, but again one checks that Φ is an algebra morphism:
$$\Phi(q_1\otimes q_2)\Phi(q_1'\otimes q_2')x = q_1(q_1'x\overline{q_2'})\overline{q_2} = (q_1q_1')x\overline{q_2q_2'} = \Phi(q_1q_1'\otimes q_2q_2')x\,.$$

3. This is even easier. Notice that the element $i\otimes i \in \mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}$ squares to the identity, whence we can form complementary projectors $p_\pm = \frac{1}{2}(1\otimes 1 \pm i\otimes i)$ whose images are commuting subalgebras isomorphic to $\mathbb{C}$. Explicitly, the isomorphism $\mathbb{C}\oplus\mathbb{C}\to\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}$ is given by $(z_1,z_2)\mapsto z_1p_+ + z_2p_-$.

$\square$

It follows at once from the first of these isomorphisms, that

$$\mathbb{C}(2) \otimes \mathbb{H} \cong \mathbb{R}(2) \otimes \mathbb{C} \otimes \mathbb{H} \cong \mathbb{R}(2) \otimes \mathbb{C}(2) \cong \mathbb{C}(4) \ ,$$

whence $C\ell(5,0) \cong \mathbb{C}(4)$, and hence

$$C\ell(0,7) \cong C\ell(5,0) \otimes C\ell(0,2) \cong \mathbb{C}(4) \otimes \mathbb{R}(2) \cong \mathbb{C}(8) \ .$$

In the same way one can show that $C\ell(6,0) \cong \mathbb{R}(8)$ and $C\ell(7,0) \cong \mathbb{R}(8) \oplus \mathbb{R}(8)$, which allows us to complete the first 8×8 corner of the table:

$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8) \oplus \mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$	$\mathbb{R}(64)$	$\mathbb{R}(64) \oplus \mathbb{R}(64)$	$\mathbb{R}(128)$
$\mathbb{H}(4)$	$\mathbb{H}(4) \oplus \mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	$\mathbb{R}(32) \oplus \mathbb{R}(32)$	$\mathbb{R}(64)$	$\mathbb{C}(64)$
$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16) \oplus \mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$
$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16) \oplus \mathbb{H}(16)$
$\mathbb{C}(2)$	$\mathbb{R}(4)$	$\mathbb{R}(4) \oplus \mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8) \oplus \mathbb{H}(8)$	$\mathbb{H}(16)$
$\mathbb{R}(2)$	$\mathbb{R}(2) \oplus \mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$\mathbb{H}(4) \oplus \mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$
$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2) \oplus \mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$

$\xrightarrow{s}$

As nice as this is, it might seem that in order to classify real Clifford algebras in arbitrary (albeit finite) dimension we have to do lots of work. Luckily this is not the case, which explains *a posteriori* why I have restricted myself to an 8×8 corner, the so-called Clifford chessboard. It turns out that the real Clifford algebras have simpler periodicities, which are an easy consequence of Theorem 2.3. We call them the **Bott periodicities**.

Corollary 2.6. *For all $n, s, t \geq 0$, the following are isomorphisms of real algebras:*

1. $C\ell(n+8,0) \cong C\ell(n,0) \otimes_{\mathbb{R}} \mathbb{R}(16)$,
2. $C\ell(0,n+8) \cong C\ell(0,n) \otimes_{\mathbb{R}} \mathbb{R}(16)$, *and*
3. $C\ell(s+4,t+4) \cong C\ell(s,t) \otimes_{\mathbb{R}} \mathbb{R}(16)$.

Proof. This follows directly from repeated application of Theorem 2.3 and the following isomorphisms:

$$C\ell(1,1)^{\otimes 4} \cong \mathbb{R}(16) \qquad \text{and} \qquad C\ell(2,0)^{\otimes 2} \otimes C\ell(0,2)^{\otimes 2} \cong \mathbb{R}(16) \ .$$

□

Theorem 2.7 (Classification theorem)**.** *The Clifford algebra $C\ell(s,t)$ is isomorphic to the real associative algebras in the following table, where $d = s+t$:*

$s-t \mod 8$	$C\ell(s,t)$
0, 6	$\mathbb{R}\left(2^{d/2}\right)$
7	$\mathbb{R}\left(2^{(d-1)/2}\right) \oplus \mathbb{R}\left(2^{(d-1)/2}\right)$
1, 5	$\mathbb{C}\left(2^{(d-1)/2}\right)$
2, 4	$\mathbb{H}\left(2^{(d-2)/2}\right)$
3	$\mathbb{H}\left(2^{(d-3)/2}\right) \oplus \mathbb{H}\left(2^{(d-3)/2}\right)$

The proof follows from the Clifford chessboard and the periodicities and it is simply a matter of bookkeeping.

2.3.1 The even subalgebra of the Clifford algebra

In the study of spinor representations it is important to identify the even subalgebra $\mathrm{C}\ell(s,t)_0$ of the Clifford algebras $\mathrm{C}\ell(s,t)$ as an *ungraded* real associative algebra. Recall that $\mathrm{C}\ell(s,t)_0$ is the fixed subalgebra under the automorphism induced by the orthogonal transformation in $\mathrm{O}(s,t)$ which sends $x \mapsto -x$ for all $x \in \mathbb{R}^{s,t}$. This means that every element in $\mathrm{C}\ell(s,t)_0$ can be written as a linear combination of products of an even number of elements in the image in $\mathrm{C}\ell(s,t)$ of $\mathbb{R}^{s,t}$.

Luckily, $\mathrm{C}\ell(s,t)_0$ can be determined from the Clifford algebra one dimension lower.

Proposition 2.8. *For all $s,t \geq 0$, we have the following isomorphisms of ungraded real associative algebras:*

$$\mathrm{C}\ell(s,t) \cong \mathrm{C}\ell(s+1,t)_0 \cong \mathrm{C}\ell(t,s+1)_0 \,.$$

In particular, we have that $\mathrm{C}\ell(s,t)_0 \cong \mathrm{C}\ell(t,s)_0$.

Proof. We will prove one of the isomorphisms and leave the other as an exercise. Let us define $\phi : \mathbb{R}^{s,t} \to \mathrm{C}\ell(s+1,t)_0$ by $\phi(x) = x e_{s+1}$, where we write $\mathbb{R}^{s+1,t} = \mathbb{R}^{s,t} \oplus \mathbb{R} e_{s+1}$. We check that ϕ is a Clifford map:

$$\phi(x)^2 = x e_{s+1} x e_{s+1} = -x^2 e_{s+1}^2 = x^2 = -\mathrm{Q}(x)\mathbf{1} \,,$$

whence it extends uniquely to an algebra homomorphism $\Phi : \mathrm{C}\ell(s,t) \to \mathrm{C}\ell(s+1,t)_0$. It is clearly surjective since it contains the image contains a generating set, and, counting dimension, we conclude Φ is an isomorphism. □

As a corollary of the classification theorem 2.7, we immediately have a classification of the $\mathrm{C}\ell(s,t)_0$.

Corollary 2.9. *The even Clifford algebra $\mathrm{C}\ell(s,t)_0$ is isomorphic to the real associative algebras in the following table, where $d = s+t$:*

$s-t \mod 8$	$\mathrm{C}\ell(s,t)_0$
1,7	$\mathbb{R}\left(2^{(d-1)/2}\right)$
0	$\mathbb{R}\left(2^{(d-2)/2}\right) \oplus \mathbb{R}\left(2^{(d-2)/2}\right)$
2,6	$\mathbb{C}\left(2^{(d-2)/2}\right)$
3,5	$\mathbb{H}\left(2^{(d-3)/2}\right)$
4	$\mathbb{H}\left(2^{(d-4)/2}\right) \oplus \mathbb{H}\left(2^{(d-4)/2}\right)$

2.4 Classification of complex Clifford algebras

Having determined the real Clifford algebras, it is a simple matter to use Proposition 2.2 and determine the complex Clifford algebras. It is however easier to derive the complex Bott periodicity directly.

Proposition 2.10. *For all $n \geq 0$ there is an isomorphism of complex associative algebras*

$$\mathbb{C}\ell(n+2) \cong \mathbb{C}\ell(n) \otimes_{\mathbb{C}} \mathbb{C}(2) \,.$$

Proof. Write $\mathbb{C}^{n+2} = \mathbb{C}^n \oplus \mathbb{C} e_1 \oplus \mathbb{C} e_2$ and define a complex linear map

$$\phi : \mathbb{C}^{n+2} \longrightarrow \mathbb{C}\ell(n) \otimes_{\mathbb{C}} \mathbb{C}(2)$$

by

$$\phi(x) = x \otimes \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad \phi(e_1) = \mathbf{1} \otimes \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \text{and} \qquad \phi(e_2) = \mathbf{1} \otimes \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

for all $x \in \mathbb{C}^n$. One checks that ϕ is Clifford and that the induced map $\Phi : \mathbb{C}\ell(n+2) \to \mathbb{C}\ell(n) \otimes_{\mathbb{C}} \mathbb{C}(2)$, being surjective (the image contains a generating set) and mapping between equidimensional spaces, is an isomorphism. □

The classification of complex Clifford algebras is then an easy corollary.

Corollary 2.11. *For every $n \geq 0$, the complex Clifford algebra $\mathbb{C}\ell(n)$ is isomorphic to*

$$\mathbb{C}\ell(n) \cong \begin{cases} \mathbb{C}\left(2^{n/2}\right) & \text{if } n \text{ is even,} \\ \mathbb{C}\left(2^{(n-1)/2}\right) \oplus \mathbb{C}\left(2^{(n-1)/2}\right) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. This follows easily from complex Bott periodicity and the "initial conditions" $\mathbb{C}\ell(0) \cong \mathbb{C}$ and $\mathbb{C}\ell(1) \cong \mathbb{C} \oplus \mathbb{C}$. □

Lecture 3: Spinor representations

Yes now I've met me another spinor...
— Suzanne Vega (with apologies)

It was Élie Cartan, in his study of representations of simple Lie algebras, who came across representations of the orthogonal Lie algebra which were not tensorial; that is, not contained in any tensor product of the fundamental (vector) representation. These are the so-called spinorial representations. His description [Car38] of the spinorial representations was quite complicated ("fantastic" according to Dieudonné's review of Chevalley's book below) and it was Brauer and Weyl [BW35] who in 1935 described these representations in terms of Clifford algebras. This point of view was further explored in Chevalley's book [Che54] which is close to the modern treatment. This lecture is devoted to the Pin and Spin groups and to a discussion of their (s)pinorial representations.

3.1 The orthogonal group and its Lie algebra

Throughout this lecture we will let (V,Q) be a real finite-dimensional quadratic vector space with Q nondegenerate. We will drop explicit mention of Q, whence the Clifford algebra shall be denoted $\mathrm{C}\ell(\mathrm{V})$ and similarly for other objects which depend on Q. We will let B denote the bilinear form defining Q.

We start by defining the group $\mathrm{O}(\mathrm{V})$ of orthogonal transformations of V:

$$\mathrm{O}(\mathrm{V}) = \{a : \mathrm{V} \to \mathrm{V} | \mathrm{Q}(av) = \mathrm{Q}(v) \quad \forall v \in \mathrm{V}\} \ . \tag{51}$$

We write $\mathrm{O}(\mathbb{R}^{s,t}) = \mathrm{O}(s,t)$ and $\mathrm{O}(n)$ for $\mathrm{O}(n,0)$. If $a \in \mathrm{O}(\mathrm{V})$, then $\det a = \pm 1$. Those $a \in \mathrm{O}(\mathrm{V})$ with $\det a = 1$ define the special orthogonal group $\mathrm{SO}(\mathrm{V})$. If V is either positive- or negative-definite then $\mathrm{SO}(\mathrm{V})$ is connected: otherwise it has two connected components. This can be inferred by the fact that the connectedness of a Lie group is controlled by that of its maximal compact subgroup, which in the case of $\mathrm{SO}(s,t)$, for $s,t>0$, is

$$\mathrm{S}\left(\mathrm{O}(s)\times\mathrm{O}(t)\right) = \{(a,b)\in\mathrm{O}(s)\times\mathrm{O}(t) | \det a = \det b\} ,$$

which has two connected components. The Lie algebra $\mathfrak{so}(\mathrm{V})$ of $\mathrm{SO}(\mathrm{V})$ is defined by

$$\mathfrak{so}(\mathrm{V}) = \{\mathrm{X} : \mathrm{V} \to \mathrm{V} | \mathrm{B}(\mathrm{X}u, v) = -\mathrm{B}(u, \mathrm{X}v) \quad \forall u, v \in \mathrm{V}\} \ . \tag{52}$$

As a vector space, $\mathfrak{so}(\mathrm{V}) \cong \Lambda^2\mathrm{V}$, where the skewsymmetric endomorphism $u \curlywedge v \in \mathfrak{so}(\mathrm{V})$ corresponding to $u \wedge v \in \Lambda^2\mathrm{V}$ is defined by

$$(u \curlywedge v)(x) = \mathrm{B}(u,x)v - \mathrm{B}(v,x)u \ . \tag{53}$$

It is easy to check that $u \curlywedge v \in \mathfrak{so}(\mathrm{V})$ as it is to compute the commutator

$$[u \curlywedge v, x \curlywedge y] = \mathrm{B}(u,x) v \curlywedge y - \mathrm{B}(u,y) v \curlywedge x - \mathrm{B}(v,x) u \curlywedge y + \mathrm{B}(v,y) u \curlywedge x \ . \tag{54}$$

The Clifford algebra $\mathrm{C}\ell(\mathrm{V})$ being associative, becomes a Lie algebra under the commutator and contains $\mathfrak{so}(\mathrm{V})$ as a Lie subalgebra via the embedding

$$\rho : \mathfrak{so}(\mathrm{V}) \to \mathrm{C}\ell(\mathrm{V}) \qquad \text{where} \qquad \rho(u \curlywedge v) = \tfrac{1}{4}(uv - vu) \ . \tag{55}$$

Indeed, it is a simple calculation using the Clifford relation $uv = -vu - 2\mathrm{B}(u,v)\mathbf{1}$ to show that

$$[\rho(u \curlywedge v), x] = \mathrm{B}(u,x)v - \mathrm{B}(v,x)u = (u \curlywedge v)(x) \ , \tag{56}$$

and hence that

$$[\rho(u \curlywedge v), \rho(x \curlywedge y)] = \mathrm{B}(u,x)\rho(v \curlywedge y) - \mathrm{B}(u,y)\rho(v \curlywedge x) - \mathrm{B}(v,x)\rho(u \curlywedge y) + \mathrm{B}(v,y)\rho(u \curlywedge x) \ , \tag{57}$$

whence ρ is an injective Lie algebra homomorphism.

Exponentiating $\mathfrak{so}(V)$ in $\mathrm{End}(V)$ generates the identity component $\mathrm{SO}_0(V)$ of $\mathrm{SO}(V)$, whereas exponentiating $\rho(\mathfrak{so}(V))$ in $C\ell(V)$ generates a covering group of $\mathrm{SO}_0(V)$. We will see this in full generality below, but let us motivate this with an example. Suppose that V contains a positive-definite plane with orthonormal basis e_1, e_2. Then relative to this basis, the restriction to this plane of $e_1 \curlywedge e_2 \in \mathfrak{so}(V)$ has matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{58}$$

whose exponential is

$$a(\theta) = \exp(\theta(e_1 \curlywedge e_2)) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{59}$$

whence, in particular, $a(2\pi)$ is the identity matrix. On the other hand, exponentiating the image of the same Lie algebra element $\rho(e_1 \curlywedge e_2) = \frac{1}{2} e_1 e_2$ in $C\ell(V)$ we obtain

$$b(\theta) = \exp(\tfrac{1}{2}\theta e_1 e_2) = \cos(\tfrac{1}{2}\theta)\mathbf{1} + \sin(\tfrac{1}{2}\theta) e_1 e_2, \tag{60}$$

using that $(e_1 e_2)^2 = -\mathbf{1}$. In particular we see that $b(2\pi) = -\mathbf{1}$, so that the periodicity of $b(\theta)$ is 4π. In other words, it suggests that the Lie group generated by exponentiating $\mathfrak{so}(V)$ in $C\ell(V)$ is a double cover of $\mathrm{SO}_0(V)$. We will see that this is indeed the case.

3.2 Pin and Spin

Definition 3.1. The **Pin group** $\mathrm{Pin}(V)$ of (V, Q) is the subgroup of (the group of units of) $C\ell(V)$ generated by $v \in V$ with $Q(v) = \pm 1$. In other words, every element of $\mathrm{Pin}(V)$ is of the form $u_1 \cdots u_r$ where $u_i \in V$ and $Q(u_i) = \pm 1$. We will write $\mathrm{Pin}(s,t)$ for $\mathrm{Pin}(\mathbb{R}^{s,t})$ and $\mathrm{Pin}(n)$ for $\mathrm{Pin}(n,0)$.

Let $v \in V \subset C\ell(V)$ and let $Q(v) \neq 0$. Then v is invertible in $C\ell(V)$ and $v^{-1} = -v/Q(v)$. We define, by analogy with the case of a Lie group, the **adjoint action** $\mathrm{Ad}_v : V \to V$, by

$$\mathrm{Ad}_v(x) = v x v^{-1} = \frac{-1}{Q(v)} v x v = \frac{-1}{Q(v)}(-xv - 2B(x,v)\mathbf{1})\,v = -x + 2\frac{B(x,v)}{Q(v)} v = -R_v x\,, \tag{61}$$

where R_v stands for the reflection on the hyperplane perpendicular to v and $x \in V$. We can extend this to a group homomorphism from the Pin group: $\mathrm{Ad}_{v_1 \cdots v_p} = \mathrm{Ad}_{v_1} \circ \cdots \circ \mathrm{Ad}_{v_p}$. Since we would prefer not to see the sign on the right-hand side of $\mathrm{Ad}_v(x)$, we define the **twisted adjoint action** by $\widetilde{\mathrm{Ad}}_v(x) = (-v)xv^{-1} = R_v x$ or more generally $\widetilde{\mathrm{Ad}}_a = \tilde{a} x a^{-1}$ for a an element of the Pin group and $a \mapsto \tilde{a}$ the grading automorphism of $C\ell(V)$, which is induced by the orthogonal transformation $v \mapsto -v$. Let $a = u_1 \cdots u_r \in \mathrm{Pin}(V)$, then $\widetilde{\mathrm{Ad}}_a = R_{u_1} \circ \cdots \circ R_{u_r}$. Since reflections are orthogonal transformations, $\widetilde{\mathrm{Ad}}$ defines a group homomorphism $\widetilde{\mathrm{Ad}} : \mathrm{Pin}(V) \to \mathrm{O}(V)$. It follows from the following classic result that $\widetilde{\mathrm{Ad}}$ is surjective.

Theorem 3.2 (Cartan–Dieudonné)**.** *Every $g \in \mathrm{O}(V)$ is the product of a finite number of reflections $g = R_{u_1} \circ \cdots \circ R_{u_r}$ along non-null lines ($Q(u_i) \neq 0$) and moreover $r \leq \dim V$.*

We will now determine the kernel of $\widetilde{\mathrm{Ad}}$. Let $a \in \mathrm{Pin}(V)$ be in the kernel of $\widetilde{\mathrm{Ad}}$. This means that $\tilde{a} v = v a$ for all $v \in V$. Let us break up $a = a_0 + a_1$ with $a_0 \in C\ell(V)_0$ and $a_1 \in C\ell(V)_1$, whence $\tilde{a} = a_0 - a_1$. Therefore $a \in \ker \widetilde{\mathrm{Ad}}$ if and only if the following pair of equations are satisfied for all $v \in V$:

$$a_0 v = v a_0 \qquad \text{and} \qquad a_1 v = -v a_1\,. \tag{62}$$

Suppose that $v \in V$ with $Q(v) \neq 0$ and consider $a_0 = \alpha + v\beta$, where α and β do not involve v. Since $\alpha \in C\ell(V)_0$ and does not involve v, then $v\alpha = \alpha v$, whereas since $\beta \in C\ell(V)_1$ and does not involve v, then $v\beta = -\beta v$. The first equation in (62) says that $\beta = 0$, whence $a_0 = \alpha$ does not involve v. Repeating this argument for all the elements of an orthonormal basis (e_i) for V, we see that a_0 does not involve any of the e_i and hence must be a multiple of the identity: $a_0 = \alpha\mathbf{1}$ for some $\alpha \in \mathbb{R}$. Similarly, write $a_1 = \gamma + v\delta$, where γ, δ do not involve v. Now we have that $\gamma v = -v\gamma$, whereas $\delta v = v\delta$. The second equation in (62) says that $\delta = 0$, whence $a_1 = \gamma$ does not involve v. Repeating this argument for the basis (e_i), we see that a_1 does not involve any of the e_i and hence must be a multiple of the identity, but $a_1 \in C\ell(V)_1$ whereas $\mathbf{1} \in C\ell(V)_0$, whence $a_1 = 0$. Hence all elements of $\mathrm{Pin}(V)$ in the kernel of $\widetilde{\mathrm{Ad}}$ are multiples of the identity. Now let $u_1 \cdots u_p = \alpha\mathbf{1}$ for $Q(u_i) = \pm 1$. Let us compute the norm of this element using the Clifford inner product (41), to arrive at

$$(63)\qquad (\alpha\mathbf{1},\alpha\mathbf{1}) = \alpha^2(\mathbf{1},\mathbf{1}) = (u_1\cdots u_p, u_1\cdots u_p) = (\mathbf{1},(-u_p)\dots(-u_1)u_1\cdot u_p) = Q(u_1)\cdots Q(u_p)(\mathbf{1},\mathbf{1})\ .$$

Since $(\mathbf{1},\mathbf{1}) \neq 0$ and $Q(u_i) = \pm 1$, it follows that $\alpha^2 = \pm 1$. Since $\alpha \in \mathbb{R}$ the only solutions to this equation are $\alpha = \pm 1$ and hence $\ker\widetilde{\mathrm{Ad}} = \{\pm\mathbf{1}\}$. In summary we have proved

Proposition 3.3. *The following sequence is exact:*

$$1 \longrightarrow \{\pm\mathbf{1}\} \longrightarrow \mathrm{Pin}(V) \xrightarrow{\widetilde{\mathrm{Ad}}} O(V) \longrightarrow 1\ .$$

Exact sequences

A sequence of groups and group homomorphisms

$$1 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 1$$

is said to be *exact* if the kernel of each homomorphism is the image of the preceding one. In the above diagram, 1 denotes the one-element group. This is both an initial and final object in the category of groups, since there is only one homomorphism into it (sending all elements to the identity) and only one homomorphism out of it (sending the identity to the identity). This explains why we have not given names to the homomorphisms $1 \to A$ and $C \to 1$. Exactness at A means that $i : A \to B$ is injective, since its kernel is the image of $1 \to A$, whence consists only of the identity. Similarly, exactness at C says that $p : B \to C$ is surjective, since the kernel of $C \to 1$ is all of C, and that is precisely the image of p. Finally, exactness at B says that the kernel of $p : B \to C$ is precisely the image of $i : A \to B$. Such an exact sequence says that B is an *extension* of C by A.

Finally, let us define the spin group.

Definition 3.4. The **spin group** of (V, Q) is the intersection

$$\mathrm{Spin}(V) = \mathrm{Pin}(V) \cap C\ell(V)_0\ .$$

Equivalently, it consists of elements $u_1 \cdots u_{2p}$, where $u_i \in V$ and $Q(u_i) = \pm 1$. We will write $\mathrm{Spin}(s,t)$ for $\mathrm{Spin}(\mathbb{R}^{s,t})$ and $\mathrm{Spin}(n)$ for $\mathrm{Spin}(n,0)$.

