Spin Geometry 2010

Tutorial Sheet 2

(Harder problems, if any, are adorned with a ✰.)

Problem 2.1. Prove this Lemma from the lectures. Let $\mathbb{K}$ stand for any of $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ and let $\mathbb{K}(n)$ denote the *real* algebra of $n \times n$ matrices with entries in $\mathbb{K}$. Then we have the following isomorphisms of real associative algebras:

$$\mathbb{K}(m) \otimes_{\mathbb{R}} \mathbb{R}(n) \cong \mathbb{K}(mn) .$$

Problem 2.2. Prove the following periodicities of real Clifford algebras:

1. $C\ell(n,0) \otimes C\ell(0,2) \cong C\ell(0,n+2)$,
2. $C\ell(s,t) \otimes C\ell(1,1) \cong C\ell(s+1,t+1)$,
3. $C\ell(n+8,0) \cong C\ell(n,0) \otimes_{\mathbb{R}} \mathbb{R}(16)$,
4. $C\ell(0,n+8) \cong C\ell(0,n) \otimes_{\mathbb{R}} \mathbb{R}(16)$, and
5. $C\ell(s+4,t+4) \cong C\ell(s,t) \otimes_{\mathbb{R}} \mathbb{R}(16)$,

where $n,s,t \geq 0$.

Problem 2.3. Use the periodicities in the lectures to prove that $C\ell(6,0) \cong \mathbb{R}(8)$ and $C\ell(7,0) \cong \mathbb{R}(8) \oplus \mathbb{R}(8)$. Similarly prove that $C\ell(1,9) \cong \mathbb{R}(32)$, $C\ell(1,10) \cong \mathbb{R}(32) \oplus \mathbb{R}(32)$ and $C\ell(2,10) \cong \mathbb{R}(64)$.

Problem 2.4. Prove the classification theorem; that is, prove the isomorphisms in the table:

$s-t \mod 8$	$C\ell(s,t)$
0, 6	$\mathbb{R}\left(2^{d/2}\right)$
7	$\mathbb{R}\left(2^{(d-1)/2}\right) \oplus \mathbb{R}\left(2^{(d-1)/2}\right)$
1, 5	$\mathbb{C}\left(2^{(d-1)/2}\right)$
2, 4	$\mathbb{H}\left(2^{(d-2)/2}\right)$
3	$\mathbb{H}\left(2^{(d-3)/2}\right) \oplus \mathbb{H}\left(2^{(d-3)/2}\right)$

Problem 2.5. Fill in the details of the proof in the lectures of the following isomorphism

$$\mathbb{C}\ell(n+2) \cong \mathbb{C}\ell(n) \otimes_{\mathbb{C}} \mathbb{C}(2) .$$

Problem 2.6. Let $e_1, \ldots, e_s, \varepsilon_1, \ldots, \varepsilon_t$ be an orthonormal basis for $\mathbb{R}^{s,t}$ and let the same symbols denote the corresponding elements of $C\ell(s,t)$. Let $\omega := e_1 \cdots e_s \varepsilon_1 \cdots \varepsilon_t$ denote the **volume element**.

1. Show that $\omega^2 = (-1)^{(s+t)(s+t-1)/2+s}\mathbf{1}$.
2. Show that if $s+t$ is odd, then ω is central; that is, it commutes with all the elements of $C\ell(s,t)$.
3. Show that if $s+t$ is odd and $\omega^2 = \mathbf{1}$ then the Clifford algebra splits as a direct sum of two subalgebras, whereas if $\omega^2 = -\mathbf{1}$ then it is the complexification which splits. Determine for which (s,t) either of the two cases happen and compare with the classification results.