Since for a reflection $R_u \in O(V)$, we have that $\det R_u = -1$, it follows that $\det\widetilde{\mathrm{Ad}}_a = 1$ for $a \in \mathrm{Pin}(V)$ if and only if $a \in \mathrm{Spin}(V)$. Since the kernel of $\widetilde{\mathrm{Ad}}$ belongs to $\mathrm{Spin}(V)$, we immediately have the following

Proposition 3.5. *The following sequence is exact:*

$$1 \longrightarrow \{\pm\mathbf{1}\} \longrightarrow \mathrm{Spin}(V) \xrightarrow{\widetilde{\mathrm{Ad}}} SO(V) \longrightarrow 1\ .$$

For V of signature (s,t) with at least one of $s,t \geq 2$, the map $\widetilde{\mathrm{Ad}} : \mathrm{Spin}(V) \to SO(V)$ is a nontrivial covering. This can be shown by exhibiting a continuous path between $\mathbf{1}$ and $-\mathbf{1}$ in $\mathrm{Spin}(V)$. Let e_1, e_2 be an orthonormal basis for a positive- or negative-definite plane. That such a plane exists is a consequence of our assumption on the signature of V. Then consider the following continuous (in fact, analytic) curve in $\mathrm{Spin}(V)$:

$$a(t) = (e_1\cos t + e_2\sin t)(e_2\sin t - e_1\cos t) = Q(e_1)\cos(2t)\mathbf{1} + \sin(2t)e_1e_2\ .$$

We see that $a(0) = Q(e_1)\mathbf{1}$, whereas $a(\pi/2) = -Q(e_1)\mathbf{1}$, whence it joins $\mathbf{1}$ to $-\mathbf{1}$.

Finally let us remark that for V either positive- or negative-definite, SO(V) and hence Spin(V) is connected, whereas for indefinite V, Spin(V) has two connected components. Let $\mathrm{Spin}_0(V)$ denote the identity component. In definite or lorentzian signatures, $\mathrm{Spin}_0(V) \to \mathrm{SO}_0(V)$ is a universal covering, but $\mathrm{Spin}_0(s,t)$ is not simply connected when both $s,t \geq 2$. The simplest interesting examples of spin covers are $\mathrm{SU}(2) \to \mathrm{SO}(3)$ and $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}_0(3,1)$.

3.3 Pinors and spinors

Informally, pinors (resp. spinors) are vectors in an irreducible representation of a Clifford algebra (resp. its even subalgebra) and, by restriction, of the corresponding Pin (resp. Spin) group. In order to define them properly we need to introduce some notation.

Definition 3.6. Let A be a real associative algebra and let $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. By a $\mathbb{K}$-**representation** of A we mean an $\mathbb{R}$-linear homomorphism

$$\rho : A \to \mathrm{End}_{\mathbb{K}}(E)$$

for some $\mathbb{K}$-vector space E. Two $\mathbb{K}$-representations $\rho : A \to \mathrm{End}_{\mathbb{K}}(E)$ and $\rho' : A \to \mathrm{End}_{\mathbb{K}}(E')$ are **equivalent** if there is a $\mathbb{K}$-linear isomorphism $f : E \to E'$ such that the following triangle commutes:

$$\begin{tikzcd} & A \arrow[dl, "\rho"'] \arrow[dr, "\rho'"] & \\ \mathrm{End}_{\mathbb{K}}(E) \arrow[rr, "\mathrm{Ad}\, f"] & & \mathrm{End}_{\mathbb{K}}(E') \end{tikzcd}$$

where $\mathrm{Ad}\, f : \mathrm{End}_{\mathbb{K}}(E) \to \mathrm{End}_{\mathbb{K}}(E')$ is defined by $\varphi \mapsto f \circ \varphi \circ f^{-1}$. In other words, for all $a \in A$, we have that $f \circ \rho(a) = \rho'(a) \circ f$.

Quaternionic vector spaces

Because $\mathbb{H}$ is not commutative, one must distinguish between left and right quaternionic vector spaces. This is largely a matter of convention, since quaternionic conjugation relates left and right vector spaces. Throughout these lectures we shall adopt the convention that $\mathbb{H}^n$ is a right quaternionic vector space. In this way, the matrix algebra $\mathbb{H}(n)$ can act $\mathbb{H}$-linearly on $\mathbb{H}^n$ from the left. (The fact that left and right multiplication commute is precisely associativity.) This defines an isomorphism of *real* algebras $\mathbb{H}(n) \cong \mathrm{End}_{\mathbb{H}}(\mathbb{H}^n)$. In fact, notice that $\mathrm{End}_{\mathbb{H}}(E)$ for a quaternionic vector space E is *only* a real algebra! This is because of the nonexistence of $\mathbb{H}$-bilinears.

In Section 1.4.4 we have already seen one example of an $\mathbb{R}$-representation of $C\ell(V)$, namely ΛV. This representation is not irreducible, however.

Definition 3.7. A **pinor representation** of Pin(V) is the restriction of an irreducible representation of $C\ell(V)$. Similarly, a **spinor representation** of Spin(V) is the restriction of an irreducible representation of $C\ell(V)_0$. (It is not hard to see that both pinor and spinor representations are irreducible, basically because there is an additive basis for $C\ell(V)$ made out of elements of Pin(V).)

The irreducible representations of $C\ell(V)$ are easy to determine from the classification of real Clifford algebras in the second lecture. Recall that as a real algebra, $C\ell(s,t)$ is isomorphic to either $\mathbb{K}(2^n)$ or $\mathbb{K}(2^n) \oplus \mathbb{K}(2^n)$ depending on the signature. The following result can be extracted from [Lan84, § XVII].

Theorem 3.8.
1. *Every irreducible $\mathbb{R}$-representation of the real algebra $\mathbb{R}(n)$ is isomorphic to $\mathbb{R}^n$, where the matrix $A \in \mathbb{R}(n)$ acts via left matrix multiplication.*
2. *Every irreducible $\mathbb{H}$-representation of the real algebra $\mathbb{H}(n)$ is isomorphic to $\mathbb{H}^n$ as a right $\mathbb{H}$-vector space and where $A \in \mathbb{H}(n)$ acts via left matrix multiplication.*

3. *Every irreducible $\mathbb{C}$-representation of the real algebra $\mathbb{C}(n)$ is isomorphic either to $\mathbb{C}^n$ with the natural action given by left matrix multiplication by $A \in \mathbb{C}(n)$ or to $\mathbb{C}^n$ with the complex conjugate action given by left matrix multiplication by $\overline{A} \in \mathbb{C}(n)$.*

This result together with the classification of real Clifford algebras, allows us to determine the pinor representations easily. First of all, we notice that because of the third isomorphism in (49), the type of the Clifford algebra does not change, only the dimension does, when we move diagonally in the table. This means that the type of the representation of $C\ell(s,t)$ only depends on $s-t$ and, moreover, because of Bott periodicity, only on $s-t \pmod 8$. Thus we need only remember one small part of the Clifford chessboard to determine the rest:

$\mathbb{C}(2)$				
$\mathbb{R}(2)$				
$\mathbb{R}\oplus\mathbb{R}$				
$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$

$\xrightarrow[s]{}$

Notice that if we colour the squares of the chessboard according to whether $C\ell(s,t)$ has one or two inequivalent irreducible representations, then we do indeed end up with a chessboard pattern.

2	1	2	1	2	
1	2	1	2	1	
2	1	2	1	2	
1	2	1	2	1	

$\xrightarrow[s]{}$

This dichotomy can also be explained by means of the volume element of $C\ell(V)$. Given an ordered orthonormal basis $(e_1^+,\dots,e_s^+,e_1^-,\dots,e_t^-)$ for $\mathbb{R}^{s,t}$, with $Q(e_i^\pm)=\pm 1$, there is associated a **volume element** of $C\ell(s,t)$ defined as the Clifford product $\omega = e_1^+\cdots e_s^+ e_1^-\cdots e_t^-$.

Lemma 3.9. *The volume element $\omega \in C\ell(s,t)$ satisfies the following properties:*

1. $\omega^2 = (-1)^{s+d(d-1)/2}\mathbf{1}$*, where $d = s+t$,*
2. ω *is central if $s+t$ is odd, and*
3. $\omega v = -v\omega$ *for all $v \in V$, if $s+t$ is even.*

It follows from the first part that the sign of ω^2 depends only on $s-t \pmod 4$:

$$\omega^2 = \begin{cases} \mathbf{1}, & s-t = 0,3 \pmod 4 \\ -\mathbf{1}, & s-t = 1,2 \pmod 4 \end{cases}$$

$-$	$-$	$+$	$+$
$-$	$+$	$+$	$-$
$+$	$+$	$-$	$-$
$+$	$-$	$-$	$+$

$\xrightarrow[s]{}$

Suppose that $s+t$ (equivalently, $s-t$) is odd, so that ω is central. Then if $\omega^2 = \mathbf{1}$ there are two pinor representations $P_\pm$, distinguished by the action of ω: $\omega = \pm 1$ on $P_\pm$. If $s-t = 3 \pmod 8$, $P_\pm$ is quaternionic, whereas if $s-t = 7 \pmod 8$, $P_\pm$ is real. If $\omega^2 = -\mathbf{1}$, so that $s-t = 1,5 \pmod 8$, there are two complex pinor representations P and $\overline{P}$, distinguished by the action of ω: $\omega = \pm i$ on P and $\overline{P}$, respectively.

In summary, the type and dimension of the pinor representations follows from the classification theorem 2.7. Similarly, the type and dimension of the spinor representations, being representations of the even subalgebra $C\ell(V)_0$, follow from Corollary 2.9. It remains to understand how the pinor and spinor representations are related, for which we need a brief scholium about representation theory.

Real, complex and quaternionic representations

Let G be a Lie group, such as Pin(V) or Spin(V). If $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, we will let $\mathrm{Rep}_{\mathbb{K}}(G)$ denote the (symmetric, monoidal) category of $\mathbb{K}$-representations of G, whose objects are $\mathbb{K}$-vector spaces E (with the usual caveat about the case $\mathbb{K} = \mathbb{H}$) together with group homomorphisms $\rho : G \to \mathrm{GL}_{\mathbb{K}}(E)$ and where a morphism between $\rho : G \to \mathrm{GL}_{\mathbb{K}}(E)$ and $\rho' : G \to \mathrm{GL}_{\mathbb{K}}(E')$ is a $\mathbb{K}$-linear map $f : E \to E'$ such that for all $g \in G$, $f \circ \rho(g) = \rho'(g) \circ f$. There are a number of functors relating these categories, which commute with the direct sum of representations, which is the categorical coproduct in $\mathrm{Rep}_{\mathbb{K}}(G)$. These functors are neatly summarised in the following (noncommutative!) diagram, borrowed from [Ada69] via [BtD85]:

(64)
$$\begin{tikzcd}
& \mathrm{Rep}_{\mathbb{C}}(G) & \\
\mathrm{Rep}_{\mathbb{R}}(G) \arrow[ur, "e^{\mathbb{C}}_{\mathbb{R}}"] & & \mathrm{Rep}_{\mathbb{H}}(G) \arrow[ul, "r^{\mathbb{H}}_{\mathbb{C}}"'] \\
& \mathrm{Rep}_{\mathbb{C}}(G) \arrow[ul, "r^{\mathbb{C}}_{\mathbb{R}}"] \arrow[ur, "e^{\mathbb{H}}_{\mathbb{C}}"'] \arrow[uu, leftrightarrow, "c"] &
\end{tikzcd}$$

where c takes a complex representation E to its complex conjugate $c(E) = \overline{E}$, $e^{\mathbb{C}}_{\mathbb{R}}$ and $e^{\mathbb{H}}_{\mathbb{C}}$ are *extension of scalars*, taking a real representation E to its complexification $e^{\mathbb{C}}_{\mathbb{R}}(E) = E \otimes_{\mathbb{R}} \mathbb{C}$ and a complex representation E to its quaternionification $e^{\mathbb{H}}_{\mathbb{C}}(E) = E \otimes_{\mathbb{C}} \mathbb{H}$, and where $r^{\mathbb{C}}_{\mathbb{R}}$ and $r^{\mathbb{H}}_{\mathbb{C}}$ are *restriction of scalars*, so that we simply view a complex representation E as a real representation $r^{\mathbb{C}}_{\mathbb{R}}(E)$ and a quaternionic representation E as a complex representation $r^{\mathbb{H}}_{\mathbb{C}}(E)$. These functors satisfy a number of identities:

(65)
$$\begin{array}{ccc}
c^2 = 1 & r^{\mathbb{C}}_{\mathbb{R}} e^{\mathbb{C}}_{\mathbb{R}} = 2 & c\, r^{\mathbb{H}}_{\mathbb{C}} = r^{\mathbb{H}}_{\mathbb{C}} \\
c\, e^{\mathbb{C}}_{\mathbb{R}} = e^{\mathbb{C}}_{\mathbb{R}} & e^{\mathbb{C}}_{\mathbb{R}} r^{\mathbb{C}}_{\mathbb{R}} = 1 + c & e^{\mathbb{H}}_{\mathbb{C}} r^{\mathbb{H}}_{\mathbb{C}} = 2 \\
e^{\mathbb{H}}_{\mathbb{C}} c = e^{\mathbb{H}}_{\mathbb{C}} & r^{\mathbb{C}}_{\mathbb{R}} c = r^{\mathbb{C}}_{\mathbb{R}} & r^{\mathbb{H}}_{\mathbb{C}} e^{\mathbb{H}}_{\mathbb{C}} = 1 + c
\end{array}$$

where $2E = E \oplus E$, $(1 + c)E = E \oplus \overline{E}$, etc.

In the following discussion, $d = s + t$ is the (real) dimension of V. We will let P and S, perhaps with decorations, denote pinor and spinor representations, respectively; although in order to compare them we must view them both as representations of Spin(V).

If d is even, then the volume element $\omega \in C\ell(s,t)_0$ and commutes with $C\ell(s,t)_0$, whence its eigenspaces in the pinor representation will correspond to the spinor representations. By contrast, if d is odd, then $\omega \not\in C\ell(s,t)_0$ and hence $C\ell(s,t) = C\ell(s,t)_0 \oplus C\ell(s,t)_0\omega \cong C\ell(s,t) \otimes_{\mathbb{R}} \mathbb{R}[\omega]$. This means that we will be able to induce a pinor representation of $C\ell(s,t)$ from a spinor representation S of $C\ell(s,t)_0$ essentially by tensoring with $\mathbb{R}[\omega]$: $P = C\ell(s,t) \otimes_{C\ell(s,t)_0} S$. If $s - t = 1, 5 \pmod 8$ then $\omega^2 = -\mathbf{1}$ so that $\mathbb{R}[\omega] \cong \mathbb{C}$, whereas if $s - t = 3, 7 \pmod 8$ then $\omega^2 = \mathbf{1}$ so that $\mathbb{R}[\omega] \cong \mathbb{R} \oplus \mathbb{R}$. We will use these facts freely in what follows.

3.3.1 $s - t = 0 \pmod 8$

Here $P \cong \mathbb{R}^{2^{d/2}}$ and $S_{\pm} \cong \mathbb{R}^{2^{(d-2)/2}}$ as vector spaces. The volume element obeys $\omega^2 = \mathbf{1}$, whence $S_{\pm}$ are the eigenspaces of ω with eigenvalues ± 1 and $P = S_+ \oplus S_-$.

3.3.2 $s - t = 1 \pmod 8$

Here $P \cong \mathbb{C}^{2^{(d-1)/2}}$ and $S \cong \mathbb{R}^{2^{(d-1)/2}}$ as vector spaces. The Clifford algebra $C\ell(s,t) \cong C\ell(s,t)_0 \otimes \mathbb{C}$, whence $P \cong e_{\mathbb{R}}^{\mathbb{C}}(S)$. It follows that $\overline{P} \cong P$ as representations of $\mathrm{Spin}(s,t)$.

3.3.3 $s - t = 2 \pmod 8$

Here $P \cong \mathbb{H}^{2^{(d-2)/2}}$ and $S, \overline{S} \cong \mathbb{C}^{2^{(d-2)/2}}$ as vector spaces. We have that $P \cong e_{\mathbb{C}}^{\mathbb{H}}(S)$, whence $r_{\mathbb{C}}^{\mathbb{H}}(P) \cong S \oplus \overline{S}$, the eigenspace decomposition under ω, which obeys $\omega^2 = -\mathbf{1}$.

3.3.4 $s - t = 3 \pmod 8$

Here $P_\pm \cong \mathbb{H}^{2^{(d-3)/2}}$ and $S \cong \mathbb{H}^{2^{(d-3)/2}}$ as vector spaces. The Clifford algebra $C\ell(s,t) \cong C\ell(s,t)_0 \oplus C\ell(s,t)_0$ and hence $P_\pm \cong S$.

3.3.5 $s - t = 4 \pmod 8$

Here $P \cong \mathbb{H}^{2^{(d-2)/2}}$ and $S_\pm \cong \mathbb{H}^{2^{(d-4)/2}}$ as vector spaces. We have that $P \cong S_+ \oplus S_-$ is the eigenspace decomposition of ω, which obeys $\omega^2 = \mathbf{1}$.

3.3.6 $s - t = 5 \pmod 8$

Here $P, \overline{P} \cong \mathbb{C}^{2^{(d-1)/2}}$ and $S \cong \mathbb{H}^{2^{(d-3)/2}}$ as vector spaces. The Clifford algebra $C\ell(s,t) \cong C\ell(s,t)_0 \otimes_{\mathbb{R}} \mathbb{C}$ and hence $P \cong r_{\mathbb{C}}^{\mathbb{H}}(S)$. It follows that $\overline{P} \cong P$ as representations of $\mathrm{Spin}(s,t)$.

3.3.7 $s - t = 6 \pmod 8$

Here $P \cong \mathbb{R}^{2^{d/2}}$ and $S, \overline{S} \cong \mathbb{C}^{2^{(d-2)/2}}$. Then $P \cong r_{\mathbb{R}}^{\mathbb{C}}(S)$, so that $e_{\mathbb{R}}^{\mathbb{C}}(P) \cong S \oplus \overline{S}$ is the eigenspace decomposition of ω, which obeys $\omega^2 = -\mathbf{1}$, acting on the complexification of P.

3.3.8 $s - t = 7 \pmod 8$

Here $P_\pm \cong \mathbb{R}^{2^{(d-1)/2}}$ and $S \cong \mathbb{R}^{2^{(d-1)/2}}$ as vector spaces. The Clifford algebra $C\ell(s,t) \cong C\ell(s,t)_0 \oplus C\ell(s,t)_0$ and hence $P_\pm \cong S$.

We can summarise and paraphrase these results by saying that in even dimensions the pinor representation (or if $s - t = 6 \pmod 8$, its complexification) decomposes into the direct sum of two equidimensional spinor representations, whereas in odd dimensions, we must distinguish several cases: if $s - t = 3, 7 \pmod 8$ then each of the two pinor representations is isomorphic to the unique spinor representation, whilst if $s - t = 1, 5 \pmod 8$ then the two complex pinor representations are isomorphic either to the complexification of the unique spinor representation, if $s - t = 1 \pmod 8$, or to the restriction of scalars of the unique quaternionic spinor representation, if $s - t = 5 \pmod 8$.

3.4 Inner products for pinors and spinors

The pinor and spinor representations have inner products which are Spin(V) invariant. In fact, the precise statement, which can be found together with a complete discussion of this topic in [Har90], requires us to review the Clifford involutions.

Clifford involutions

There are three natural involutions of the Clifford algebra $C\ell(V)$:

1. the grading automorphism $\alpha \mapsto \tilde{\alpha}$, which extends the orthogonal transformation $v \mapsto -v$ on V, e.g., $\widetilde{u_1 \cdots u_p} = (-1)^p u_1 \cdots u_p$;
2. the **check involution** $\alpha \to \check{\alpha}$, which is the antiautomorphism of $C\ell(V)$ defined by reversing the order of the generators in every monomial, e.g., $(u_1 \dots u_p)\check{} = u_p \cdots u_1$; and
3. the **hat involution** $\alpha \mapsto \hat{\alpha}$, obtained by combining the previous two.

If $\alpha \in C\ell(V)$ comes from $\Lambda^p V$ under the isomorphism $C\ell(V) \cong \Lambda V$, then $\tilde{\alpha}$, $\check{\alpha}$ and $\hat{\alpha}$ will be $\pm\alpha$ according the following signs:

$p \mod 4$	0	1	2	3
$\tilde{}$	+	−	+	−
$\check{}$	+	−	−	+
$\hat{}$	+	+	−	−

Notice that on $C\ell(V)_0$, the hat and check involutions agree. This is called the **canonical involution** of $C\ell(V)_0$.

The following theorem can be found in [Har90, Chapter 13].

Theorem 3.10. *There exists an inner product $\langle -, - \rangle$ on every spinor representation S such that*

$$\langle ax, y\rangle = \langle x, \hat{a}y\rangle \qquad \text{for all } a \in C\ell(V)_0 \text{ and } x, y \in S. \tag{66}$$

There exist inner products $\hat{\varepsilon}$ and $\check{\varepsilon}$ on the pinor representation P (possibly taking the direct sum of the two irreducible pinor representations when appropriate) such that

$$\check{\varepsilon}(ax, y) = \check{\varepsilon}(x, \check{a}y) \qquad \text{and} \qquad \hat{\varepsilon}(ax, y) = \hat{\varepsilon}(x, \hat{a}y). \tag{67}$$

Moreover all seven types of inner products (real symmetric, real symplectic, complex symmetric, complex symplectic, complex hermitian, quaternionic hermitian and quaternionic skewhermitian) appear!

The spinor representations (with these Spin(V)-invariant inner products) are behind most of the isomorphisms between the following low-dimensional Lie groups:

$$\begin{array}{lll} \mathrm{Spin}(2) \cong \mathrm{U}(1) & \mathrm{Spin}(6) \cong \mathrm{SU}(4) & \mathrm{Spin}(5,1)_0 \cong \mathrm{SL}(2,\mathbb{H}) \\ \mathrm{Spin}(3) \cong \mathrm{Sp}(1) & \mathrm{Spin}(2,1)_0 \cong \mathrm{SL}(2,\mathbb{R}) & \mathrm{Spin}(2,2)_0 \cong \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \\ \mathrm{Spin}(4) \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1) & \mathrm{Spin}(3,1)_0 \cong \mathrm{SL}(2,\mathbb{C}) & \mathrm{Spin}(3,2)_0 \cong \mathrm{Sp}(4,\mathbb{R}) \\ \mathrm{Spin}(5) \cong \mathrm{Sp}(2) & \mathrm{Spin}(4,1)_0 \cong \mathrm{Sp}(1,1) & \mathrm{Spin}(4,2)_0 \cong \mathrm{SU}(2,2) \end{array} \tag{68}$$

In particular, notice the sequence $\mathrm{Spin}_0(2,1) \cong \mathrm{SL}(2,\mathbb{R})$, $\mathrm{Spin}_0(3,1) \cong \mathrm{SL}(2,\mathbb{C})$, $\mathrm{Spin}_0(5,1) \cong \mathrm{SL}(2,\mathbb{H})$, which would suggest that $\mathrm{Spin}_0(9,1)$ would be isomorphic to $\mathrm{SL}(2,\mathbb{O})$ if the octonions were associative and such a group could be defined.

Lecture 4: Spin manifolds

Thus, the existence of a spinor structure appears, on physical grounds, to be a reasonable condition to impose on any cosmological model in general relativity.
— Robert Geroch, 1969

In this lecture we will discuss the notion of a spin structure on a finite-dimensional smooth manifold. We start with some basic notions, just in case the intended audience includes people with little background in differential geometry.

4.1 What is a manifold?

We start with a familiar definition from topology.

Definition 4.1. A (topological) n**-dimensional manifold** is a Hausdorff topological space with a countable basis and which is locally homeomorphic to $\mathbb{R}^n$; that is, every point in M has a neighbourhood which is homeomorphic to $\mathbb{R}^n$.

We shall be interested in doing calculus on manifolds, and this requires introducing a differentiable structure. We assume that we know how to do calculus on $\mathbb{R}^n$ and the point of a differentiable structure is to enable us to do calculus on spaces which are locally "like" $\mathbb{R}^n$ in a way that it is as independent as possible on the precise form of the local homeomorphisms. Please note that we will consider only infinitely differentiable (or *smooth*) structures. This is not necessary, but it is certainly sufficient for our purposes.

Definition 4.2. A **smooth structure** on an n-dimensional manifold M is an **atlas** of **coordinate charts** $\{(U_\alpha,\phi_\alpha)\}_{\alpha\in I}$, for I some indexing set, where $\{U_\alpha\}$ is an open cover of M and $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ are homeomorphisms whose **transition functions** on nonempty overlaps $U_{\alpha\beta} := U_\alpha \cap U_\beta$

$$g_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_{\alpha\beta}) \longrightarrow \phi_\alpha(U_{\alpha\beta})$$

are diffeomorphisms between open subsets of $\mathbb{R}^n$; that is, $g_{\alpha\beta}$ are infinitely differentiable with infinitely differentiable inverses. Two smooth structures $\{(U_\alpha,\phi_\alpha)\}$ and $\{(V_\beta,\psi_\beta)\}$ are **equivalent** if their union is also an atlas. A maximal atlas consists of the union of all atlases in one such equivalence class. A topological manifold with a maximal atlas is called a **smooth manifold**.

Remark 4.3. Notice that not every topological manifold admits a smooth structure and that there are topological manifolds admitting more than one inequivalent smooth structures. For example, it is known that $\mathbb{R}^4$ admits an uncountably infinite number of smooth structures, but for us in this course $\mathbb{R}^n$ will always have the standard smooth structure, unless otherwise explicitly stated.

Let M be a smooth manifold. A function $f : M \to \mathbb{R}$ is **smooth** if for each $\alpha \in I$, $f \circ \phi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}$ is smooth as a function of n real variables. Similarly a function $f : M \to \mathbb{R}^p$ is **smooth** if each component function $f_i : M \to \mathbb{R}$, for $i = 1,\dots,p$, is smooth. A map $f : M^n \to N^p$ between smooth manifolds of dimensions n and p, respectively, is **smooth** if for every $m \in M$, $\psi_\beta \circ f : V_\beta \to \mathbb{R}^p$ is smooth for some (and hence all) coordinate charts (V_β,ψ_β) containing the point $f(m)$. Smooth manifolds form the objects of a category whose morphisms are the smooth maps between them. The isomorphisms in that category are the called **diffeomorphisms**: namely, smooth maps $f : M \to N$ with a smooth inverse.

We could say a lot more about calculus on manifolds, but perhaps this suffices for now.

4.2 Fibre bundles

The definition of a spin structure on a smooth manifold is phrased in the language of fibre bundles and we introduce this language in this section.

4.2.1 Basic notions

Let G be a Lie group and let F be a smooth manifold with a smooth G-action: $G \times F \to F$. We will assume that G acts effectively so that if $(g, f) \mapsto f$ for all $f \in F$, then $g = 1$, the identity. It is often convenient to write the action as a map $\rho : G \to \mathrm{Aut}(F)$, where Aut(F) is the automorphism group of the fibre. In the most general case, $\mathrm{Aut}(F) = \mathrm{Diff}(F)$ is the group of diffeomorphisms, but we will be working mostly with vector bundles, for which F is a vector space and $\mathrm{Aut}(F) = \mathrm{GL}(F)$, whence ρ is a representation of G.

Definition 4.4. A **fibre bundle** (with **structure group** G and **typical fibre** F as above) over M is a smooth surjection $\pi : E \to M$ together with a local triviality condition: every $m \in M$ has a neighbourhood U and a diffeomorphism $\phi_U : \pi^{-1}U \longrightarrow U \times F$ such that the following triangle commutes:

$$\begin{tikzcd} \pi^{-1}U \arrow[rr, "\phi_U"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

and such that on nonempty overlaps $U \cap V$

$$\phi_U \circ \phi_V^{-1}\big|_{\{m\}\times F} = \rho(g_{UV}(m))$$

for some the **transition functions** $g_{UV} : U \cap V \to G$. The manifold M is called the **base** of the fibre bundle, whereas E is called the **total space**. For each $m \in M$, the **fibre** $\pi^{-1}m = \{e \in E | \pi(e) = m\}$ over m is a submanifold of E which is diffeomorphic to F.

The **trivial** bundle with typical fibre F is simply the Cartesian product $M \times F \xrightarrow{\mathrm{pr}_1} M$, in which case we can take the ϕ_U to be the restriction to U of the identity diffeomorphism. Fibre bundles are of course locally trivial, but can be *twisted in the large*. The maps ϕ_U in the general case are called **local trivialisations**. Fibre bundles (with structure group G) over a fixed smooth manifold M are the objects of a category, where a morphism between two fibre bundles $\pi : E \to M$ and $\pi' : E' \to M$ over M is a G-equivariant fibre-preserving smooth map $\varphi : E \to E'$ such that the following triangle commutes:

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd} \tag{69}$$

The restriction of having the same structure group can be lifted and we can equally well consider morphisms between fibre bundles with different structure groups where now the fibre-preserving map $\varphi : E \to E'$ intertwines between the G and G′ actions on the fibres. In any case, if φ is a diffeomorphism, then the two bundles are said to be **equivalent**. A fibre bundle is said to be **trivial** if it is equivalent to the trivial bundle $M \times F$.

A smooth map $s : M \to E$ is a **section** of the fibre bundle $\pi : E \to M$ if $\pi \circ s = \mathrm{id}_M$. In other words, it is a smooth assignment to every $m \in M$ of a point $s(m)$ on its fibre. A fibre bundle may admit no sections: principal fibre bundles (see below) do not unless they are trivial, vector bundles (see below) always do. Local triviality means that local sections always exist, they are locally defined maps from $M \to F$. Our notation for sections of E is $C^\infty(M, E)$, unless otherwise stated.

4.2.2 Construction from local data

A fibre bundle gives rise to some local data from where it can then be reconstructed, up to equivalence. Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ be an open cover of M and let $\phi_\alpha : \pi^{-1}U_\alpha \to U_\alpha \times F$ be local trivialisations with transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$, defined by $\rho(g_{\alpha\beta}(m)) = \phi_\alpha \circ \phi_\beta^{-1}\big|_{\{m\}\times F}$ as above. Notice that the $\{g_{\alpha\beta}\}$ satisfy the following conditions:

1. $g_{\alpha\alpha}(m) = 1$ for all $m \in U_\alpha$,

2. $g_{\alpha\beta}(m)g_{\beta\alpha}(m) = 1$ for all $m \in U_{\alpha\beta}$, and
3. the **cocycle condition**

$$g_{\alpha\beta}(m)g_{\beta\gamma}(m)g_{\gamma\alpha}(m) = 1 \qquad \text{for all } m \in U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma. \tag{70}$$

Notice that if we do not demand that α, β and γ be different, then the cocycle condition implies the other two. We will refer to them collectively as the "cocycle conditions". Notice as well that we are using that G acts effectively on F, otherwise the right-hand sides of these equations would not necessarily be the identity in G, but anything in the kernel of the action ρ. Now the local trivialisations ϕ_α glue to define a diffeomorphism

$$E \cong \left(\bigsqcup_{\alpha\in I} U_\alpha \times F \right) \Big/ \sim \qquad \text{where } (m, f) \sim (m, \rho(g_{\alpha\beta}(m)) f), \text{ for all } m \in U_{\alpha\beta} \text{ and } f \in F. \tag{71}$$

From now on we shall drop ρ from the notation and simply write gf for $g \in G$ acting on $f \in F$. Notice that the cocycle conditions above are, respectively, the reflexive, symmetry and transitivity conditions for the equivalence relation $\sim$.

This allows us to construct fibre bundles by gluing local data. Indeed, if we are given an open cover $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ for M and functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$ on overlaps satisfying the cocycle conditions then we get a fibre bundle by defining E by (71) and the surjection $\pi : E \to M$ by the projection $\mathrm{pr}_1 : U_\alpha \times F \to U_\alpha$, which is respected by the equivalence relation. The resulting bundle is a fibre bundle trivialised over $\mathfrak{U}$.

4.2.3 Vector and principal bundles

As mentioned above, a general fibre bundle will have as structure group the diffeomorphism group of the typical fibre, but there are important examples where G is much smaller. For example, if we take F to be a vector space and G to act linearly, then we have a **vector bundle**. Similarly if $F \cong G$ itself and G acts on G by left multiplication, then we have a **principal** G-**bundle**. In this latter case, there is a well-defined *right* action of G on the total space E of the principal bundle: $(m, g) \mapsto (m, gg')$ which is fibre-preserving and clearly G-equivariant, since associativity of the group multiplication says that left and right multiplications commute. Since right multiplication of G on itself is simply transitive, we have that $M \cong E/G$ is the quotient of E by this right action of G. One often sees this as the starting point for a definition of a principal bundle.

We can go back and forth between vector and principal bundles via two natural constructions:

$$\text{vector bundles} \quad \underset{\text{associated bundle}}{\overset{\text{bundle of frames}}{\rightleftarrows}} \quad \text{principal bundles}$$

Given a vector bundle $E \xrightarrow{\pi} M$ with typical fibre a vector space F, let us define a principal bundle $GL(E) \xrightarrow{\Pi} M$ by declaring the fibre $GL(E)_m$ to be the set of frames of the vector space E_m. This is a principal homogeneous space (or *torsor*) of the general linear group in that any two frames are related by a unique invertible linear transformation. This means that as a set $GL(E)_m \cong GL(E_m)$, but the isomorphism is not natural: it depends on choosing a reference frame. Nevertheless, in terms of a local trivialisation $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in I}$ for E, with transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to GL(F)$ we define

$$GL(E) = \left(\bigsqcup_{\alpha\in I} U_\alpha \times GL(F) \right) \Big/ \sim \qquad \text{where } (m, g) \sim (m, g_{\alpha\beta}(m) g), \text{ for all } m \in U_{\alpha\beta} \text{ and } g \in GL(F).$$

This is then a principal GL(F)-bundle called the **frame bundle** of E.

Conversely, given a principal G-bundle $P \xrightarrow{\Pi} M$ and a finite-dimensional representation $\rho : G \to GL(F)$ on a vector space F, we have a right G-action on $P \times F$ given by $(p, f)g = (pg, \rho(g^{-1}) f)$ for $p \in P$, $f \in F$ and $g \in G$. This action is free because G acts freely on P, and hence the quotient $E = (P \times F)/G$ can be given a smooth structure. Since $P/G \cong M$, we see that this is a fibre bundle with typical fibre F. The surjection $\pi : E \to M$ is induced from the surjection $\Pi : P \to M$ which is preserved by the equivalence relation by virtue of $\Pi(pg) = \Pi(p)$. Alternatively, in terms of a local trivialisation $\{U_\alpha, \phi_\alpha\}_{\alpha\in I}$ for P with transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$, we define

$$E = \left(\bigsqcup_{\alpha\in I} U_\alpha \times F\right) \Big/ \sim \qquad \text{with } (m, f) \sim (m, g_{\alpha\beta}(m) f) \text{ for } m \in U_{\alpha\beta} \text{ and } f \in F$$

and $\pi : E \to M$ defined by $\pi[(m, f)] = m$. This vector bundle, often denoted $P \times_G F \to M$ is called an **associated vector bundle** of $P \to M$, associated to the representation ρ.

In summary, the two constructions above relate fibre bundles which are locally trivialisable over the same cover and the corresponding transition functions are simply related. It is largely a matter of choice whether one decides to work with principal bundles and their associated bundles or with vector bundles and their bundles of frames. For the most part we will choose the former.

4.2.4 Equivalence classes of principal bundles

From the above discussion, the emerging picture is one of principal G-bundles defined by data consisting of a trivialising cover $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ and functions $g_{\alpha\beta} : U_{\alpha\beta} \to G$ on double overlaps satisfying the cocycle conditions (70). Different choices of $\mathfrak{U}$ and of cocycles $\{g_{\alpha\beta}\}$ can still give rise to equivalent bundles.

From the definition of the $g_{\alpha\beta}$ in terms of local trivialisations $g_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$ one can see that it is still possible to compose ϕ_α with functions $g_\alpha : U_\alpha \to G$ to give rise to a new trivialisation $\phi'_\alpha = g_\alpha \circ \phi_\alpha$ and hence to new transition functions $g'_{\alpha\beta} = g_\alpha \circ g_{\alpha\beta} \circ g_\beta^{-1}$ which still satisfy the cocycle conditions. We say that two cocycles $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ are equivalent if $g'_{\alpha\beta} = g_\alpha \circ g_{\alpha\beta} \circ g_\beta^{-1}$ for some "cochain" $\{g_\alpha\}$. We will let $H^1(\mathfrak{U}, G)$ denote the set of equivalence classes of cocycles. It classifies the principal G-bundles trivialised on $\mathfrak{U}$ up to equivalence.

Remark 4.5. Those familiar with sheaf cohomology will recognise $H^1(\mathfrak{U}, G)$ as the first Čech cohomology *set* of the sheaf of germs of smooth functions $M \to G$ relative to the open cover $\mathfrak{U}$. For G nonabelian, this will fail to be a group and be only a pointed set, with distinguished element the isomorphism class of the trivial bundle.

Now suppose that we are given two principal G-bundles defined by local data $(\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}, \{g_{\alpha\beta}\})$ and $(\mathfrak{V} = \{V_\alpha\}_{\alpha\in J}, \{g'_{\alpha\beta}\})$. In order to compare them we would like to define the two bundles relative to the same trivialising cover. This is done by passing to a common *refinement* of $\mathfrak{U}$ and $\mathfrak{V}$. More precisely let $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ be an open cover for M. We say that an open cover $\mathfrak{V} = \{V_\beta\}_{\beta\in J}$ **refines** $\mathfrak{U}$ if there is a reindexing map $j : J \to I$ such that for every $\beta \in J$, $V_\beta \subseteq U_{j(\beta)}$. Now any two open covers $\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}$ and $\mathfrak{V} = \{V_\beta\}_{\beta\in J}$ have a common refinement. For example, we can take $\mathfrak{W} = \{U_\alpha \cap V_\beta\}_{(\alpha,\beta)\in I\times J}$. It is clearly again an open cover and it is clear that it refines both $\mathfrak{U}$ and $\mathfrak{W}$: the reindexing functions $I \times J \to I$ and $I \times J \to J$ are the cartesian projections. (This makes the set of open covers into a **directed set**.) We can then restrict the cocycles $\{g_{\alpha\beta}\}$ defined on $\mathfrak{U}$ and $\{g'_{\alpha\beta}\}$ defined on $\mathfrak{V}$ to $\mathfrak{W}$ and in effect consider them as cocycles on $\mathfrak{W}$ where they can be compared as above. So that two principal bundles defined by local data $(\mathfrak{U} = \{U_\alpha\}_{\alpha\in I}, \{g_{\alpha\beta}\})$ and $(\mathfrak{V} = \{V_\alpha\}_{\alpha\in J}, \{g'_{\alpha\beta}\})$ are equivalent if the restriction of their cocycles to some refinement $\mathfrak{W}$ are equivalent, whence they define the same class in $H^1(\mathfrak{W}, G)$. The way to formalise this is to define $H^1(M, G) = \varinjlim_{\mathfrak{U}} H^1(\mathfrak{U}, G)$, the direct limit of the restriction maps $H^1(\mathfrak{U}, G) \to H^1(\mathfrak{V}, G)$ for $\mathfrak{V}$ a refinement of $\mathfrak{U}$. Then we see that two principal G-bundles are equivalent if they define the same class in $H^1(M, G)$, which then becomes the set of equivalence classes of principal G-bundles on M. It is a pointed set with the trivial bundle as distinguished element.

4.3 Fibre bundles on riemannian manifolds

We will now specialise to riemannian manifolds.

4.3.1 Orientability and the orthonormal frame bundle

Tangent bundle

Let M be a smooth manifold and $m \in \mathrm{M}$ a point. Then by a curve through m we mean a smooth function $t \mapsto c(t)$, with $c(0) = m$. Its velocity at m is the derivative with respect to t evaluated at $t = 0$: $c'(0)$. The space of the velocities at m of all curves though m defines the **tangent space** $\mathrm{T}_m\mathrm{M}$ of M at m. It is a vector space. The union $\mathrm{TM} := \bigcup_{m\in \mathrm{M}} \mathrm{T}_m\mathrm{M}$ can be given the structure of a smooth manifold in such a way that the map $\pi : \mathrm{TM} \to \mathrm{M}$ which sends $v \in \mathrm{T}_m\mathrm{M}$ to m is a surjection making it into a vector bundle over M. Sections of the tangent bundle are called **vector fields** and the space of vector fields on M is denoted $\mathscr{X}(\mathrm{M})$.

A **riemannian manifold** (M, g) is a manifold M together with a **metric** g, which is a smoothly varying family of nondegenerate symmetric bilinear forms on the tangent spaces of M. Notice that we do not demand that g be positive-definite.

Remark 4.6. Although every (paracompact) smooth manifold admits a positive-definite metric, the existence of indefinite metrics often imposes topological restrictions on M. For example, if M is compact (and orientable?) then it admits a lorentzian metric (i.e., one of signature $(1, n-1)$ or $(n-1, 1)$ for $n > 1$) if and only if its Euler characteristic vanishes – a result due to Geroch.

The tangent bundle of a smooth n-dimensional manifold has structure group $\mathrm{GL}(n,\mathbb{R})$, but for a riemannian manifold, the existence of orthonormal frames implies that it is equivalent to a vector bundle with structure group $\mathrm{O}(s, t)$ if the metric has signature (s, t). If $\mathfrak{U} = \{\mathrm{U}_\alpha\}_{\alpha\in \mathrm{I}}$ is a trivialising cover for the tangent bundle, we let $g_{\alpha\beta} : \mathrm{U}_{\alpha\beta} \to \mathrm{O}(s, t)$ be the transition functions for the bundle $\mathrm{O(M)} \to \mathrm{M}$ of orthonormal frames.

Example 4.7. Let $\mathrm{S}^n \subset \mathbb{R}^{n+1}$ be the unit sphere. For $x \in \mathrm{S}^n$, the tangent space $\mathrm{T}_x\mathrm{S}^n$ is given by those vectors in $\mathbb{R}^{n+1}$ which are perpendicular to x. The orthonormal frame bundle is $\mathrm{O}(n+1) \to \mathrm{S}^n$. Indeed, given $x \in \mathrm{S}^n$ and an orthonormal frame $e_1,\dots,e_n$ for $\mathrm{T}_x\mathrm{S}^n$, the $(n+1)\times(n+1)$-matrix whose first n columns are given by the e_i and whose last column is given by x is orthogonal. Conversely, given $a \in \mathrm{O}(n+1)$, the map $\pi : \mathrm{O}(n+1) \to \mathrm{S}^n$ defined by setting $\pi(a)$ to be the last column of a is such that the fibre at $\pi(a)$ is the set of orthonormal frames for the perpendicular subspace to $\pi(a)$ in $\mathbb{R}^{n+1}$.

A riemannian manifold (M, g) is **oriented** if we can restrict consistently to oriented orthonormal frames or, in other words, whether we can reduce the structure group of TM from $\mathrm{O}(s, t)$ to $\mathrm{SO}(s, t)$. Concretely, this means being able to choose transition functions for the orthonormal frame bundle which lie in $\mathrm{SO}(s, t)$, perhaps relative to a refinement of the trivialising cover. So given $\{g_{\alpha\beta}\}$ taking values in $\mathrm{O}(s, t)$ we ask whether we can find $\{g'_{\alpha\beta}\}$ taking values in $\mathrm{SO}(s, t)$. Let $f_{\alpha\beta}(m) = \det g_{\alpha\beta}(m)$ for $m \in \mathrm{U}_{\alpha\beta}$. Since an orthogonal matrix, independently of the signature, has determinant ± 1, the $f_{\alpha\beta}(m)$ take values in the group $\{\pm 1\}$ of order 2. The cocycle condition for $\{g_{\alpha\beta}\}$ imply the cocycle condition for $\{f_{\alpha\beta}\}$, whence this defines a principal fibre bundle with structure group $\mathbb{Z}_2$. Orientability of M is equivalent to the triviality of this bundle. Indeed, if (and only if) $f_{\alpha\beta}(m) = f_\alpha(m) f_\beta(m)$ for some $\mathbb{Z}_2$-valued "cochain" $f_\alpha : \mathrm{U}_\alpha \to \mathbb{Z}_2$, then we can define $g'_{\alpha\beta}(m) = g_\alpha(m) g_{\alpha\beta}(m) g_\beta^{-1}(m)$ for some $g_\alpha : \mathrm{U}_\alpha \to \mathrm{O}(s, t)$ with $\det g_\alpha(m) = f_\alpha(m)$, whence $g'_{\alpha\beta} : \mathrm{U}_{\alpha\beta} \to \mathrm{SO}(s, t)$ and still satisfies the cocycle conditions. The cohomology class defined by $\{f_{\alpha\beta}\}$ in $\mathrm{H}^1(\mathrm{M}, \mathbb{Z}_2)$ which measures the failure of M to being orientable is called the **first Stiefel–Whitney class** of M. Its vanishing is tantamount to orientability. Since $\mathrm{H}^1(\mathrm{M}, \mathbb{Z}_2) \cong \mathrm{Hom}(\pi_1(\mathrm{M}), \mathbb{Z}_2)$, if M is simply connected then it is automatically orientable. Even if M is not orientable, there is a double cover, namely the total space of the principal $\mathbb{Z}_2$-bundle defined by the class of $\{f_{\alpha\beta}\}$ in $\mathrm{H}^1(\mathrm{M}, \mathbb{Z}_2)$, which is oriented and locally isometric to (M, g).

Remark 4.8. Readers familiar with Čech cohomology will recognise the obstruction of orientability as image of the class in $\mathrm{H}^1(\mathrm{M}, \mathrm{O}(s, t))$ corresponding to the orthonormal frame bundle under the last map in the long exact cohomology sequence

$$\mathrm{H}^1(\mathrm{M}, \mathrm{SO}(s, t)) \longrightarrow \mathrm{H}^1(\mathrm{M}, \mathrm{O}(s, t)) \longrightarrow \mathrm{H}^1(\mathrm{M}, \mathbb{Z}_2)$$

coming from the exact sheaf sequence which is induced from the exact sequence of groups

$$1 \longrightarrow \mathrm{SO}(s,t) \longrightarrow \mathrm{O}(s,t) \xrightarrow{\det} \mathbb{Z}_2 \longrightarrow 1$$

Notice that since $\mathrm{O}(s,t)$ and $\mathrm{SO}(s,t)$ are nonabelian groups, there are no $\mathrm{H}^{p>1}$, whence the exact cohomology sequence ends there.

Example 4.9. For $n \geq 2$, the sphere S^n is simply connected, whence it is orientable. In fact, the oriented orthonormal frame bundle is $\mathrm{SO}(n+1) \to \mathrm{S}^n$, with the map given again by the last column of the matrix.

4.3.2 The Clifford bundle and the obstruction to defining a pinor bundle

Any functorial construction on vector spaces — e.g., $\oplus$, $\otimes$, Hom,... — gives rise to a similar construction on vector bundles, and in particular any such construction on representations of G gives rise to similar constructions on associated vector bundles to any principal G-bundle. On a riemannian manifold (M, g) each tangent space becomes a quadratic vector space, relative to the quadratic form induced from the inner product defined by the metric. Hence one should expect that any functorial construction on quadratic vector spaces should globalise to a similar construction on a riemannian manifold. One such construction is the Clifford algebra, which gives rise to a **Clifford bundle** $\mathrm{C}\ell(\mathrm{TM})$. As a vector bundle, $\mathrm{C}\ell(\mathrm{TM}) \cong \Lambda\mathrm{TM}$, but $\mathrm{C}\ell(\mathrm{TM})$ is actually a bundle of Clifford algebras. Alternatively we can define it from a local trivialisation of the orthonormal frame bundle $\mathrm{O}(\mathrm{M})$:

$$\mathrm{C}\ell(\mathrm{TM}) = \left(\bigsqcup_{\alpha\in\mathrm{I}} \mathrm{U}_\alpha \times \mathrm{C}\ell(s,t)\right)\Big/ \sim \qquad \text{with } (m,c) \sim (m, \mathrm{C}\ell(g_{\alpha\beta}(m))c) \text{ for } m \in \mathrm{U}_{\alpha\beta} \text{ and } c \in \mathrm{C}\ell(s,t),$$

where $\mathrm{C}\ell(g_{\alpha\beta}(m))$ is the Clifford algebra automorphism derived functorially from the orthogonal transformation $g_{\alpha\beta}(m)$. Since $\mathrm{C}\ell(g_{\alpha\beta}(m))$ are automorphisms of the Clifford algebra, the Clifford product on $\mathrm{C}\ell(\mathrm{TM})$ is well-defined.

A natural question, given the existence of the Clifford bundle, is whether there is a vector bundle associated to the pinor representation of the Clifford algebra. If $\mathrm{P}(s,t)$ is a pinor representation of $\mathrm{C}\ell(s,t)$ one could try to build such a bundle from local data as follows

$$\mathrm{P} \stackrel{?}{=} \left(\bigsqcup_{\alpha\in\mathrm{I}} \mathrm{U}_\alpha \times \mathrm{P}(s,t)\right)\Big/ \sim \qquad \text{with } (m,p) \sim (m, g_{\alpha\beta}(m)p) \text{ for } m \in \mathrm{U}_{\alpha\beta} \text{ and } p \in \mathrm{C}\ell(s,t),$$

except that $\mathrm{O}(s,t)$ does not act on $\mathrm{P}(s,t)$ and hence we don't know what $g_{\alpha\beta}(m)p$ is.

Since $\mathrm{Pin}(s,t)$ does act on $\mathrm{P}(s,t)$, we could try to define

$$\mathrm{P} \stackrel{?}{=} \left(\bigsqcup_{\alpha\in\mathrm{I}} \mathrm{U}_\alpha \times \mathrm{P}(s,t)\right)\Big/ \sim \qquad \text{with } (m,p) \sim (m, \widetilde{g_{\alpha\beta}(m)}p) \text{ for } m \in \mathrm{U}_{\alpha\beta} \text{ and } p \in \mathrm{C}\ell(s,t),$$

where $\widetilde{g_{\alpha\beta}(m)} \in \mathrm{Pin}(s,t)$ is a lift of $g_{\alpha\beta}(m) \in \mathrm{O}(s,t)$. In other words, $\widetilde{\mathrm{Ad}}_{\widetilde{g_{\alpha\beta}(m)}} = g_{\alpha\beta}(m)$, where $\widetilde{\mathrm{Ad}} : \mathrm{Pin}(s,t) \to \mathrm{O}(s,t)$ is the surjection in Proposition 3.3. For the above definition to make sense, $\sim$ must be an equivalence relation and this is tantamount to the cocycle condition for $\widetilde{g_{\alpha\beta}(m)} : \mathrm{U}_{\alpha\beta} \to \mathrm{Pin}(s,t)$:

$$\widetilde{g_{\alpha\beta}(m)}\,\widetilde{g_{\beta\gamma}(m)}\,\widetilde{g_{\gamma\alpha}(m)} = 1 \qquad \text{for all } m \in \mathrm{U}_{\alpha\beta\gamma}.$$

Applying $\widetilde{\mathrm{Ad}}$ to the cocycle conditions, we obtain the cocycle conditions for the $g_{\alpha\beta}(m) : \mathrm{U}_{\alpha\beta} \to \mathrm{O}(s,t)$, which are satisfied, hence

$$f_{\alpha\beta\gamma}(m) := \widetilde{g_{\alpha\beta}(m)}\,\widetilde{g_{\beta\gamma}(m)}\,\widetilde{g_{\gamma\alpha}(m)} \in \ker\widetilde{\mathrm{Ad}} = \mathbb{Z}_2$$

and hence defines maps $f_{\alpha\beta\gamma} : \mathrm{U}_{\alpha\beta\gamma} \to \mathbb{Z}_2$. Moreover $f_{\alpha\beta\gamma}$ is itself a cocycle, in that in quadruple overlaps

$$f_{\beta\gamma}(m)\, f_{\alpha\beta\delta}(m)\, f_{\alpha\gamma\delta}(m)\, f_{\beta\gamma\delta}(m) = 1 \qquad \text{for all } m \in \mathrm{U}_{\alpha\beta\gamma\delta}.$$

Since $\widetilde{\mathrm{Ad}}$ has nontrivial kernel, the lift $\widetilde{g_{\alpha\beta}}(m)$ is not unique and any other lift is related to this by some cochain $f_{\alpha\beta} : \mathrm{U}_{\alpha\beta} \to \mathbb{Z}_2$. This changes the cocycle $f_{\alpha\beta\gamma}$ by a coboundary

$$f_{\alpha\beta\gamma}(m) \mapsto f'_{\alpha\beta\gamma}(m) := f_{\alpha\beta\gamma}(m) f_{\alpha\beta}(m) f_{\beta\gamma}(m) f_{\alpha\gamma}(m) \; .$$

In particular, $f'_{\alpha\beta\gamma}$ still satisfies the cocycle condition on quadruple overlaps and its class in $\mathrm{H}^2(\mathfrak{U}, \mathbb{Z}_2)$, and hence in $\mathrm{H}^2(\mathrm{M}, \mathbb{Z}_2)$, is unchanged. If (and only if) this class vanishes, will we be able to lift the $g_{\alpha\beta}$ to $\mathrm{Pin}(s,t)$ in such a way that the cocycle conditions are satisfied. Indeed, the class of $f_{\alpha\beta\gamma}$ in $\mathrm{H}^2(\mathrm{M}, \mathbb{Z}_2)$ vanishes if on some "good" cover $\mathfrak{U}$, $f_{\alpha\beta\gamma}(m) = f_{\alpha\beta}(m) f_{\beta\gamma}(m) f_{\alpha\gamma}(m)$ for some $f_{\alpha\beta} : \mathrm{U}_{\alpha\beta} \to \mathbb{Z}_2$. This being the case, then $g'_{\alpha\beta}(m) = \widetilde{g_{\alpha\beta}(m)} f_{\alpha\beta}(m)$ is our desired $\mathrm{Pin}(s,t)$-valued cocycle. The class in $\mathrm{H}^2(\mathrm{M}, \mathbb{Z}_2)$ defined by the $f_{\alpha\beta\gamma}$ is essentially the **second Stiefel–Whitney class** of M, and the pinor bundle can be defined if and only if this class vanishes.

We can view the same class appearing in the more traditional approach to defining a spin structure, to which we now turn.

4.3.3 Spin structures

Let (M, g) be an orientable riemannian manifold of signature (s,t) and let $\mathrm{SO}(\mathrm{M}) \to \mathrm{M}$ denote the bundle of oriented orthonormal frames.

Definition 4.10. A **spin structure** on (M, g) is a principal $\mathrm{Spin}(s,t)$-bundle $\mathrm{Spin}(\mathrm{M}) \to \mathrm{M}$ together with a bundle morphism

$$\begin{tikzcd} \mathrm{Spin}(\mathrm{M}) \arrow[rr, "\varphi"] \arrow[dr] & & \mathrm{SO}(\mathrm{M}) \arrow[dl] \\ & \mathrm{M} & \end{tikzcd}$$

which restricts fibrewise to the covering homomorphism $\widetilde{\mathrm{Ad}} : \mathrm{Spin}(s,t) \to \mathrm{SO}(s,t)$ of Proposition 3.5.

Spin structures need not exist and even if they do they need not be unique. To understand the obstruction let us try to build a spin bundle starting with a trivialisation $(\mathfrak{U}, \{g_{\alpha\beta}\})$ of $\mathrm{SO}(\mathrm{M})$. We choose $\widetilde{g_{\alpha\beta}(m)} \in \mathrm{Spin}(s,t)$ such that under $\widetilde{\mathrm{Ad}} : \mathrm{Spin}(s,t) \to \mathrm{SO}(s,t)$, $\widetilde{g_{\alpha\beta}(m)} \mapsto g_{\alpha\beta}(m)$. This choice is not unique, of course: any other choice $\widetilde{g'_{\alpha\beta}(m)}$ is related to $\widetilde{g_{\alpha\beta}(m)}$ by multiplication with some $f_{\alpha\beta}(m) \in \ker \widetilde{\mathrm{Ad}} = \mathbb{Z}_2$: $\widetilde{g'_{\alpha\beta}(m)} = \widetilde{g_{\alpha\beta}(m)} f_{\alpha\beta}(m)$. We would build the spin bundle $\mathrm{Spin}(\mathrm{M})$ as usual by

$$\mathrm{Spin}(\mathrm{M}) \stackrel{?}{=} \left(\bigsqcup_{\alpha \in \mathrm{I}} \mathrm{U}_\alpha \times \mathrm{Spin}(s,t) \right) \Big/ \sim \qquad \text{with } (m,s) \sim (m, \widetilde{g_{\alpha\beta}(m)} s) \text{ for } m \in \mathrm{U}_{\alpha\beta} \text{ and } s \in \mathrm{Spin}(s,t),$$

except that, for this to make sense, the $\widetilde{g_{\alpha\beta}(m)}$ should satisfy the cocycle condition. As in the case of the construction of the pinor bundle, the obstruction is the class of $f_{\alpha\beta\gamma}(m) = \widetilde{g_{\alpha\beta}(m)}\,\widetilde{g_{\beta\gamma}(m)}\,\widetilde{g_{\gamma\alpha}(m)}$ in $\mathrm{H}^2(\mathrm{M}, \mathbb{Z}_2)$, which is again the second Stiefel–Whitney class of M. If and only if this class vanishes does (M, g) admit a spin structure. Assuming the class vanishes, then one can ask whether the spin structure is unique. Spin structures are in bijective correspondence with the inequivalent lifts $\widetilde{g_{\alpha\beta}}$ of $g_{\alpha\beta}$. As mentioned above, any two lifts are related by multiplication by $f_{\alpha\beta} : \mathrm{U}_{\alpha\beta} \in \mathbb{Z}_2$. The cocycle conditions of the two lifts implies the cocycle condition of $f_{\alpha\beta}$, whence it defines a class in $\mathrm{H}^1(\mathrm{M}, \mathbb{Z}_2)$. If (and only if) this class is trivial, so that $f_{\alpha\beta}(m) = f_\alpha(m) f_\beta(m)$ for some $f_\alpha : \mathrm{U}_\alpha \to \mathbb{Z}_2$, do the two lifts yield equivalent spin bundles. In summary, spin structures are classified by $\mathrm{H}^1(\mathrm{M}, \mathbb{Z}_2) \cong \mathrm{Hom}(\pi_1(\mathrm{M}), \mathbb{Z}_2)$, whence it usually comes down to assigning signs to noncontractible loops consistently.

Remark 4.11. Readers familiar with Čech cohomology will recognise the obstruction of the existence of a spin structure as the image of the class of $\mathrm{SO}(\mathrm{M})$ in $\mathrm{H}^1(\mathrm{M}, \mathrm{SO}(s,t))$ under the connecting map in the long exact cohomology sequence

$$\mathrm{H}^1(\mathrm{M}, \mathbb{Z}_2) \longrightarrow \mathrm{H}^1(\mathrm{M}, \mathrm{Spin}(s,t)) \longrightarrow \mathrm{H}^1(\mathrm{M}, \mathrm{SO}(s,t)) \longrightarrow \mathrm{H}^2(\mathrm{M}, \mathbb{Z}_2)$$

coming from the exact sheaf sequence which is induced from the exact sequence of groups

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(s,t) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{SO}(s,t) \longrightarrow 1$$

Again notice that since $\mathrm{SO}(s,t)$ and $\mathrm{Spin}(s,t)$ are (in general) nonabelian, there are no $\mathrm{H}^{p>1}$, whence the exact cohomology sequence ends there. Indeed a principal $\mathrm{SO}(s,t)$-bundle admits a Spin lift if and only its image in $\mathrm{H}^2(\mathrm{M},\mathbb{Z}_2)$ under the connecting homomorphism vanishes and the "difference" of any two lifts lives in $\mathrm{H}^1(\mathrm{M},\mathbb{Z}_2)$.

Example 4.12. For $n \geq 2$, the sphere S^n admits a unique spin structure, and indeed $\mathrm{Spin}(\mathrm{S}^n) = \mathrm{Spin}(n+1)$ and the bundle morphism $\mathrm{Spin}(\mathrm{S}^n) \to \mathrm{SO}(\mathrm{S}^n)$ is the covering homomorphism $\mathrm{Spin}(n+1) \to \mathrm{SO}(n+1)$.

Example 4.13. The circle S^1 has two inequivalent spin structures which, in some quarters at least, go by the names of *Ramond* and *Neveu–Schwarz*. (This is *not* a joke.)

Example 4.14. A compact Riemann surface Σ of genus g admits 2^{2g} inequivalent spin structures. The second Stiefel–Whitney class vanishes because if it the reduction $\mod 2$ of the Euler class and the Euler characteristic is even ($= 2-2g$). The inequivalent spin structures are classified by homomorphisms $\mathrm{Hom}(\pi_1(\Sigma),\mathbb{Z}_2)$. Now the fundamental group of Σ is generated by $2g$ elements $\mathrm{A}_1,\dots,\mathrm{A}_g,\mathrm{B}_1,\dots,\mathrm{B}_g$ subject to the relation $[\mathrm{A}_1,\mathrm{B}_1][\mathrm{A}_2,\mathrm{B}_2]\cdots[\mathrm{A}_g,\mathrm{B}_g] = 1$, where $[\mathrm{A},\mathrm{B}] = \mathrm{ABA}^{-1}\mathrm{B}^{-1}$ is the (group-theoretical) commutator of A,B. Every homomorphism is determined by what it does on generators, subject to the relation being satisfied. Clearly, though, since $\mathbb{Z}_2$ is abelian, any homomorphism from the free group generated by the A_i and the B_i automatically preserves the relation. Thus every spin structure is specified by associating a sign to every generator. For the case of genus 1, there are four spin structures which, in some quarters at least, are called *Neveu–Schwarz/Neveu–Schwarz*, *Neveu–Schwarz/Ramond*, *Ramond/Neveu–Schwarz* and *Ramond/Ramond*. (This is not a joke either and moreover illustrates the multiplicative nature of the spin structures.)

Given a spin structure $\mathrm{Spin}(\mathrm{M}) \to \mathrm{M}$ we can now construct **spinor bundles** as associated vector bundles. Let $\mathrm{S}(s,t)$ denote a spinor representation of $\mathrm{Spin}(s,t)$ and define $\mathrm{S}(\mathrm{M}) \to \mathrm{M}$ to be the vector bundle with total space

$$\mathrm{S}(\mathrm{M}) = \left(\mathrm{Spin}(\mathrm{M}) \times \mathrm{S}(s,t)\right) \Big/ \mathrm{Spin}(s,t)\ .$$

Depending on signature we might also have **half-spinor bundles** $\mathrm{S}_\pm(\mathrm{M})$ associated to the half-spinor representations $\mathrm{S}(s,t)_\pm$.

FIXME: I am not very happy with this lecture. I will eventually update this to include a small discussion of Čech cohomology with coefficients in a sheaf, to allow me at the very least to use the language freely.

Lecture 5: Connections on principal and vector bundles

The beauty and profundity of the geometry of fibre bundles were to a large extent brought forth by the (early) work of Chern. I must admit, however, that the appreciation of this beauty came to physicists only in recent years.
— CN Yang, 1979

The aim of this lecture is the construction of a connection on the spin bundle and hence on the associated spinor bundles, but first we will discuss the rudiments of the theory of Ehresmann and Koszul connections on principal and vector bundles, respectively. This is, of course, the language of gauge theory and I will borrow freely from my own preliminary lecture notes on this subject.

5.1 Connections on principal bundles

The push-forward and the pull-back

Let $f : M \to N$ be a smooth map between manifolds. The **push-forward**

$$Tf : TM \to TN$$

is the collection of fibre-wise linear maps $f_* : T_mM \to T_{f(m)}N$ defined as follows. Let $v \in T_mM$ be represented as the velocity of a curve $t \mapsto \gamma(t)$ through m; that is, $\gamma(0) = m$ and $\gamma'(0) = v$. Then $f_*(v) \in T_{f(m)}N$ is the velocity at $f(m)$ of the curve $t \mapsto f(\gamma(t))$; that is, $f_*\gamma'(0) = (f\circ\gamma)'(0)$. If $g : N \to Q$ is another smooth map between manifolds, then so is their composition $g \circ f : M \to Q$. The chain rule for derivatives says that $T(g \circ f) = Tg \circ Tf$. Since the push-forward of the identity diffeomorphism 1_M is the identity diffeomorphism 1_{TM}, we see that T is indeed a functor from the category of smooth manifolds and smooth maps to itself.
Dual to the tangent bundle TM is the cotangent bundle T^*M, where $T^*_mM = \mathrm{Hom}(T_mM, \mathbb{R})$. Its sections are called **one-forms** and the space of one-forms on M is denoted $\Omega^1(M)$. The dual of the push-forward is the **pull-back** $f^* : T^*N \to T^*M$, defined for a one-form α by $(f^*\alpha)(v) = \alpha(f_*v)$. Notice that $f^* : T^*_{f(m)}N \to T^*_mM$. It is also functorial, but now reversing the order $(g\circ f)^* = f^* \circ g^*$. (It's a contravariant functor.) Unlike the case of the push-forward, the pull-back defines a map on sections also denoted $f^* : \Omega^1(N) \to \Omega^1(M)$. We also use the notation $\Omega^k(M)$ to denote the sections of the k-th exterior power : Λ^kT^*M of the cotangent bundle. If $k = 0$, $\Omega^0(M) = C^\infty(M)$.

Let $\pi : P \to M$ be a principal G-bundle and let $m \in M$ and $p \in \pi^{-1}(m)$. The **vertical** subspace $V_p \subset T_pP$ consists of those vectors tangent to the fibre at p; in other words, $V_p = \ker \pi_* : T_pP \to T_mM$. A vector field $v \in \mathscr{X}(P)$ is **vertical** if $v(p) \in V_p$ for all p. The Lie bracket of two vertical vector fields is again vertical. The vertical subspaces define a G-invariant distribution (in the sense of Frobenius) $V \subset TP$: indeed, since $\pi \circ R_g = \pi$, we have that $\pi_* \circ (R_g)_*\pi_*$, whence $(R_g)_*V_p = V_{pg}$.

We can understand the vertical space also as the image of the Lie algebra $\mathfrak{g}$ of G under the G-action. If we fix $p \in P$, then the action gives a map $G \to P$ defined by $g \mapsto pg$, whose push-forward at the identity defines a map $\sigma_p : \mathfrak{g} \to T_pP$; explicitly,

$$\sigma_p(X) = \frac{d}{dt}\big(p\exp(tX)\big)\Big|_{t=0} .$$

Since $\pi(p\exp(tX)) = \pi(p)$, it follows that $\sigma_p(X) \in V_p$. Since the action of G is free, the map in one-to-one and hence counting dimension we see that $\sigma_p : \mathfrak{g} \to V_p$ is an isomorphism.

Lemma 5.1.

$$(R_g)_*\sigma_p(X) = \sigma_{pg}\left(\mathrm{Ad}_{g^{-1}}X\right) .$$

Proof. By definition, at $p \in \mathrm{P}$, we have

$$\begin{aligned}(\mathrm{R}_g)_* \sigma_p(\mathrm{X}) &= \frac{d}{dt}\mathrm{R}_g\left(p\exp(t\mathrm{X})\right)\Big|_{t=0}\\ &= \frac{d}{dt}\left(p\exp(t\mathrm{X})g\right)\Big|_{t=0}\\ &= \frac{d}{dt}\left(pgg^{-1}\exp(t\mathrm{X})g\right)\Big|_{t=0}\\ &= \frac{d}{dt}\left(pg\exp(t\mathrm{Ad}_{g^{-1}}\mathrm{X})\right)\Big|_{t=0}\\ &= \sigma_{pg}\left(\mathrm{Ad}_{g^{-1}}\mathrm{X}\right).\end{aligned}$$

□

In the absence of any extra structure, there is no natural complement to V_p in $\mathrm{T}_p\mathrm{P}$. This is in a sense what a connection provides.

5.1.1 Connections as horizontal distributions

A **connection** (in the sense of Ehresmann) on P is a smooth choice of **horizontal** subspaces $\mathrm{H}_p \subset \mathrm{T}_p\mathrm{P}$ complementary to V_p:

$$\mathrm{T}_p\mathrm{P} = \mathrm{V}_p \oplus \mathrm{H}_p$$

and such that $(\mathrm{R}_g)_*\mathrm{H}_p = \mathrm{H}_{pg}$. In other words, a connection is a G-invariant distribution $\mathrm{H} \subset \mathrm{TP}$ complementary to V.

Example 5.2. A G-invariant riemannian metric on P gives rise to a connection, simply by defining $\mathrm{H}_p = \mathrm{V}_p^\perp$. This simple observation underlies the Kałuża–Klein programme relating gravity on P to gauge theory on M. It also underlies many geometric constructions, since it is often the case that 'nice' metrics will give rise to 'nice' connections and viceversa.

5.1.2 The connection one-form

The horizontal subspace $\mathrm{H}_p \subset \mathrm{T}_p\mathrm{P}$, being a linear subspace, is cut out by $k = \dim \mathrm{G}$ linear equations $\mathrm{T}_p\mathrm{P} \to \mathbb{R}$. In other words, H_p is the kernel of k one-forms at p, the components of a one-form ω at p with values in a k-dimensional vector space. There is a natural such vector space, namely the Lie algebra $\mathfrak{g}$ of G, and since ω annihilates horizontal vectors it is defined by what it does to the vertical vectors, and we do have a natural map $\mathrm{V}_p \to \mathfrak{g}$ given by the inverse of σ_p. This prompts the following definition.

The **connection one-form** of a connection $\mathrm{H} \subset \mathrm{TP}$ is the $\mathfrak{g}$-valued one-form $\omega \in \Omega^1(\mathrm{P};\mathfrak{g})$ defined by

$$\omega(v) = \begin{cases}\mathrm{X} & \text{if } v = \sigma(\mathrm{X})\\ 0 & \text{if } v \text{ is horizontal.}\end{cases}$$

Proposition 5.3. *The connection one-form obeys*

$$\mathrm{R}_g^*\omega = \mathrm{Ad}_{g^{-1}} \circ \omega\,.$$

Proof. Let $v \in \mathrm{H}_p$, so that $\omega(v) = 0$. By the G-invariance of H, $(\mathrm{R}_g)_* v \in \mathrm{H}_{pg}$, whence $\mathrm{R}_g^*\omega$ also annihilates v and the identity is trivially satisfied. Now let $v = \sigma_p(\mathrm{X})$ for some $\mathrm{X} \in \mathfrak{g}$. Then, using Lemma 5.1,

$$\mathrm{R}_g^*\omega(\sigma(\mathrm{X})) = \omega\left((\mathrm{R}_g)_*\sigma(\mathrm{X})\right) = \omega\left(\sigma\left(\mathrm{Ad}_{g^{-1}}\mathrm{X}\right)\right) = \mathrm{Ad}_{g^{-1}}\mathrm{X}\,.$$

□

Conversely, given a one-form $\omega \in \Omega^1(\mathrm{P};\mathfrak{g})$ satisfying the identity in Proposition 5.3 and such that $\omega(\sigma(\mathrm{X})) = \mathrm{X}$, the distribution $\mathrm{H} = \ker\omega$ defines a connection on P.

We say that a form on P is **horizontal** if it annihilates the vertical vectors. Notice that if ω and ω' are connection one-forms for two connections H and H′ on P, their difference $\omega - \omega' \in \Omega^1(\mathrm{P};\mathfrak{g})$ is horizontal. We will see later that this means that it defines a section through a bundle on M associated to P.

5.1.3 The horizontal projection

Given a connection $H \subset TP$, we define the **horizontal projection** $h : TP \to TP$ to be the projection onto the horizontal distribution along the vertical distribution. It is a collection of linear maps $h_p : T_pP \to T_pP$, for every $p \in P$, defined by

$$h_p(v) = \begin{cases} v & \text{if } v \in H_p, \text{ and} \\ 0 & \text{if } v \in V_p. \end{cases}$$

In other words, $\operatorname{im} h = H$ and $\ker h = V$. Since both H and V are invariant under the the action of G, the horizontal projection is equivariant:

$$h \circ (R_g)_* = (R_g)_* \circ h\,.$$

We will let $h^* : T^*P \to T^*P$ denote the dual map, whence if, say, $\alpha \in \Omega^1(P)$ is a one-form, $h^*\alpha = \alpha \circ h$. More generally if $\beta \in \Omega^k(P)$, then $(h^*\beta)(v_1, \dots, v_k) = \beta(hv_1, \dots, hv_k)$. However...

> Despite the notation, h^* is *not* the pull-back by a smooth map! In particular, h^* will *not* commute with the exterior derivative d!

5.1.4 The curvature 2-form

Let $\omega \in \Omega^1(P;\mathfrak{g})$ be the connection one-form for a connection $H \subset TP$. The 2-form $\Omega := h^* d\omega \in \Omega^2(P;\mathfrak{g})$ is called the **curvature (2-form)** of the connection. We will derive more explicit formulae for Ω later on, but first let us interpret the curvature geometrically.

By definition,

$$\begin{aligned} \Omega(u,v) &= d\omega(hu,hv) \\ &= (hu)\omega(hv) - (hv)\omega(hu) - \omega([hu,hv]) \\ \text{(since } h^*\omega = 0) \qquad &= -\omega([hu,hv])\,; \end{aligned}$$

whence $\Omega(u,v) = 0$ if and only if $[hu,hv]$ is horizontal. In other words, the curvature of the connection measures the failure of integrability of the horizontal distribution $H \subset TP$.

> **Frobenius integrability**
>
> A distribution $D \subset TP$ is said to be integrable if the Lie bracket of any two sections of D lies again in D. The theorem of Frobenius states that a distribution is integrable if every $p \in P$ lies in a unique submanifold of P whose tangent space at p agrees with the subspace $D_p \subset T_pP$. These submanifolds are said to *foliate* P. As we have just seen, a connection $H \subset TP$ is integrable if and only if its curvature 2-form vanishes.
> In contrast, the vertical distribution $V \subset TP$ is always integrable, since the Lie bracket of two vertical vector fields is again vertical, and Frobenius's theorem guarantees that P is foliated by submanifolds whose tangent spaces are the vertical subspaces. These submanifolds are of course the fibres of $\pi : P \to M$.

The integrability of a distribution has a dual formulation in terms of differential forms. A horizontal distribution $H = \ker\omega$ is integrable if and only if (the components of) ω generate a differential ideal, so that $d\omega = \Theta \wedge \omega$, for some $\Theta \in \Omega^1(P;\operatorname{End}(\mathfrak{g}))$. Since Ω measures the failure of integrability of H, the following formula should not come as a surprise.

Proposition 5.4 (Structure equation)**.**

$$\Omega = d\omega + \tfrac{1}{2}[\omega,\omega]\,,$$

where $[-,-]$ *is the* symmetric *bilinear product consisting of the Lie bracket on* $\mathfrak{g}$ *and the wedge product of one-forms.*

Proof. We need to show that

$$d\omega(hu, hv) = d\omega(u, v) + [\omega(u), \omega(v)] \tag{72}$$

for all vector fields $u, v \in \mathscr{X}(P)$. We can treat this case by case.

- Let u, v be horizontal. In this case there is nothing to show, since $\omega(u) = \omega(v) = 0$ and $hu = u$ and $hv = v$.

- Let u, v be vertical. Without loss of generality we can take $u = \sigma(X)$ and $v = \sigma(Y)$, for some $X, Y \in \mathfrak{g}$. Then equation (72) becomes

$$\begin{aligned} 0 &\stackrel{?}{=} d\omega(\sigma(X), \sigma(Y)) + [\omega(\sigma(X)), \omega(\sigma(Y))] \\ (\omega(\sigma(X)) = X, \text{etc}) \qquad &= \sigma(X)Y - \sigma(Y)X - \omega([\sigma(X), \sigma(Y)]) + [X, Y] \\ &= -\omega([\sigma(X), \sigma(Y)]) + [X, Y] \\ ([\sigma(X), \sigma(Y)] = \sigma([X, Y])) \qquad &= -\omega(\sigma([X, Y])) + [X, Y] \,, \end{aligned}$$

  which is clearly true.

- Finally, let u be horizontal and $v = \sigma(X)$ be vertical, whence equation (72) becomes

$$d\omega(u, \sigma(X)) = 0 \,,$$

  which in turn reduces to

$$\omega([u, \sigma(X)]) = 0 \,.$$

  In other words, we have to show that the Lie bracket of a vertical and a horizontal vector field is again horizontal. But this is simply the infinitesimal version of the G-invariance of H.

□

An immediate consequence of this formula is the

Proposition 5.5 (Bianchi identity)**.**

$$h^* d\Omega = 0 \,.$$

Proof. This is simply a calculation using the structure equation:

$$\begin{aligned} h^* d\Omega &= h^* d\left(d\omega + \tfrac{1}{2}[\omega, \omega]\right) \\ &= h^* \left(\tfrac{1}{2}[d\omega, \omega] - \tfrac{1}{2}[\omega, d\omega]\right) \\ &= h^*[d\omega, \omega] \\ &= [h^* d\omega, h^*\omega] \\ &= 0 \,. \end{aligned}$$

□

5.2 Connections on vector bundles

A connection on a principal bundle allows us to define a covariant derivative (a.k.a. a Koszul connection) on sections of any associated vector bundle. If $E \to M$ is a vector bundle, we let $C^\infty(M, E)$ denote the space of smooth sections. If $s \in C^\infty(M, E)$ and $f \in C^\infty(M)$, then $fs \in C^\infty(M, E)$, where $(fs)(m) = f(m)s(m)$. This makes $C^\infty(M, E)$ into a $C^\infty(M)$-module. In fact, a celebrated theorem of Swann's (based on a theorem of Serre's in algebraic geometry) says that the category of smooth vector bundles on a (compact) manifold M is equivalent to the category of finitely-generated projective $C^\infty(M)$-modules.

5.2.1 Koszul connections

Notation

If $E \to M$ is a vector bundle, we let $\Omega^k(M,E)$ denote the space of sections of the vector bundle $\Lambda^k T^*M \otimes E$. If F is a vector space then $\Omega^k(M,F)$ denotes the F-valued k-forms on M, but they can also be interpreted as an example of the previous notation, where $E = M \times F$ is a trivial bundle.

Definition 5.6. A **Koszul connection** on a vector bundle $\pi : E \to M$ is a map $\nabla : C^\infty(M,E) \to \Omega^1(M,E)$ satisfying the following property:

$$\nabla(fs) = df \otimes s + f\nabla s \qquad \text{for all } f \in C^\infty(M) \text{ and } s \in C^\infty(M,E).$$

In other words, if $\xi \in \mathscr{X}(M)$ is a vector field then $\nabla_\xi : C^\infty(M,E) \to C^\infty(M,E)$ satisfies the following properties:

$$\nabla_{f\xi} s = f\nabla_\xi s \qquad \nabla_{\xi+\chi} s = \nabla_\xi s + \nabla_\chi s \qquad \text{and} \qquad \nabla_\xi(fs) = \xi(f)s + f\nabla_\xi s\,,$$

for all $\xi, \chi \in \mathscr{X}(M)$, $f \in C^\infty(M)$ and $s \in C^\infty(M,E)$.

We will now show how a connection on a principal bundle $P \to M$ defines a Koszul connection on any associated vector bundle $P \times_G F \to M$, but first we need to understand better the relation between forms on P and forms on M.

5.2.2 Basic forms

A k-form $\alpha \in \Omega^k(P)$ is **horizontal** if $h^*\alpha = \alpha$. A horizontal form which in addition is G-invariant is called **basic**. It is a basic fact (no pun intended) that α is basic if and only if $\alpha = \pi^*\bar\alpha$ for some k-form $\bar\alpha$ on M (hence the name). This story extends to forms on P taking values in a vector space F admitting a representation $\varrho : G \to GL(F)$ of G. Let α be such a form. Then α is **horizontal** if $h^*\alpha = \alpha$ and it is **invariant** if for all $g \in G$,

$$R_g^*\alpha = \varrho(g^{-1}) \circ \alpha\,.$$

If α is both horizontal and invariant, it is said to be **basic**. Basic forms are in one-to-one correspondence with forms on M with values in the associated bundle $P \times_G F$. Indeed, let

$$\Omega^k_G(P,F) = \left\{\bar\zeta \in \Omega^k(P,F) \,\middle|\, h^*\bar\zeta = \bar\zeta \quad\text{and}\quad R_g^*\bar\zeta = \varrho(g^{-1}) \circ \bar\zeta\right\} \tag{73}$$

denote the basic forms on P with values in F. Then we have an isomorphism $\Omega^k_G(P,F) \cong \Omega^k(M, P \times_G F)$. The case $k = 0$ is particularly important. This is the an isomorphism between G-equivariant functions $P \to F$ (which are vacuously horizontal) and sections of $P \times_G F$.

It is instructive to prove the general result, though. To this end we need to introduce one more object.

Every principal fibre bundle admits local sections. In fact, a local trivialisation $\mathfrak{U} = \{U_\alpha\}$,

$$\begin{tikzcd} \pi^{-1}U_\alpha \arrow[rr, "\psi_\alpha"] \arrow[dr, "\pi"'] & & U_\alpha \times G \arrow[dl, "\mathrm{pr}_1"] \\ & U_\alpha & \end{tikzcd}$$

defines local sections $s_\alpha : U_\alpha \to \pi^{-1}U_\alpha$ by $\psi_\alpha(s_\alpha(m)) = (m,e)$, where e is the identity in G. Conversely, local sections $s_\alpha : U_\alpha \to \pi^{-1}U_\alpha$ define a trivialisation by $\psi_\alpha(s_\alpha(m)g) = (m,g)$. On overlaps, these sections are related by the transition functions of the bundle. Indeed, if $m \in U_\alpha \cap U_\beta$, then

$$\psi_\alpha(s_\beta(m)) = (\psi_\alpha \circ \psi_\beta^{-1} \circ \psi_\beta)(s_\beta(m)) = (\psi_\alpha \circ \psi_\beta^{-1})(m,e) = (m, g_{\alpha\beta}(m))\,, \tag{74}$$

whence $s_\beta(m) = s_\alpha(m)g_{\alpha\beta}(m)$. Using the local sections we can now prove the following:

Proposition 5.7. *We have an isomorphism of* $C^\infty(M)$*-modules*

$$\Omega^k_G(P,F) \cong \Omega^k(M, P\times_G F) .$$

Proof. We will only give the construction and let the verification to the reader. If $\bar\zeta \in \Omega^k_G(P,F)$, let $\zeta_\alpha = s_\alpha^* \bar\zeta \in \Omega^k(U_\alpha, F)$. Then one shows that on $U_{\alpha\beta}$, $\zeta_\beta(m) = \varrho(g_{\alpha\beta}(m)^{-1})\zeta_\alpha(m)$, whence the $\{\zeta_\alpha\}$ define a section of $\Omega^k(M, P\times_G F)$. Conversely, if $\{\zeta_\alpha \in \Omega^k(U_\alpha, F)\}$ satisfy $\zeta_\beta(m) = \varrho(g_{\alpha\beta}(m)^{-1})\zeta_\alpha(m)$ for $m \in U_{\alpha\beta}$, we define $\bar\zeta_\alpha(p) = \varrho(g_\alpha^{-1}) \circ \pi^*\zeta_\alpha$, where $g_\alpha : \pi^{-1}(m) \to G$ is defined by $\psi_\alpha(p) = (\pi(p), g_\alpha(p))$. The $\bar\zeta_\alpha$ are basic by construction and one simply checks that on $\pi^{-1}U_{\alpha\beta}$, $\bar\zeta_\alpha = \bar\zeta_\beta$. □

5.2.3 The covariant derivative

The exterior derivative $d : \Omega^k(P,F) \to \Omega^{k+1}(P,F)$ obeys $d^2 = 0$ and defines a complex: the F-**valued de Rham complex**. The invariant forms do form a subcomplex, but the basic forms do not, since $d\alpha$ need not be horizontal even if α is. Projecting onto the horizontal forms defines the **exterior covariant derivative**

$$d^\nabla : \Omega^k_G(P,F) \to \Omega^{k+1}_G(P,F) \qquad \text{by} \qquad d^\nabla\alpha = h^* d\alpha .$$

The price we pay is that $(d^\nabla)^2 \neq 0$ in general, so we no longer have a complex. Indeed, the failure of d^∇ defining a complex is again measured by the curvature of the connection.

Let us start by deriving a more explicit formula for the exterior covariant derivative on sections of $P\times_G F$. Every section $\zeta \in \Omega^0(M, P\times_G F)$ defines an equivariant function $\bar\zeta \in \Omega^0_G(P,F)$ obeying $R_g^*\bar\zeta = \varrho(g^{-1})\circ\bar\zeta$ and whose covariant derivative is given by $d^\nabla\bar\zeta = h^* d\bar\zeta$. Applying this to a vector field $u = u_V + hu \in \mathscr{X}(P)$,

$$(d^\nabla\bar\zeta)(u) = d\bar\zeta(hu) = d\bar\zeta(u - u_V) = d\bar\zeta(u) - u_V(\bar\zeta) .$$

The derivative $u_V\bar\zeta$ at a point p only depends on the value of u_V at that point, whence we can take $u_V = \sigma(\omega(u))$, so that

$$u_V\bar\zeta = \sigma(\omega(u))\bar\zeta = \frac{d}{dt}\Big|_{t=0} R^*_{g(t)}\bar\zeta \qquad \text{for } g(t) = \exp(t\omega(u)).$$

By equivariance,

$$u_V\bar\zeta = \frac{d}{dt}\Big|_{t=0} \varrho(g(t)^{-1})\circ\bar\zeta = -\varrho(\omega(u))\circ\bar\zeta ,$$

where we also denote by $\varrho : \mathfrak{g} \to \mathfrak{gl}(F)$ the representation of the Lie algebra. In summary,

$$(d^\nabla\bar\zeta)(u) = d\bar\zeta(u) + \varrho(\omega)(u)\circ\bar\zeta$$

or, abstracting u,

$$d^\nabla\bar\zeta = d\bar\zeta + \varrho(\omega)\circ\bar\zeta . \tag{75}$$

This form is clearly horizontal by construction, and it is also invariant:

$$\begin{aligned}
R_g^* d^\nabla\bar\zeta &= R_g^* h^* d\bar\zeta \\
\text{(since H is invariant)} \quad &= h^* R_g^* d\bar\zeta \\
\text{(since } d \text{ commutes with pull-backs)} \quad &= h^* d R_g^*\bar\zeta \\
\text{(equivariance of } \bar\zeta) \quad &= h^* d\left(\varrho(g^{-1})\circ\bar\zeta\right) \\
&= \varrho(g^{-1})\circ h^* d\bar\zeta \\
&= \varrho(g^{-1})\circ d^\nabla\bar\zeta .
\end{aligned}$$

As a result, it is a basic form and hence comes from a 1-form $\nabla\zeta \in \Omega^1(M, P\times_G F)$. In this way, we have defined a Koszul connection

$$\nabla : C^\infty(M, P\times_G F) \to \Omega^1(M, P\times_G F) .$$

This story extends to k-forms in the obvious way. Let $\alpha \in \Omega^k(M, P\times_G F)$ and represent it by a basic form $\bar\alpha \in \Omega^k_G(P,F)$. Define $d^\nabla\bar\alpha = h^* d\bar\alpha$. Then one can show that

$$d^\nabla\bar\alpha = d\bar\alpha + \varrho(\omega)\wedge\bar\alpha \in \Omega^{k+1}_G(P,F) ,$$

where $\wedge$ denotes both the wedge product of forms and the composition of the components of $\varrho(\omega)$ with $\bar\alpha$, whence it defines an element $d^\nabla\alpha \in \Omega^{k+1}(M, P\times_G F)$. Contrary to the exterior derivative, $(d^\nabla)^2\bar\zeta \neq 0$ in general. Instead, for $\bar\zeta \in \Omega^0_G(P,F)$, we have

$$\begin{aligned} (d^\nabla)^2\bar\zeta &= h^* d h^* d\bar\zeta \\ &= h^* d\left(d\bar\zeta + \varrho(\omega)\circ\bar\zeta\right) \\ &= h^*\left(\varrho(d\omega)\circ\bar\zeta - \varrho(\omega)\wedge d\bar\zeta\right) \\ \text{(since } h^*\omega = 0\text{)} \qquad &= \varrho(h^* d\omega)\circ\bar\zeta \\ &= \varrho(\Omega)\circ\bar\zeta . \end{aligned}$$

More generally, if $\bar\alpha \in \Omega^k_G(P,F)$, we have

$$(d^\nabla)^2\bar\alpha = \varrho(\Omega)\wedge\bar\alpha ,$$

whence the curvature measures the obstruction of the exterior covariant derivative to define a complex.

5.2.4 Gauge fields

We often need to do explicit calculations with objects in the base manifold of a fibre bundle and we need to have an expression for the covariant derivative of, say, a section of $P\times_G F$ explicitly and not just in terms of the G-equivariant functions $P \to F$. This requires the introduction of locally defined 1-forms which go by the name of gauge fields. More precisely, the connection 1-form ω on a principal fibre bundle pulls back to the base via any local section. In particular we can use the local sections s_α associated to a trivialisation to define $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$ by $A_\alpha = s_\alpha^*\omega$. One can show that on overlaps $U_{\alpha\beta}$,

$$A_\alpha(m) = g_{\alpha\beta}(m)A_\beta(m)g_{\alpha\beta}(m)^{-1} - dg_{\alpha\beta}g_{\alpha\beta}^{-1} , \tag{76}$$

in a notation appropriate to matrix groups. Conversely given $A_\alpha \in \Omega^1(U_\alpha, \mathfrak{g})$ subject to equation (76) on overlaps, we can define $\omega_\alpha \in \Omega^1(\pi^{-1}U_\alpha, \mathfrak{g})$ by

$$\omega_\alpha = \mathrm{Ad}_{g_\alpha^{-1}}\circ\pi^* A_\alpha + g_\alpha^{-1} dg_\alpha , \tag{77}$$

where the second term on the right-hand side is the pullback by g_α of the left-invariant Maurer–Cartan 1-form on G, again in a notation appropriate to matrix groups. One checks that on $\pi^{-1}U_{\alpha\beta}$, $\omega_\alpha = \omega_\beta$, whence it does define a global one-form $\omega \in \Omega^1(P, \mathfrak{g})$. One finally verifies that it is a connection one-form.

This means that we have now three ways to think of connections on a principal fibre bundle: as invariant horizontal distributions, as connection one-forms or as gauge fields. Each way has its virtue and it's convenient to understand all three and how they are related.

Back to the covariant derivative, letting $E = P\times_G F$, we define $d^\nabla : \Omega^k(M,E) \to \Omega^{k+1}(M,E)$ by the commutativity of the following diagram:

$$\begin{tikzcd} \Omega^k_G(P,F) \arrow[r, "h^* d"] \arrow[d, leftrightarrow] & \Omega^{k+1}_G(P,F) \arrow[d, leftrightarrow] \\ \Omega^k(M,E) \arrow[r, "d^\nabla"] & \Omega^{k+1}(M,E) \end{tikzcd}$$

For example, if $\{\sigma_\alpha : U_\alpha \to F\}$ defines a section $\sigma \in C^\infty(M,E)$, then on U_α, $d^\nabla\sigma$ is represented by

$$d^\nabla\sigma_\alpha = d\sigma_\alpha + \varrho(A_\alpha)\sigma_\alpha .$$

Then on overlaps, we have

$$d^{\nabla}\sigma_{\alpha} = \varrho(g_{\alpha\beta})\, d^{\nabla}\sigma_{\beta} \;, \tag{78}$$

which earns the derivative d^{∇} the adjective 'covariant'.

Often we write simply $\nabla\sigma$ for $d^{\nabla}\sigma$ when σ is a section. The curvature 2-form R^{∇} associated to ∇ is the section of $\Omega^2(M, \operatorname{End} E)$ given by $R^{\nabla} = d^{\nabla} \circ \nabla$, or explicitly,

$$R^{\nabla}(X,Y)\sigma = \nabla_{[X,Y]}\sigma - \nabla_X\nabla_Y\sigma + \nabla_Y\nabla_X\sigma \;, \tag{79}$$

for all $X, Y \in \mathscr{X}(M)$.

Lecture 6: The spin connection

On the tangent bundle of a riemannian manifold (M, g) there is a privileged connection called the Levi-Civita connection. Thinking of the tangent bundle as an associated vector bundle to the bundle $O(M)$ of orthonormal frames, we will see that this connection is induced from a connection on $O(M)$, which restricts to a connection on $SO(M)$ when (M, g) is orientable and lifts to a connection on any spin bundle $Spin(M)$ if (M, g) is spin. That being the case, it defines a connection on the spinor bundles which is usually called the spin connection.

6.1 The Levi-Civita connection

Let (M, g) be a riemannian manifold. We summarise here the basic definitions and results of the riemannian geometry of (M, g).

Theorem 6.1 (The fundamental theorem of riemannian geometry)**.** *There is a unique connection on the tangent bundle* TM *which is*

1. metric-compatible*:*

$$\nabla_X g = 0 \qquad \textit{equivalently} \qquad X g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)\ ,$$

2. *and* torsion-free*:*

$$\nabla_X Y - \nabla_Y X = [X, Y]\ ,$$

where X, Y, Z *are vector fields on* M *and* $[X, Y]$ *denotes the Lie bracket of vector fields.*

Proof. The proof consists in finding an explicit formula for the connection in terms of the metric. Let $X, Y, Z \in \mathscr{X}(M)$. The metric compatibility condition says that

$$\begin{aligned} Xg(Y,Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ Yg(Z,X) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\ Zg(X,Y) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y)\ , \end{aligned}$$

whereas the vanishing of the torsion allows to rewrite the middle equation as

$$Yg(Z,X) = g(\nabla_Y Z, X) + g(Z, \nabla_X Y) + g(Z, [X,Y])\ .$$

We now compute

$$Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) = 2g(\nabla_X Y, Z) + g(Y, \nabla_X Z - \nabla_Z X) + g(\nabla_Y Z - \nabla_Z Y, X) + g(Z, [X,Y])$$

and use the torsionless condition once again to arrive at the **Koszul formula**

(80) $$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(Y,[X,Z]) - g([Y,Z],X) - g(Z,[X,Y])$$

which determines $\nabla_X Y$ uniquely. □

The connection so defined is called the **Levi-Civita connection**. Its curvature, defined by

(81) $$R(X,Y)Z = \nabla_{[X,Y]} Z - \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z\ ,$$

gives rise to the **Riemann curvature tensor**

$$R(X,Y,Z,W) := g(R(X,Y)Z, W)\ .$$

Proposition 6.2. *The curvature satisfies the following identities*

1. symmetry conditions*:*

$$R(X,Y)Z = -R(Y,X)Z \qquad \textit{and} \qquad R(X,Y,Z,W) = -R(X,Y,W,Z) ,$$

2. algebraic Bianchi identity*:*

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0 ,$$

3. differential Bianchi identity*:*

$$\nabla_X R(Y,Z) + \nabla_Y R(Z,X) + \nabla_Z R(X,Y) = 0 .$$

A tensor satisfying the symmetry conditions and the algebraic Bianchi identity is called an **algebraic curvature tensor**.

If we fix $X,Y \in \mathscr{X}(M)$, the curvature defines a linear map $Z \mapsto R(X,Z)Y$, whose trace is the **Ricci (curvature) tensor** $r(X,Y)$.

Proposition 6.3. *The Ricci tensor is symmetric:* $r(X,Y) = r(Y,X)$.

The trace (relative to the metric g) of the Ricci tensor is called the **scalar curvature** of (M,g) and denoted s.

Definition 6.4. A riemannian manifold (M,g) is said to be **Einstein** if $r(X,Y) = \lambda g(X,Y)$ for some $\lambda \in \mathbb{R}$. Clearly $\lambda = s/n$ where n is the dimension of M. It is said to be **Ricci-flat** if $r = 0$ and **flat** if $R = 0$.

If $h,k \in C^\infty(M, S^2T^*M)$ are two symmetric tensors, their **Kulkarni–Nomizu product** $h \odot k$ is the algebraic curvature tensor defined by

(82) $$(h \odot k)(X,Y,Z,W) = h(X,Z)k(Y,W) + h(Y,W)k(X,Z) - h(X,W)k(Y,Z) - h(Y,Z)k(X,W) ,$$

for all $X,Y,Z,W \in \mathscr{X}(M)$.

Proposition 6.5. *The Riemann curvature tensor can be decomposed as*

$$R = \frac{s}{2n(n-1)} g \odot g + \frac{1}{n-2}(r - \frac{s}{n} g) \odot g + W$$

where W *is the* **Weyl (curvature) tensor**.

The Weyl tensor is the "traceless" part of the Riemann tensor. It is conformally invariant and if it vanishes, (M,g) is said to be *conformally flat.* If (M,g) is Einstein, then the middle term in R is absent. If only the first term is present then (M,g) is said to have *constant sectional curvature.*

6.2 The connection one-forms on O(M), SO(M) and Spin(M)

The Levi-Civita connection of a riemannian manifold induces a connection one-form ω on the orthonormal frame bundle and, if orientable, also on the oriented orthonormal frame bundle. Indeed, let us assume that M is orientable and let $\mathscr{E} : U \subset M \to SO(M)$ be local orthonormal frame, i.e., a local section of SO(M). Then we may pull ω back to a gauge field $\mathscr{E}^*\omega$ on U with values in $\mathfrak{so}(s,t)$, for (M,g) of signature (s,t). We can describe the gauge field explicitly as follows. Let (e_i) denote the elements in the frame $\mathscr{E}$. Being orthonormal, their inner products are given by $g(e_i,e_j) = \varepsilon_i\delta_{ij}$, where $\varepsilon_i = \pm 1$. Then we have

$$\mathscr{E}^*\omega = \tfrac{1}{2}\sum_{i,j} \omega_{ij}\varepsilon_i\varepsilon_j e_i \curlywedge e_j ,$$

where $\omega_{ij} \in \Omega^1(U)$ is defined by

(83) $$\omega_{ij}(X) = g(\nabla_X e_i, e_j)$$

for all $X \in \mathscr{X}(M)$ and $e_i \curlywedge e_j \in \mathfrak{so}(s,t)$ are the skewsymmetric endomorphisms defined by (53). It is convenient in calculations to introduce the **dual frame** $e^i = \varepsilon_i e_i$, where now $g(e_i, e^j) = \delta_{ij}$, and in terms of which

$$\mathscr{E}^*\omega = \tfrac{1}{2}\sum_{i,j}\omega_{ij} e^i \curlywedge e^j \ .$$

If $\mathscr{E}'$ is another local frame $\mathscr{E}' : U' \to SO(M)$, so that on $U \cap U'$, $\mathscr{E}' = \mathscr{E}h$ for some $h : U \cap U' \to SO(s,t)$, then on $U \cap U'$,

$$\mathscr{E}'^*\omega = h\mathscr{E}^*\omega h^{-1} - dhh^{-1} \ ,$$

whence it does indeed give rise to a gauge field.

Now let

$$\begin{tikzcd} \mathrm{Spin}(M) \arrow[rr, "\varphi"] \arrow[dr] & & SO(M) \arrow[dl] \\ & M & \end{tikzcd}$$

denote a spin bundle. The connection 1-form ω on $SO(M)$ pulls back to a connection 1-form $\varphi^*\omega$ on $\mathrm{Spin}(M)$, called the **spin connection**. Now given a local section $\mathscr{E}$ of $SO(M)$, let $\widetilde{\mathscr{E}}$ denote a local section of $\mathrm{Spin}(M)$ such that $\varphi \circ \widetilde{\mathscr{E}} = \mathscr{E}$. Then the gauge field associated to $\varphi^*\omega$ via $\widetilde{\mathscr{E}}$ coincides with the one associated to ω via $\mathscr{E}$:

$$\widetilde{\mathscr{E}}^*\varphi^*\omega = (\varphi \circ \widetilde{\mathscr{E}})^*\omega = \mathscr{E}^*\omega \ . \tag{84}$$

If $\varrho : \mathrm{Spin}(s,t) \to GL(F)$ is any representation, then on sections of the associated vector bundle $\mathrm{Spin}(M) \times_{\mathrm{Spin}(s,t)} F$ we have a covariant derivative

$$d^\nabla = d + \tfrac{1}{2}\sum_{i,j}\omega_{ij}\varrho(e^i \curlywedge e^j) \ , \tag{85}$$

where we also denote by $\varrho : \mathfrak{so}(s,t) \to \mathfrak{gl}(F)$ the representation of the Lie algebra.

We shall be interested primarily in the spinor representations of $\mathrm{Spin}(s,t)$, which are induced by restriction from pinor representations of $C\ell(s,t)$. This means that the associated bundle $\mathrm{Spin}(M) \times_{\mathrm{Spin}(s,t)} F$ is (perhaps a subbundle of) a bundle $C\ell(TM) \times_{C\ell(s,t)} P$ of Clifford modules. In this case, it is convenient to think of the gauge field as taking values in the Clifford algebra. If we let $\rho : \mathfrak{so}(s,t) \to C\ell(s,t)$ denote the embedding defined in (55), then

$$\rho(\mathscr{E}^*\omega) = \tfrac{1}{4}\sum_{i,j}\omega_{ij} e^i e^j \ , \tag{86}$$

where $e^i e^j \in C\ell(s,t)$. The curvature two-form of this connection is given by

$$\rho(\mathscr{E}^*\Omega) = \tfrac{1}{4}\sum_{i,j}\Omega_{ij} e^i e^j \ , \tag{87}$$

where $\Omega_{ij}(X,Y) = g(R(X,Y)e_i, e_j)$ for all $X, Y \in \mathscr{X}(M)$, with $R(X,Y)$ defined by (81).

The Clifford algebra-valued covariant derivative is compatible with Clifford action in the following sense. Suppose that $\theta \in C\ell(TM)$ and ψ is a section of a bundle of Clifford modules associated to $C\ell(TM)$. Then for all vector fields $X \in \mathscr{X}(M)$, we have that

$$\nabla_X(\theta \cdot \psi) = \nabla_X\theta \cdot \psi + \theta \cdot \nabla_X\psi \ , \tag{88}$$

where $\nabla_X\theta$ agrees with the action of the Levi-Civita connection on θ viewed as a section of ΛTM.

6.3 Parallel spinor fields

We can now define the notion of a parallel spinor field as a (nonzero) section of a spinor bundle which is covariantly constant. On a trivialising neighbourhood U of M, where Spin(M) is trivialised by a local section $\widetilde{\mathscr{E}}$ lifing a local orthonormal frame $\mathscr{E}$, a spinor field is given by a function $\psi : \mathrm{U} \to \mathrm{S}(s,t)$ taking values in the spinor representation, which we think of as the restriction to $\mathrm{Spin}(s,t)$ of an irreducible $\mathrm{C}\ell(s,t)$-module. Depending on (s,t), it may very well be the case that the $\mathrm{S}(s,t)$ so defined is not irreducible, in which case $\mathrm{S}(s,t) = \mathrm{S}(s,t)_+ \oplus \mathrm{S}(s,t)_-$ decomposes into two half-spinor irreducible representations of $\mathrm{Spin}(s,t)$. The covariant derivative of ψ is given by

$$d^\nabla \psi = d\psi + \tfrac{1}{4}\sum_{i,j} \omega_{ij} e^i e^j \psi \,, \tag{89}$$

and we say that ψ is **covariantly constant** (or **parallel**) if $d^\nabla \psi = 0$. The fact (78) that d^∇ is covariant means that this equation is well-defined on global section of the spinor bundle.

Differentiating $d^\nabla\psi$ again we obtain an integrability condition for the existence of parallel spinor fields, namely

$$d^\nabla d^\nabla \psi = \tfrac{1}{4}\sum_{i,j} \Omega_{ij} e^i e^j \psi = 0 \,. \tag{90}$$

This equation is equivalent to

$$\mathrm{R}(\mathrm{X},\mathrm{Y})\psi = 0 \,, \tag{91}$$

where $\mathrm{R}(\mathrm{X},\mathrm{Y}) \in \mathrm{C}\ell(\mathrm{TM})$ acts on ψ via Clifford multiplication. Relative to the local orthonormal frame $\mathscr{E} = (e_i)$, we have

$$\mathrm{R}(e_i,e_j)\cdot\psi = 0 \implies \sum_{k,\ell} \mathrm{R}_{ijk\ell} e^k e^\ell \psi = 0 \,. \tag{92}$$

If we multiply the above equation with e^j and sum over j, we obtain the following:

$$\begin{aligned}
0 &= \sum_{j,k,\ell} \mathrm{R}_{ijk\ell} e^j e^k e^\ell \psi \\
&= \sum_{j,k,\ell} \mathrm{R}_{ijk\ell} \left(e^{jk\ell} - g^{jk} e^\ell + g^{j\ell} e^k \right) \psi \\
&= \sum_{j,k,\ell} \mathrm{R}_{ijk\ell} \left(e^{jk\ell} + 2 g^{j\ell} e^k \right) \psi \,.
\end{aligned}$$

The first term vanishes by the algebraic Bianchi identity and the second term yields the Ricci tensor, whence the integrability condition becomes

$$\sum_j \mathrm{R}_{ij} e^j \psi = 0 \,. \tag{93}$$

More invariantly, this says the following. The Ricci tensor defines an endomorphism R of the tangent bundle called the **Ricci operator**, by $g(\mathrm{R}(\mathrm{X}),\mathrm{Y}) = r(\mathrm{X},\mathrm{Y})$. Then the above integrability condition says that $\mathrm{R}(\mathrm{X})\psi = 0$ for all $\mathrm{X} \in \mathscr{X}(\mathrm{M})$. Hitting this equation again with $\mathrm{R}(\mathrm{X})$, we see that $g(\mathrm{R}(\mathrm{X}),\mathrm{R}(\mathrm{X})) = 0$ for all X. If g is positive-definite, then $\mathrm{R}(\mathrm{X}) = 0$ and (M, g) is Ricci-flat. In indefinite signature, the image of the Ricci operator consists of null vectors, whence we could call such manifolds *Ricci-null*.

In the next lecture we will reformulate the question of which spin manifolds admit parallel spinor fields in terms of holonomy.

Lecture 7: Holonomy groups

Knowing the importance of groups in mathematics, it is quite natural to try to capture some part of Riemannian geometry in a group.

— Marcel Berger, 2003

In this lecture we will discuss the rudiments of the theory of holonomy groups for principal and vector bundles and in particular the relevant case of the holonomy group of the Levi-Civita connection on a riemannian manifold. As we will see in the next lecture, both the problems of determining the class of manifolds admitting parallel and Killing spinor fields will be solved in terms of riemannian holonomy groups.

7.1 Parallel transport in principal fibre bundles

Let $\pi : P \to M$ be a fixed principal G-bundle with connection $H \subset TP$. Let ω denote the connection 1-form. A smooth curve $\tilde{\gamma} : [0,1] \to P$ is said to be **horizontal** if the velocity vector is everywhere horizontal: $\dot{\tilde{\gamma}}(t) \in H_{\tilde{\gamma}(t)}$ for all t. This is equivalent to $\omega(\dot{\tilde{\gamma}}(t)) = 0$. Let $\gamma(t) = \pi(\tilde{\gamma}(t))$ denote the projection of the curve onto M. Assume that the curve is small enough so that the image of γ lies inside some trivialising neighbourhood U_α. Then $\psi_\alpha(\tilde{\gamma}(t)) = (\gamma(t), g(t))$, where $g(t)$ is a smooth curve on G. The condition $\omega(\dot{\tilde{\gamma}}(t)) = 0$ translates into the following ordinary differential equation for the curve $g(t)$. Indeed, using equation (77) and noticing that $\pi_* \dot{\tilde{\gamma}} = \dot{\gamma}$, we arrive at

$$\mathrm{ad}_{g(t)^{-1}} A_\alpha(\dot{\gamma}(t)) + g(t)^{-1}\dot{g}(t) = 0\,, \tag{94}$$

where A_α is the gauge field on U_α corresponding to the connection and where again we use notation appropriate to matrix groups. Indeed, for matrix groups we can rewrite this equation further as a matrix differential equation:

$$\dot{g}(t) + A_\alpha(\dot{\gamma}(t))g(t) = 0\,. \tag{95}$$

Being a first-order ordinary differential equations with smooth coefficients, equation (94) (equivalently (95)) has a unique solution for specified initial conditions, so that if we specify $g(0)$ then $g(1)$ is determined uniquely. This then defines a map $\Pi_\gamma : P_{\gamma(0)} \to P_{\gamma(1)}$ from the fibre over $\gamma(0)$ to the fibre over $\gamma(1)$, associated to the curve $\gamma : [0,1] \to M$. Rephrasing, given the curve γ, there is a unique horizontal lift $\tilde{\gamma}$ once we specify $\tilde{\gamma}(0) \in P_{\gamma(0)}$ and $\Pi_\gamma \tilde{\gamma}(0) = \tilde{\gamma}(1)$ is simply the endpoint of this horizontal curve. The map Π_γ is called **parallel transport along** γ **with respect to the connection** H.

Lemma 7.1. *Parallel transport is* G*-equivariant:* $\Pi_\gamma \circ R_a = R_a \circ \Pi_\gamma$.

Proof. This follows from the observation that if $\tilde{\gamma}(t)$ is a horizontal lift of $\gamma(t)$, then so is $\tilde{\gamma}(t)a$. □

Now let γ be a loop, so that $\gamma(0) = \gamma(1)$. Parallel transport along γ defines a group element $g_\gamma \in G$ defined by $g_\gamma = g(1)g(0)^{-1}$. To show that this element is well-defined, we need to show that it does not depend on the initial point $g(0)$. Indeed, suppose we choose a different starting point $\overline{g}(0)$. Then there is some group element $h \in G$ such that $\overline{g}(0) = g(0)h$. From the lemma $\overline{g}(t) := g(t)h$ is the horizontal lift with initial condition $\overline{g}(0)$. Therefore the final point of the curve is $\overline{g}(1) = g(1)h$, whence $\overline{g}(1)\overline{g}(0)^{-1} = g(1)g(0)^{-1}$ and g_γ is well-defined. This procedure defines a map from piecewise-smooth loops based at $m = \gamma(0)$ to G, whose image is a subgroup of G called the **holonomy group of the connection at** m denoted

$$\mathrm{Hol}(m) = \left\{ g_\gamma \middle| \gamma : [0,1] \to M\,,\ \gamma(1) = \gamma(0) = m \right\}\,. \tag{96}$$

The holonomy group is indeed a subgroup of G; that is, it is closed under inverses and multiplication. More precisely, if $g_\gamma \in \mathrm{Hol}(m)$, then let $\gamma^{-1}(t) := \gamma(1-t)$ be the curve with the same image as γ but traced backward and let $(\gamma(1-t), g(1-t))$ be its horizontal lift. Then $g_{\gamma^{-1}} = g(0)g(1)^{-1} = (g(1)g(0)^{-1})^{-1} = g_\gamma^{-1}$. Similarly, if g_{γ_1} and g_{γ_2} are elements in $\mathrm{Hol}(m)$, then so is their product. Indeed let (γ_1, g_1) be a horizontal lift of γ_1 (in a trivialisation) and (γ_2, g_2) a horizontal lift of γ_2, so that $g_{\gamma_1} = g_1(1)g_1(0)^{-1}$ and $g_{\gamma_2} = g_2(1)g_2(0)^{-1}$. It then follows that $g_{\gamma_1}g_{\gamma_2} = g_1(1)g_1(0)^{-1}g_2(1)g_2(0)^{-1}$. Let us consider the piecewise smooth curve

$$\gamma(t) = \begin{cases} \gamma_2(2t)\,, & t \in [0, \frac{1}{2}] \\ \gamma_1(2t-1)\,, & t \in [\frac{1}{2}, 1]\,. \end{cases}$$

A continuous horizontal lift of this curve is given (in a trivialisation) by $(\gamma(t), g(t))$ where

$$g(t) = \begin{cases} g_2(2t)\,, & t \in [0, \frac{1}{2}] \\ g_1(2t-1)\,, & t \in [0, \frac{1}{2}] \end{cases}$$

where, for continuity, we choose the horizontal lift of γ_1 in such a way that $g_1(0) = g_2(1)$. Then

$$g_{\gamma_1}g_{\gamma_2} = g_1(1)g_1(0)^{-1}g_2(1)g_2(0)^{-1} = g_1(1)g_2(0)^{-1} = g(1)g(0)^{-1} = g_\gamma\,.$$

Furthermore, if $m, m' \in \mathrm{M}$ belong to the same connected component, the holonomy groups $\mathrm{Hol}(m)$ and $\mathrm{Hol}(m')$ are conjugate in G and hence isomorphic. For a manifold the notion of connected component agrees with that of path component, hence there is a curve $\delta : [0,1] \to \mathrm{M}$ such that $\delta(0) = m$ and $\delta(1) = m'$. Let $\delta^{-1} : [0,1] \to \mathrm{M}$ be the curve $\delta^{-1}(t) = \delta(1-t)$. Then there is a one-to-one correspondence between loops based at m and based at m'. Indeed, if γ' is a loop based at m' then the composition $\gamma = \delta^{-1} \circ \gamma' \circ \delta$ is a loop based at m; and viceversa. Arguments similar to the ones above show that the element g_γ of the holonomy group at m is given by $hg_{\gamma'}h^{-1}$ where h is the group element corresponding to $\delta(0)$ in the trivialisation. This shows that $\mathrm{Hol}(m)$ and $\mathrm{Hol}(m')$ are conjugate subgroups of G, so that if M is connected there is a sense in which we can discuss the holonomy group of the connection, up to isomorphism, without having to specify the base point.

Considering only null-homotopic loops, we arrive at a normal subgroup of the holonomy group called the **restricted holonomy group** and denoted $\mathrm{Hol}_0(m)$. It can be shown that it is the identity component of the holonomy group. We have a surjective homomorphism $\pi_1(\mathrm{M}, m) \to \mathrm{Hol}(m)/\mathrm{Hol}_0(m)$, which is not generally an isomorphism.

7.2 Parallel transport on vector bundles

Let $\mathrm{E} = \mathrm{P} \times_{\mathrm{G}} \mathrm{F} \to \mathrm{M}$ be an associated vector bundle to $\mathrm{P} \to \mathrm{M}$ and let ∇ be the Koszul connection on sections of E induced from the connection on P. If $\gamma : [0,1] \to \mathrm{M}$ is a curve on M, then we define the parallel transport $\Pi_\gamma : \mathrm{E}_{\gamma(0)} \to \mathrm{E}_{\gamma(1)}$ as follows. We can use γ to pull the bundle E back to a bundle $\gamma^{-1}\mathrm{E} \to [0,1]$, whose fibre at $t \in [0,1]$ is the fibre of E at $\gamma(t)$. Vector bundles over the interval are trivial, so that sections of $\gamma^{-1}\mathrm{E}$ are functions $[0,1] \to \mathrm{F}$, where F is the typical fibre. Let $f_0 \in \mathrm{E}_{\gamma(0)}$ and let $f : [0,1] \to \mathrm{F}$ satisfy $\nabla_{\dot\gamma(t)} f = 0$, subject to $f(0) = f_0$. Then $\Pi_\gamma f_0 = f(1) \in \mathrm{E}_{\gamma(1)}$. Explicitly, the parallel transport equation $\nabla_{\dot\gamma(t)} f = 0$ becomes the ordinary differential equation

$$\frac{d}{dt} f(t) + \varrho(\mathrm{A}(\dot\gamma(t)))f(t) = 0\,, \tag{97}$$

where $\varrho : \mathfrak{g} \to \mathfrak{gl}(\mathrm{F})$ and A is the gauge field, where we have assumed that the image of γ lies inside a trivialising neighbourhood. By considering loops we define the notion of (restricted) holonomy group just as for principal fibre bundles.

We can recover the connection from the parallel transport by the following limiting procedure analogous to the usual definition of the derivative of a real variable:

$$\nabla_{\dot\gamma(t)} f = \lim_{h\to 0} \frac{1}{h}\left(\Pi_{-h} f(\gamma(t+h)) - f(\gamma(t))\right)\,, \tag{98}$$

where $\Pi_{-h} : \mathrm{E}_{\gamma(t+h)} \to \mathrm{E}_{\gamma(t)}$ is the parallel transport along γ from $t+h$ to t.

7.3 The holonomy principle

The holonomy principle is arguably the most important conceptual result in the theory of holonomy. Let $E \to M$ be a vector bundle with connection over a connected manifold M. A section σ of $E \to M$ is said to be **invariant under parallel transport** if for every curve $\gamma : [0,1] \to M$ we have that $\Pi_\gamma \sigma(\gamma(0)) = \sigma(\gamma(1))$. Taking γ to be a loop, we see that $\sigma(\gamma(0))$ is invariant under $\mathrm{Hol}(\gamma(0))$. Conversely, given $\sigma(m)$ invariant under $\mathrm{Hol}(m)$, we define $\sigma(m') = \Pi_\gamma \sigma(m)$, where $\gamma : [0,1] \to M$ is a curve with $\gamma(0) = m$ and $\gamma(1) = m'$. This does not depend on the choice of curve γ precisely because $\sigma(m)$ is invariant under the holonomy group. From equation (98), it follows that if σ is invariant under parallel transport, it is covariantly constant: $\nabla\sigma = 0$. If M is simply connected, then the converse also holds. This follows from the following

Theorem 7.2 (Ambrose–Singer). *Let* $E \to M$ *be a vector bundle with connection with* M *connected. Then the Lie algebra* $\mathfrak{hol}(m)$ *of the holonomy group* $\mathrm{Hol}(m)$ *is the Lie subalgebra of* $\mathfrak{gl}(E_m)$ *spanned by the curvature endomorphisms* $R^\nabla(X,Y)$ *and all its covariant derivatives* $\nabla_{Z_1}\cdots\nabla_{Z_k} R^\nabla(X,Y)$ *for* $X, Y, Z_i \in \mathscr{X}(M)$.

Indeed, fix $m \in M$ and suppose that $\nabla\sigma = 0$. Then $R^\nabla(X,Y)\sigma(m) = 0$ for all $X, Y \in T_m M$. Taking a further covariant derivative ∇_Z, say, we see that

$$0 = \nabla_Z(R^\nabla(X,Y)\sigma) = (\nabla_Z R^\nabla(X,Y))\sigma + R^\nabla(X,Y)\nabla_Z\sigma \ ,$$

but the last term vanishes because σ is covariantly constant, whence the endomorphism $\nabla_Z R^\nabla(X,Y)$ annihilates $\sigma(m)$. Continuing in this way and using the Theorem we see that $\sigma(m)$ is invariant under the Lie algebra of the holonomy group $\mathrm{Hol}(m)$, whence under the restricted holonomy group $\mathrm{Hol}_0(m)$. If M is simply-connected, then the holonomy group agrees with the restricted holonomy group, and hence $\sigma(m)$ is invariant under $\mathrm{Hol}(m)$.

We can summarise the above in the following

Theorem 7.3 (Holonomy principle). *Let* M *be a 1-connected manifold and* $E \to M$ *be a vector bundle with connection. Then there is a one-to-one correspondence between*

1. *sections of* E *which are invariant under parallel transport,*
2. $\mathrm{Hol}(m)$*-invariant vectors in* E_m*, for some* $m \in M$*, and*
3. *covariantly constant sections of* E.

If M *is connected but not simply-connected, then there may be covariantly constant sections which are only* $\mathrm{Hol}_0(m)$*-invariant, but not* $\mathrm{Hol}(m)$*-invariant.*

The holonomy principle allows us to turn questions concerning covariantly constant objects into algebraic questions about the holonomy representation.

7.4 Riemannian holonomy groups

Let (M, g) be a connected riemannian manifold of signature (s,t). Let ∇ denote the Levi-Civita connection on the tangent bundle TM. Since g is covariantly constant, it follows from the holonomy principle that the holonomy group is contained inside the orthogonal group, or more precisely, $\mathrm{Hol}(m) \subset O(T_m M) \cong O(s,t)$ and in particular at the level of the Lie algebras, $\mathfrak{hol}(m) \cong \mathfrak{so}(T_m M) \cong \mathfrak{so}(s,t)$. A natural question is whether any Lie subalgebra of $\mathfrak{so}(s,t)$ can appear as the holonomy Lie algebra of a riemannian manifold. To this day this problem has only been solved in the positive-definite and lorentzian signatures. In this section we will recall the positive-definite case and explain why the indefinite case is so much harder.

The vector space $T_m M$ is naturally a representation of $\mathrm{Hol}(m)$, called the **holonomy representation**. It is clear that for a riemannian product $(M, g) = (M_1, g_1) \times (M_2, g_2)$ the holonomy representation is reducible. (Since M is assumed connected, the (ir)reducibility of the holonomy representation does not depend on the point m.) The de Rham decomposition theorem below provides a partial converse to this *in positive-definite signature.*

Theorem 7.4 (De Rham)**.** *Let* (M, g) *be a 1-connected, complete, positive-definite riemannian manifold. If its holonomy representation is reducible, then* (M, g) *is a riemannian product.*

A sketch of a proof can be found in [Bes87, §10.44]. This result essentially reduces the classification of positive-definite riemannian holonomy groups to representation theory. The classification problem was was eventually solved by Berger, although later refined by a number of people including Simons, Alekseevsky and Bryant. A recent survey of this story can be found in [Bry00b], which also describes the more general problem for torsion-free affine connection (not necessarily metric), recently solved by Merkulov and Schwachhöfer. The torsion-free condition is what makes this problem nontrivial, since a classical theorem of Nomizu's states that any group can appear if we drop the torsion-free condition.

Back to the riemannian holonomy problem, the difference in indefinite signature is that reducibility is not enough to decompose the space. We say that a subspace $W \subset T_m M$ is **nondegenerate** if the restriction of the metric to W is non-degenerate, and **degenerate** otherwise. Clearly in positive-definite signature all subspaces are nondegenerate, but this is not the case in indefinite signature: a null line, for instance, provides an example of a degenerate subspace. In a riemannian product $(M, g) = (M_1, g_1) \times (M_2, g_2)$, the embedding at $m_2 \in M_2$ of the tangent space $T_{m_1} M_1$ into $T_{(m_1,m_2)} M$ is a nondegenerate subspace, and similarly for the embedding $T_{m_2} M_2 \subset T_{(m_1,m_2)} M$ at $m_1 \in M_1$. Hence it may happen that the holonomy representation is reducible, yet the manifold is not a riemannian product. Let us say that the holonomy representation is **decomposable** if it is reducible and if each invariant subspace is nondegenerate. We can then state the following extension of the de Rham decomposition theorem due to Wu [Wu64].

Theorem 7.5 (Wu)**.** *Let* (M, g) *be a 1-connected, complete, riemannian manifold. If its holonomy representation is decomposable, then* (M, g) *is a riemannian product.*

This means that it is not enough to restrict to irreducible holonomy representations in order to classify indefinite riemannian holonomy groups. Indeed, a result of Bérard-Bergery and Ikemakhen [BI93] says that the only irreducible lorentzian holonomy group is the Lorentz group itself, yet there exist indecomposable lorentzian manifolds with reduced holonomy. It is this which makes the indefinite case much harder. To date only the lorentzian problem has been solved completely. It is described in a recent survey by Leistner and Galaev [GL08].

Let us review the positive-definite classification, since this will play an important rôle in the rest of the lectures. The classification breaks up naturally into two classes of irreducible manifolds. The first class consists of those for which the curvature is parallel with respect to the Levi-Civita connection: $\nabla R = 0$. Such manifolds are said to be **locally symmetric** and if complete and simply connected, they are (riemannian) **symmetric spaces**. Symmetric spaces were classified by Élie Cartan. He found two types, each type being a pair consisting of a compact and a noncompact space and all indexed by simple Lie algebras. Pairs of the first type are $(G, G^{\mathbb{C}})$, where G is a 1-connected compact simple Lie group and $G^{\mathbb{C}}$ is the corresponding complex Lie group. Typical examples are (the simply-connected version of) $(SO(n), SO(n, \mathbb{C}))$ and $(SU(n), SL(n, \mathbb{C}))$. The second type consists of pairs $(G/H, G^*/H)$ where G is a 1-connected noncompact simple Lie group, H the connected maximal compact subgroup and G^* the compact form. A typical example is (the simply-connected version of) $(SL(n, \mathbb{R})/SO(n), SU(n)/SO(n))$. The holonomy group is G in the first type and H in the second type.

The second class, which in some sense is the most interesting for our purposes, consists of a finite list known as **Berger's table**; although the original list contained one more case which Alekseevsky showed was necessarily symmetric.

Theorem 7.6. *Let* (M, g) *be a complete, 1-connected, non-symmetric positive-definite riemannian manifold. Then its holonomy representation is one of the following:*

$n = \dim M$	$H \subset SO(n)$	*Geometry*
n	$SO(n)$	*generic*
$2m$	$U(m)$	*Kähler*
$2m$	$SU(m)$	*Calabi–Yau*
$4m$	$Sp(m)$	*hyperkähler*
$4m$	$Sp(m)\cdot Sp(1)$	*quaternionic Kähler*
7	G_2	*exceptional*
8	$Spin(7)$	*exceptional*

where $Sp(m)\cdot Sp(1)$ *is the image of* $Sp(m)\times Sp(1) \subset Spin(4m)$ *under* $Spin(4m) \to SO(4m)$.

One can understand these subgroups better in terms of the objects that they leave invariant which, by the holonomy principle, translates into the existence of covariantly constant fields on a riemannian manifold with that holonomy. We will look in detail at the Kähler, Calabi–Yau and G_2-holonomy cases.

7.4.1 Kähler manifolds

Every $U(m)$ subgroup of $SO(2m)$ arises as the subgroup of automorphisms of $\mathbb{R}^{2m}$ which commute with an orthogonal complex structure. Recall that a complex structure on $\mathbb{R}^{2m}$ is any endomorphism $J : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ obeying $J^2 = -\mathbf{1}$. The reason for the name is that J allows us to multiply by complex numbers: $a + ib \in \mathbb{C}$ acts like $a\mathbf{1} + bJ$. This makes $\mathbb{R}^{2m}$ into a complex vector space. A linear transformation which commutes with J commutes with complex multiplication, whence it is complex linear. The complex structure determines an embedding $GL(m,\mathbb{C})$ in $GL(2m,\mathbb{R})$: namely, those invertible linear transformations commuting with J.

A complex structure J is said to be orthogonal if it preserves the inner product $\langle -,-\rangle$ defining the $SO(2m)$ subgroup; that is, $\langle Jx, Jy\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{R}^{2m}$. In particular, we have that $\omega(x, y) := \langle Jx, y\rangle$ is a symplectic structure. Any two of $\langle -,-\rangle$, J and ω determines the third. It also defines a positive-definite hermitian structure on $\mathbb{R}^{2m}$ by

$$H(x, y) = \langle x, y\rangle + i\omega(x, y) . \tag{99}$$

It follows that $H(Jx, y) = -iH(x, y)$ and $H(x, Jy) = iH(x, y)$ and $\overline{H(x, y)} = H(y, x)$, whence H is indeed a hermitian structure. It is positive definite since so is $\langle -,-\rangle$. Any orthogonal linear transformation commuting with J preserves H and, conversely, if it preserves H, then it preserves its real and imaginary parts separately, whence it is orthogonal and commutes with J. Hence the $U(m)$ subgroup defined above is precisely the subgroup leaving H invariant. It can be thought of as the intersection with $SO(2m)$ of the $GL(m,\mathbb{C})$ subgroup defined by J.

On a $2m$-dimensional riemannian manifold with $U(m)$ holonomy, the holonomy principle guarantees the existence of a parallel complex structure J and a parallel symplectic form ω, called the **Kähler form**. Since $J^2 = -\mathbf{1}$, its eigenvalues are $\pm i$. The complexified tangent bundle $T^{\mathbb{C}}M$ decomposes into a direct sum of eigenbundles of J:

$$T^{\mathbb{C}}M = T^+M \oplus T^-M , \tag{100}$$

where $T^{\pm}M$ is the J-eigenbundle with eigenvalue $\pm i$. It follows from the fact that J is parallel, that $T^{\pm}M$ are integrable distributions in the sense of Frobenius. In other words, if X, Y are smooth sections of T^+M, then so is their Lie bracket $[X, Y]$, and similarly for T^-M. A hard theorem of Newlander and Nirenberg [NN57] then shows that M is a complex manifold; that is, there are coordinate charts homeomorphic to open subsets of $\mathbb{C}^m$ such that the transition functions on nonempty overlaps are biholomorphic. That means that we have local complex coordinates z^α and that there is a well-defined notion of holomorphicity. In turn this refines the de Rham complex, allowing us to define a notion of (p, q)**-form**, as a differential form of the form (summation convention in force!)

$$f_{\alpha_1\ldots\alpha_p\beta_1\ldots\beta_q}\, dz^{\alpha_1} \wedge \cdots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \cdots \wedge d\bar{z}^{\beta_q} , \tag{101}$$

where $f_{\alpha_1\ldots\alpha_p\beta_1\ldots\beta_q}$ are *smooth* functions on the domain of definition of the coordinates z^α. The notion of being a (p, q)-form is preserved on overlaps due to the holomorphicity of the transition functions. For example, the Kähler form $\omega(X,Y) = g(JX,Y)$ is a $(1,1)$-form. Its normalised powers $\frac{1}{k!}\omega^k$ are (k,k)-forms and, in particular, $\frac{1}{m!}\omega^m$ is the volume form corresponding to the metric g.

Let $\Omega^{(p,q)}(M)$ denote the $C^\infty(M)$-module of (p,q)-forms. This is a bigraded refinement of the de Rham complex in that

$$\Omega^r(M) = \bigoplus_{p+q=r} \Omega^{(p,q)}(M) . \tag{102}$$

If $f \in C^\infty(M)$, then $df \in \Omega^1(M) = \Omega^{(1,0)}(M) \oplus \Omega^{(0,1)}(M)$. Let us denote the component in $\Omega^{(1,0)}(M)$ by ∂f and the component in $\Omega^{(0,1)}(M)$ by $\bar\partial f$. More generally one has that $d = \partial + \bar\partial$, where $\partial : \Omega^{(p,q)}(M) \to \Omega^{(p+1,q)}(M)$ and $\bar\partial : \Omega^{(p,q)}(M) \to \Omega^{(p,q+1)}(M)$. Since $d^2 = 0$, it follows by looking at degrees that $\partial^2 = 0$, $\bar\partial^2 = 0$ and $\partial\bar\partial + \bar\partial\partial = 0$. A great deal more could be said about Kähler manifolds, but this is not the place.

7.4.2 Calabi–Yau manifolds

Since $SU(m) \subset U(m)$, manifolds with $SU(m)$ holonomy are Kähler. They are Ricci-flat and, in fact, they are characterised also in this way. Calabi–Yau manifolds have a parallel complex structure J and Kähler form ω, but in addition also a parallel **complex volume form** $\Theta \in \Omega^{(m,0)}(M)$. This follows from the fact that $SU(m)$ is the intersection of $U(m)$ with $SL(m,\mathbb{C})$ and $SL(m,\mathbb{C})$ is the subgroup of $GL(m,\mathbb{C})$ which acts trivially on the top exterior power of $\mathbb{C}^m$. The complex volume form Θ is in particular holomorphic: $\bar\partial\Theta = 0$. Let $\overline{\Theta} \in \Omega^{(0,m)}$ be its complex conjugate. Then $\Theta \wedge \overline{\Theta}$ is the volume form corresponding to the metric g.

7.4.3 Manifolds of G_2 holonomy

The group G_2 can be defined in several ways. It is the subgroup of Spin(8) fixed under the triality automorphism. Therefore any two representations of Spin(8) related by triality are equivalent when restricted to G_2. In particular, consider the three 8-dimensional representations: the vector V and the two half-spinor representations $S_\pm$. They are equivalent as representations of G_2. With hindsight, let us denote this representation (which is not irreducible) by $\mathbb{O}$. The Clifford action $V \otimes S_+ \to S_-$ becomes, under G_2, a *non-associative* multiplication $\mathbb{O} \otimes \mathbb{O} \to \mathbb{O}$, turning $\mathbb{O}$ into an $\mathbb{R}$-algebra. This is nothing but the algebra of **octonions** and G_2 is the group of automorphisms of the octonions: the map $V \otimes S_+ \to S_-$ is Spin(8) equivariant, hence $\mathbb{O} \times \mathbb{O} \to \mathbb{O}$ is G_2-equivariant. Since G_2 acts under automorphisms, it preserves the identity $1 \in \mathbb{O}$. Thus we see that $\mathbb{O}$ breaks up under G_2 into a direct sum of irreducible representations $\mathbb{R}1 \oplus \mathrm{Im}\mathbb{O}$. The holonomy representation $G_2 \subset SO(7)$ is precisely the action of G_2 on $\mathrm{Im}\mathbb{O}$. Octonion multiplication restricts to a bilinear map $\mathrm{Im}\mathbb{O} \otimes \mathrm{Im}\mathbb{O} \to \mathbb{O}$ and this, in turn, defines a G_2-invariant tensor $\varphi : \mathrm{Im}\mathbb{O} \otimes \mathrm{Im}\mathbb{O} \otimes \mathrm{Im}\mathbb{O} \to \mathbb{R}$ by $\varphi(x,y,z) = B(xy,z)$ for all $x,y,z \in \mathrm{Im}\mathbb{O}$. Here $B(-,-)$ is the inner product on $\mathbb{O}$, defined by $B(x,y) = \mathrm{Re}(x\overline{y})$, where $\overline{y}$ is the octonionic conjugate of y. When $x,y \in \mathrm{Im}\mathbb{O}$, then $B(x,y) = -\mathrm{Re}\,xy$. The octonion algebra is not associative, but nevertheless φ is totally skewsymmetric. The holonomy principle guarantees that on a manifold of G_2 holonomy, there is a parallel 3-form, also denoted φ. Its Hodge dual $\star\varphi$ is a parallel 4-form and $\frac{1}{7}\varphi \wedge \star\varphi$ is equal to the volume form corresponding to the metric. These are the only parallel forms on a generic manifold of G_2 holonomy. Manifolds of G_2 holonomy are also Ricci-flat.

7.4.4 Ricci-flatness

We saw in the previous lecture that Ricci-flatness was an integrability condition for the existence of parallel spinors in a positive-definite riemannian manifold. It is thus natural to ask whether a reduction of the holonomy group implies Ricci-flatness. We saw above that Calabi–Yau and G_2-holonomy manifolds are Ricci flat. It turns out that hyperkähler and Spin(7)-holonomy manifolds are also Ricci-flat. In contrast, quaternionic Kähler manifolds are Einstein but never Ricci-flat – indeed, a Ricci-flat quaternionic Kähler manifold is hyperkähler. Similarly, the Calabi conjecture (proved by Yau) says that a Ricci-flat Kähler manifold is Calabi–Yau. There are examples of noncompact positive-definite riemannian manifolds with $SO(n)$ holonomy, but to this day there is no known Ricci-flat compact manifold which has $SO(n)$-holonomy. This is perhaps the last remaining mystery in the holonomy of positive-definite riemannian manifolds.

Lecture 8: Parallel and Killing spinor fields

Killing spinors are lethal.
— Claude LeBrun, LMS Durham Symposium 2001

In this lecture we will characterise manifolds admitting spinor fields satisfying some natural differential equations. We will first revisit parallel (or covariantly constant) spinor fields, which were already discussed in §6.3, from the point of view of the holonomy representation. We will then introduce the notion of a (real) Killing spinor field, as a special case of a "twistor" spinor field.

8.1 Manifolds admitting parallel spinor fields

Recall that a covariantly constant spinor field ψ — that is, one obeying $d^\nabla\psi = 0$ — is invariant under parallel transport and hence its value at any point m is a spinor which is invariant under (the spin lift of) the holonomy group $\mathrm{Hol}(m)$. We also learnt that in positive-definite signature, a spin manifold admits parallel spinor fields only if it is Ricci-flat. This means that if $\mathrm{Hol}(m)$ leaves invariant a (nonzero) spinor, the manifold must be Ricci-flat. As we discussed in the last lecture, there are four Ricci-flat holonomy representations: $\mathrm{SU}(n) \subset \mathrm{SO}(2n)$, $\mathrm{Sp}(n) \subset \mathrm{SO}(4n)$, $G_2 \subset \mathrm{SO}(7)$ and $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$. Curiously, as shown by Wang [Wan89], each of these representations preserve a nonzero spinor. His results can be summarised in the following table. The column labelled "Parallel spinors" lists the dimension of the space of parallel spinors. In even dimensions, this is further refined according to chirality, in such a way that (n_+, n_-) means that the space of positive (resp. negative) parallel half-spinors has (real) dimension n_+ (resp. n_-). Of course, changing the orientation of the manifold interchanges n_+ and n_-.

Table 1: Irreducible, simply-connected manifolds admitting parallel spinors

Holonomy representation	Geometry	Parallel spinors
$\mathrm{SU}(2n+1) \subset \mathrm{SO}(4n+2)$	Calabi–Yau	$(1,1)$
$\mathrm{SU}(2n) \subset \mathrm{SO}(4n)$	Calabi–Yau	$(2,0)$
$\mathrm{Sp}(n) \subset \mathrm{SO}(4n)$	hyperkähler	$(k+1,0)$
$G_2 \subset \mathrm{SO}(7)$	exceptional	1
$\mathrm{Spin}(7) \subset \mathrm{SO}(8)$	exceptional	$(1,0)$

We will concentrate on two examples: $\mathrm{SU}(3) \subset \mathrm{SO}(6)$ and $G_2 \subset \mathrm{SO}(7)$.

8.1.1 Calabi–Yau 3-folds

We start with the following lemma.

Lemma 8.1. *The spin representation gives an isomorphism* $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$.

Proof. First of all we remark that $\mathrm{Spin}(6) \subset C\ell(6)_0 \cong \mathbb{C}(4)$. Thus we have an injective homomorphism $\iota : \mathrm{Spin}(6) \to \mathrm{GL}(4,\mathbb{C})$, which is the spin representation. Since $\mathrm{Spin}(6)$ is compact, its image in $\mathrm{GL}(4,\mathbb{C})$ must lie inside a maximal compact subgroup of $\mathrm{GL}(4,\mathbb{C})$: namely, a copy of $\mathrm{U}(4)$. Since $\mathrm{Spin}(6)$ is simple, its image must be inside $\mathrm{SU}(4)$. Finally, since $\dim \mathrm{Spin}(6) = \dim \mathrm{SU}(4) = 15$, and since both $\mathrm{Spin}(6)$ and $\mathrm{SU}(4)$ are connected, ι is an isomorphism. □

This means that the spinor representation of $\mathrm{Spin}(6)$ is the defining representation of $\mathrm{SU}(4)$ on $\mathbb{C}^4$. A nonzero spinor is a vector $\psi \in \mathbb{C}^4$. Without loss of generality we can assume that $\psi = (z,0,0,0)$ for some $0 \neq z \in \mathbb{C}$. It is then clear that the subgroup of $\mathrm{SU}(4)$ leaving that vector invariant is an $\mathrm{SU}(3)$ subgroup, which is the image under ι of an $\mathrm{SU}(3)$ subgroup of $\mathrm{Spin}(6)$. Since $-\mathbf{1} \in \mathrm{Spin}(6)$ does not leave ψ invariant, it does not belong to $\mathrm{SU}(3)$ whence its image under $\widetilde{\mathrm{Ad}} : \mathrm{Spin}(6) \to \mathrm{SO}(6)$ is an $\mathrm{SU}(3)$ subgroup of $\mathrm{SO}(6)$. This is precisely the holonomy representation $\mathrm{SU}(3) \subset \mathrm{SO}(6)$ in Berger's table. The complex conjugate spinor $\overline{\psi}$ has the opposite chirality to ψ and is also left invariant by the same $\mathrm{SU}(3)$ subgroup, whence the $(1,1)$ in the corresponding entry in the table.

8.1.2 Manifolds of G_2 holonomy

Let $\mathbb{O}$ denote the real division algebra of octonions, obtained from the quaternions by the Cayley–Dickson doubling construction. It is a normed algebra with a positive-definite inner product $B(x,y) = \operatorname{Re}(x\overline{y})$. The octonions are *not* associative, but they are alternating, which means that the subalgebra generated by any two elements is associative. In particular, if $x, y \in \mathbb{O}$, then $x(xy) = x^2y$ and $(yx)x = yx^2$. This is equivalent to associator $(xy)z - x(yz)$ being totally skewsymmetric in x, y, z. Consider now the linear maps defined by $\ell : x \mapsto \ell_x$ and $r : x \mapsto r_x$, where ℓ_x and r_x are, respectively, left and right multiplication by $x \in \mathbb{O}$.

Lemma 8.2. *The linear maps* $\ell, r : \operatorname{Im}\mathbb{O} \to \operatorname{End}_{\mathbb{R}}(\mathbb{O})$ *defined above are Clifford.*

Proof. We will prove the lemma for ℓ and leave r as an exercise. First of all, notice that the alternating property of $\mathbb{O}$ says that for all $x, y, z \in \mathbb{O}$,

$$x(yz) - (xy)z = -y(xz) + (yx)z \, , \tag{103}$$

whence

$$\begin{aligned} \ell_x\ell_y z + \ell_y\ell_x z &= x(yz) + y(xz) \\ &= (xy)z + (yx)z && \text{by equation (103)} \\ &= (xy + yx)z \, . \end{aligned}$$

But notice that since $x, y \in \operatorname{Im}\mathbb{O}$, $xy + yx \in \mathbb{R} \subset \mathbb{O}$ and is indeed equal to $-2B(x,y)$, whence we conclude that

$$\ell_x\ell_y + \ell_y\ell_x = -2B(x,y)\mathbf{1} \, .$$

□

This means that ℓ and r extend to representations of the Clifford algebra $C\ell(\operatorname{Im}\mathbb{O}) \cong C\ell(7)$. Indeed, the isomorphism $C\ell(7) \cong \mathbb{R}(8) \oplus \mathbb{R}(8)$ is the Clifford extension of the Clifford map $x \mapsto (\ell_x, r_x)$. The spinor representation of $\mathrm{Spin}(7)$ is obtained by restricting either of these two Clifford modules to $\mathrm{Spin}(7) \subset C\ell(7)$. This defines a map $\mathrm{Spin}(7) \to \mathrm{GL}(8,\mathbb{R})$ whose image, since $\mathrm{Spin}(7)$ is compact and connected, lies inside $\mathrm{SO}(8)$, for some $\mathrm{SO}(8)$ subgroup of $\mathrm{GL}(8,\mathbb{R})$. Indeed, it is the $\mathrm{SO}(8)$ which preserves the octonionic inner product. This follows form the fact that $\mathbb{O}$ is a normed algebra, whence $B(xy,xy) = B(x,x)B(y,y)$, whence if $B(x,x) = 1$ then both ℓ_x and r_x are isometries.

Let ψ be a nonzero spinor, which we may take to correspond to $1 \in \mathbb{R} \subset \mathbb{O}$. The subgroup of $\mathrm{Spin}(7)$ which fixes ψ is a G_2 subgroup of $\mathrm{Spin}(7)$ which does not contain $-\mathbf{1}$ and hence projects under $\widetilde{\mathrm{Ad}} : \mathrm{Spin}(7) \to \mathrm{SO}(7)$ to a G_2 subgroup of $\mathrm{SO}(7)$, which is precisely the holonomy representation $G_2 \subset \mathrm{SO}(7)$. Any other spinor left invariant by this G_2 subgroup is proportional to ψ.

8.1.3 Some comments about indefinite signature

In physical applications it is often necessary to determine the lorentzian (or even higher index) spin manifolds admitting parallel spinors. There is a classification of lorentzian holonomy groups due to Leistner and Galaev [GL08], as well as earlier results of Bryant [Bry00a] and myself [FO00]. A lorentzian spin n-dimensional manifold admits parallel spinors if its holonomy representation is $G \ltimes \mathbb{R}^{n-2} \subset \mathrm{SO}_0(n-1,1)$, where $G \subset \mathrm{SO}(n-2)$ is one of the riemannian holonomy representations admitting parallel spinors. The subgroup $G \ltimes \mathbb{R}^{n-2}$ of $\mathrm{SO}(n-1,1)$ is such that G acts on $\mathbb{R}^{n-2}$ via the holonomy representation $G \subset \mathrm{SO}(n-2)$, and the abelian normal subgroup $\mathbb{R}^{n-2}$ acts as *null rotations* on $\mathbb{R}^{n-1,1}$. The situation for higher index is much less clear and still the subject of investigation.

8.2 Manifolds admitting (real) Killing spinor fields

On a spin manifold one can define natural equations satisfied by spinor fields other than $d^\nabla\psi = 0$. In this section we will discuss the Killing spinor equation which is a special case of the twistor spinor equation, about which we will not say anything beyond its definition.

8.2.1 The Dirac operator

Let $\mathscr{E} = (e_i)$ be a local frame and let (e^i) denote the dual frame, so that $g(e^i, e_j) = \delta^i_j$. The **Dirac operator** is the differential operator D acting on a spinor field ψ as

$$D\psi = \sum_i e^i \cdot \nabla_{e_i}\psi \,, \tag{104}$$

where the dot $(\cdot)$ stands for Clifford action. More invariantly, it is defined as the composition of the following two maps

$$C^\infty(M, S(M)) \xrightarrow{d^\nabla} C^\infty(M, T^*M \otimes S(M)) \xrightarrow{\mathrm{cl}} C^\infty(M, S(M)) \tag{105}$$

where the first map is the covariant derivative and the second map is the fibrewise Clifford action $T^*M \otimes S(M) \to S(M)$.

Example 8.3. The original Dirac operator was defined on four-dimensional Minkowski spacetime. Relative to flat coordinates x^μ and the associated frame, the Dirac operator takes the form

$$D\psi = \sum_\mu \Gamma^\mu \cdot \nabla_{\frac{\partial}{\partial x^\mu}} \psi = \sum_\mu \Gamma^\mu \frac{\partial \psi}{\partial x^\mu} \,, \tag{106}$$

where $\psi : \mathbb{R}^{3,1} \to \mathbb{C}^4$ and $\Gamma^\mu = \sum_\nu \eta^{\mu\nu}\Gamma_\nu$ and Γ_μ are the 4×4 gamma matrices representing the Clifford action by the frame vectors $\frac{\partial}{\partial x^\mu}$.

Spinors which are annihilated by the Dirac operator are known as **harmonic spinors**. The origin of the name is due to the fact that squaring the original Dirac operator, one gets the laplacian:

$$D^2\psi = -\eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \psi = \Box\psi \,. \tag{107}$$

In the general case, squaring the Dirac operator results in a curvature-dependent correction:

$$D^2\psi = \nabla^*\nabla\psi + \frac{s}{4}\psi \,, \tag{108}$$

where s is the scalar curvature and $\nabla^*\nabla$ is the covariant laplacian.

An immediate corollary of this calculation is the following theorem due to Lichnerowicz.

Theorem 8.4 (Lichnerowicz)**.** *If* (M, g) *is a compact positive-definite riemannian spin manifold with* $s \geq 0$ *and* $s > 0$ *at at least one point, then* (M, g) *admits no nonzero harmonic spinor fields; whereas if* $s \equiv 0$ *then a harmonic spinor field is parallel.*

Proof. Indeed, let $(-,-)$ denote the invariant inner product on the spinor bundle, and consider the integral

$$\int_M \left(\psi, D^2\psi\right) = \int_M |d^\nabla\psi|^2 + \frac{1}{4}\int_M s|\psi|^2 \,.$$

Let $D\psi = 0$, so that the LHS vanishes. Then if $s \geq 0$, the RHS is positive-semidefinite and in particular we see that $d^\nabla\psi = 0$. This being the case, ψ is determined uniquely by its value at any point, so that in particular if it vanishes anywhere, it must vanish everywhere. If $s > 0$ at at least one point, then it $s > 0$ is a neighbourhood of that point and hence $\psi = 0$ in a neighbourhood of that point and hence $\psi = 0$ everywhere. $\square$

8.2.2 The Penrose operator and twistor spinor fields

Let $W \subset T^*M \otimes S(M)$ denote the subbundle defined as the kernel of the Clifford action $T * M \otimes S(M) \to S(M)$. Let $\pi : T^*M \otimes S(M) \to W$ denote the projection onto W along S(M). The **Penrose operator** $P : C^\infty(M, S(M)) \to C^\infty(M, W)$ is defined as the composition

$$C^\infty(M, S(M)) \xrightarrow{d^\nabla} C^\infty(M, T^*M \otimes S(M)) \xrightarrow{\pi} C^\infty(M, W) \,. \tag{109}$$

Explicitly, we can write for all spinor fields ψ and all vector fields X,

$$P_X\psi = \nabla_X\psi + \tfrac{1}{n} X \cdot D\psi \,, \tag{110}$$

where $n = \dim M$. Spinor fields in the kernel of the Penrose operator are known as **twistor fields**.

8.2.3 Killing spinor fields

A **Killing spinor field** is a special type of twistor field ψ which satisfies the stronger equation

$$\nabla_X\psi = \lambda X\cdot\psi \ , \tag{111}$$

for some constant $\lambda \in \mathbb{C}$ called the **Killing constant**. A calculation similar to that in section 6.3 reveals that the integrability condition for the existence of Killing spinor fields is

$$(R(X) - 4\lambda^2(n-1)X)\cdot\psi = 0 \ ,$$

for all vector fields X and where $X \mapsto R(X)$ is the Ricci operator and $n = \dim M$. In positive-definite signature, it says that $R(X) = 4\lambda^2(n-1)X$ for all vector fields X, or equivalently after taking the inner product with a second vector field Y, that $r(X,Y) = 4\lambda^2(n-1)g(X,Y)$, whence (M,g) is Einstein. In indefinite signature this is no longer the case, but we can take the Clifford trace of the above equation to conclude that

$$s\psi = 4\lambda^2 n(n-1)\psi \ ,$$

whence if ψ is not identically zero, the scalar curvature is constrained in terms of λ: namely, $s = 4\lambda^2 n(n-1)$. Since the scalar curvature is real, we see that λ^2 is real, whence it is either real or pure imaginary. The nature of the Killing constant gives rise to two different kinds of Killing spinor fields: **real** and **imaginary**, respectively. They each have a very different flavour and in the rest of this lecture we will concentrate on the real case. Furthermore via a homothety (i.e., rescaling the metric by a constant positive number) we can further assume that $\lambda = \pm\frac{1}{2}$. Finally, we will concentrate on positive-definite signature, whence we will be interested in characterising those positive-definite riemannian spin manifolds admitting nonzero spinor fields ψ satisfying

$$\nabla_X\psi = \pm\tfrac{1}{2}X\cdot\psi \ , \tag{112}$$

for all vector fields X.

Bär's cone construction [Bär93] will relate such Killing spinor fields to parallel spinor fields in an auxiliary geometry. To at least demonstrate the plausibility of such a construction, let us first of all notice that a spinor field obeying equation (112) is actually parallel with respect to the connection $\mathscr{D}_X = \nabla_X \mp \frac{1}{2}X\cdot$. The connection one-form associated with $\mathscr{D}$ is given, relative to a local frame $\mathscr{E} = (e_i)$, by

$$\tfrac{1}{4}\sum_{i,j}\omega_{ij}e^i e^j \mp \tfrac{1}{2}\sum_i \theta_i e^i \ , \tag{113}$$

where $\omega_{ij}(X) = g(\nabla_X e_i, e_j)$ and $\theta_i(X) = g(X, e_i)$. But now notice that $\frac{1}{4}[e^i, e^j]$ and $\mp\frac{1}{2}e_i$ in $C\ell(n)$ span an $\mathfrak{so}(n+1)$ subalgebra of $C\ell(n)$, whence the above connection one-form is $\mathfrak{so}(n+1)$-valued, which suggests that it could very well be the spin connection of an $(n+1)$-dimensional manifold. This manifold is the metric cone as we now review.

8.2.4 The cone construction

Let (M,g) be an n-dimensional riemannian manifold and let $\widetilde{M} = \mathbb{R}^+ \times M$. We parametrise $\mathbb{R}^+$ by $r > 0$ and define a metric $\tilde{g}$ on $\widetilde{M}$ by $\tilde{g} = dr^2 + r^2 g$. The riemannian manifold $(\widetilde{M}, \tilde{g})$ thus constructed is the **metric cone** of (M,g). (M,g) embeds isometrically into $(\widetilde{M},\tilde{g})$ as the submanifold at $r = 1$. Generically the metric on $\widetilde{M}$ cannot be extended smoothly to $r = 0$. The exception occurs when (M,g) is the round n-sphere, in which case the cone is $\mathbb{R}^{n+1}\setminus\{0\}$ with the flat euclidean metric, since in that case the flat metric is clearly regular at the origin and can be extended there.

The cone $\widetilde{M}$ admits a homothetic action by R^+, where $e^t \in \mathbb{R}^+$ acts by rescaling the "radial" coordinate: $(r,x) \mapsto (e^t r, x)$. The conformal Killing vector generating this action is the **Euler vector** $\xi = \frac{1}{r}\frac{\partial}{\partial r}$. A vector field $X \in \mathscr{X}(M)$ admits a unique lift to $\widetilde{M}$, also denoted X with a little abuse of notation, such that it is orthogonal to ξ and such that it maps to X under the natural projection $\widetilde{M} \to M$, sending (r,x) to x. Let $\widetilde{\nabla}$ denote the Levi-Civita connection on $\widetilde{M}$.

Lemma 8.5. *Let* $X, Y \in \mathscr{X}(\widetilde{M})$ *be lifts of vector fields on* M*. Then*

$$\widetilde{\nabla}_\xi \xi = \xi\,, \qquad \widetilde{\nabla}_\xi X = X\,, \qquad \widetilde{\nabla}_X \xi = X \qquad \textit{and} \qquad \widetilde{\nabla}_X Y = \nabla_X Y - g(X,Y)\xi\,.$$

Remark 8.6. In fact, a result of Gibbons and Rychenkova [GR98] states that a riemannian manifold is a metric cone if and only if there exists a vector field ξ such that $\nabla_V \xi = V$ for all vector fields V, where ∇ is the Levi-Civita connection.

Now given a local frame $\mathscr{E} = (e_i)$ for M, we extend it to a local frame $\widetilde{\mathscr{E}} = (\widetilde{e}_0 = \frac{\partial}{\partial r}, \widetilde{e}_i = \frac{1}{r} e_i)$ for $\widetilde{M}$. The connection coefficients of $\widetilde{\nabla}$ relative to $\widetilde{\mathscr{E}}$ are given in terms of the connection coefficients of ∇ relative to $\mathscr{E}$ by the following formulae.

Lemma 8.7. *Let* $\widetilde{\omega}_{ab} = \widetilde{g}(\widetilde{\nabla}\widetilde{e}_a, \widetilde{e}_b)$ *be the connection 1-form in* $\widetilde{M}$ *relative to the local frame* $\widetilde{\mathscr{E}}$*. Then*

$$\widetilde{\omega}_{ab}(\tfrac{\partial}{\partial r}) = 0\,, \qquad \widetilde{\omega}_{0i}(e_j) = \delta_{ij} \qquad \textit{and} \qquad \widetilde{\omega}_{ij}(e_k) = \omega_{ij}(e_j)\,.$$

Since $\mathbb{R}^+$ is contractible, the cone $\widetilde{M}$ is homotopy equivalent to M, whence if M is spin, so is $\widetilde{M}$. Furthermore, if M is spin, the embedding (at $r = 1$) of M into $\widetilde{M}$ sets up a bijective correspondence between the spin structures on M and on $\widetilde{M}$. From now on we assume that both M and $\widetilde{M}$ are spin, with corresponding spin structures. Now let $\widetilde{\psi}$ be a spinor field on $\widetilde{M}$. Its covariant derivative can be computed from equation (89) and the previous lemma and one finds

$$\widetilde{\nabla}_{\frac{\partial}{\partial r}} \widetilde{\psi} = \tfrac{\partial}{\partial r}\widetilde{\psi} \qquad \text{and} \qquad \widetilde{\nabla}_{e_k}\widetilde{\psi} = \nabla_{e_k}\widetilde{\psi} + \tfrac{1}{2}\widetilde{e}_0\widetilde{e}_k\widetilde{\psi}\,.$$

Therefore a parallel spinor field $\widetilde{\psi}$ on $\widetilde{M}$ satisfies $\frac{\partial}{\partial r}\widetilde{\psi} = 0$ and $\nabla_{e_k}\widetilde{\psi} = \frac{1}{2}\widetilde{e}_k\widetilde{e}_0\widetilde{\psi}$. The restriction of $\widetilde{\psi}$ to $r = 1$ is a spinor field on M which satisfies the second of the above equations. To understand this equation intrinsically, we recall the isomorphism $C\ell(n) \cong C\ell(n+1)_0$ given in Proposition 2.8. In fact, there are two possible isomorphisms, distinguished by a sign: $e_i \mapsto \varepsilon\widetilde{e}_i\widetilde{e}_0$, for $\varepsilon^2 = 1$. It is now that we must make a distinction between even- and odd-dimensional M. Consider the volume element $e_1 \cdots e_n \in C\ell(n)$. Its image in $C\ell(n+1)_0$ under the above isomorphism is given by

$$e_1 \cdots e_n \mapsto \begin{cases} -\varepsilon\widetilde{e}_0\widetilde{e}_1 \cdots \widetilde{e}_n & n \text{ odd} \\ \widetilde{e}_1 \cdots \widetilde{e}_n & n \text{ even.} \end{cases} \tag{114}$$

If $n = \dim M$ is odd, then there are two inequivalent Clifford modules, each determined by the action of the volume element $e_1 \cdots e_n$ in $C\ell(n)$, which goes over to $-\varepsilon$ times the action of the volume element $\widetilde{e}_0\widetilde{e}_1 \cdots \widetilde{e}_n$ in $C\ell(n+1)$. This means that ε can be fixed in order to relate Killing spinor fields on M (with respect to one choice of Clifford module) to the chirality of the parallel spinor field on $\widetilde{M}$. Hence the sign of the Killing constant and the chirality of the parallel spinor field are correlated. On the other hand, if n is even, then ε is not fixed and for every parallel spinor field on $\widetilde{M}$ we obtain a Killing spinor field on M with either sign of the Killing constant, simply by making the right choice of ε.

8.2.5 The classification

We have just reduced the problem of which riemannian manifolds admit real Killing spinor fields to which metric cones admit parallel spinors. We will assume that (M, g) is complete and admits real Killing spinor fields. Then since it is Einstein, Myers Theorem [CE75, Theorem 1.26] implies that it is compact. Then a result of Gallot's [Gal79, Proposition 3.1] says that if (M, g) is in addition simply connected, the cone $(\widetilde{M}, \widetilde{g})$ is either irreducible or flat. If the latter, (M, g) is the round sphere; if the former it is one of the geometries in Table 8.1.

Every geometry in Table 8.1 admits parallel forms, constructed via the holonomy principle from the invariants under the holonomy representation and indeed constructed out of the parallel spinors. Since in addition the manifold in question is a cone, and hence we have at our disposal also the Euler vector field ξ, we can construct a number of geometric structures on the manifold M, which are listed in Table 8.2.5, where $N_\pm$ is the dimension of the space of Killing spinor fields with Killing constant $\pm\frac{1}{2}$.

Table 2: Simply-connected, complete riemannian manifolds with real Killing spinor fields

dim	Geometry	Cone	(N_+, N_-)
n	round sphere	flat	$(2^{\lfloor n/2 \rfloor}, 2^{\lfloor n/2 \rfloor})$
$4k-1$	3-Sasaki	hyperkähler	$(k+1, 0)$
$4k-1$	Sasaki–Einstein	Calabi–Yau	$(2, 0)$
$4k+1$	Sasaki–Einstein	Calabi–Yau	$(1, 1)$
6	nearly Kähler	G_2	$(1, 1)$
7	weak G_2	Spin(7)	$(1, 0)$

For example, if the cone is Calabi–Yau, then we have a parallel complex structure J. The vector field $\chi = J\xi$ is orthogonal to ξ and it is the lift of a vector field on M, which we also denote χ. It is easy to show that χ is a Killing vector and has unit norm. Its dual one-form θ is (the restriction to $r = 1$ of) the contraction of the Euler vector into the Kähler form on the cone. The covariant derivative $\nabla\chi$ defines a skewsymmetric endomorphism T of the TM such that $T(X) = \nabla_X \chi$. The fact that J is parallel means that

$$(\nabla_X T)(Y) = \theta(Y)X - g(X, Y)\chi \ . \tag{115}$$

The triple (χ, θ, T) defines a **Sasakian structure** on M, whence M is Sasaki–Einstein.

For another example, consider the case of a G_2-holonomy cone. We have a parallel 3-form ϕ into which we contract the Euler vector field ξ to define a 2-form $\omega \in \Omega^2(M)$: $\omega(X, Y) = \phi(\xi, X, Y)$, evaluated at $r = 1$. This defines an endomorphism J of TM by $g(J(X), Y) = \omega(X, Y)$. One can show that J is an orthogonal almost complex structure. It is not parallel, but it satisfies $(\nabla_X J)(X) = 0$ for all vector fields $X \in \mathscr{X}(M)$. This defines a (non-Kähler) **nearly Kähler** structure on M.

These geometries defined via Killing spinors are presently under very active investigation, largely due to their rôle in the gauge/gravity correspondence (see, e.g., [AFOHS98, MP99]).

References

[ABS64] Michael Atiyah, Raoul Bott, and Arnold Shapiro, *Clifford modules*, Topology **3** (1964), 3–38.

[Ada69] J. Frank Adams, *Lectures on Lie groups*, W. A. Benjamin, Inc., New York-Amsterdam, 1969. MR MR0252560 (40 #5780)

[AFOHS98] B S Acharya, J M Figueroa-O'Farrill, C M Hull, and B Spence, *Branes at conical singularities and holography*, Adv. Theor. Math. Phys. **2** (1998), 1249–1286.

[Bär93] C Bär, *Real Killing spinors and holonomy*, Comm. Math. Phys. **154** (1993), 509–521.

[Bes87] Arthur L Besse, *Einstein manifolds*, Springer-Verlag, 1987.

[BI93] L Bérard Bergery and A Ikemakhen, *On the holonomy of lorentzian manifolds*, Proc. Symp. Pure Math. **54** (1993), 27–40.

[Bry00a] R L Bryant, *Pseudo-Riemannian metrics with parallel spinor fields and vanishing Ricci tensor*, Global analysis and harmonic analysis (Marseille-Luminy, 1999) (J P Bourgignon, T Branson, and O Hijazi, eds.), Sémin. Congr., vol. 4, Soc. Math. France, Paris, 2000, pp. 53–94.

[Bry00b] Robert Bryant, *Recent advances in the theory of holonomy*, Astérisque (2000), no. 266, Exp. No. 861, 5, 351–374, Séminaire Bourbaki, Vol. 1998/99.

[BtD85] Theodor Bröcker and Tammo tom Dieck, *Representations of compact Lie groups*, Graduate Texts in Mathematics, vol. 98, Springer-Verlag, New York, 1985.

[BW35] Richard Brauer and Hermann Weyl, *Spinors in n Dimensions*, Amer. J. Math. **57** (1935), no. 2, 425–449.

[Car38] Élie Cartan, *Leçons sur la théorie des spineurs*, Paris: Hermann & Cie. 96 p. , 1938.

[CE75] J Cheeger and D Ebin, *Comparison theorems in Riemannian Geometry*, North-Holland, Amsterdam, 1975.

[Che54] Claude C. Chevalley, *The algebraic theory of spinors*, Columbia University Press, New York, 1954.

[Cli78] William Kingdon Clifford, *Applications of Grassmann's Extensive Algebra*, American Journal of Mathematics **1** (1878), no. 4, 350–358.

[FO00] J M Figueroa-O'Farrill, *Breaking the M-waves*, Class. Quant. Grav. **17** (2000), 2925–2947.

[Gal79] S Gallot, *Equations différentielles caractéristiques de la sphère*, Ann. Sci. École Norm. Sup. **12** (1979), 235–267.

[GL08] Anton Galaev and Thomas Leistner, *Holonomy groups of Lorentzian manifolds: classification, examples, and applications*, Recent developments in pseudo-Riemannian geometry, ESI Lect. Math. Phys., Eur. Math. Soc., Zürich, 2008, pp. 53–96.

[GR98] G W Gibbons and P Rychenkova, *Cones, tri-sasakian structures and superconformal invariance*, Phys. Lett. **B443** (1998), 138–142.

[Har90] F. Reese Harvey, *Spinors and calibrations*, Academic Press, 1990.

[KS87] Bertram Kostant and Shlomo Sternberg, *Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras*, Ann. Physics **176** (1987), no. 1, 49–113.

[Lan84] Serge Lang, *Algebra*, second ed., Addison-Wesley Publishing Company Advanced Book Program, Reading, MA, 1984.

[LM89] H. Blaine Lawson and Marie-Louise Michelsohn, *Spin geometry*, Princeton University Press, 1989.

[MP99] D R Morrison and M R Plesser, *Non-spherical horizons, I*, Adv. Theor. Math. Phys. **3** (1999), 1–81.

[NN57] A. Newlander and L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. of Math. (2) **65** (1957), 391–404.

[Wan89] MY Wang, *Parallel spinors and parallel forms*, Ann. Global Anal. Geom. **7** (1989), no. 1, 59–68.

[Wu64] H Wu, *On the de Rham decomposition theorem*, Illinois J. Math. **8** (1964), 291–311.