Spin Geometry 2010

Tutorial Sheet 3

(Harder problems, if any, are adorned with a ✰.)

Problem 3.1. Prove the two-dimensional Cartan–Dieudonné theorem for both euclidean and lorentzian signatures. In other words, show that every element of $\mathrm{O}(2)$ can be written as the product of one or two reflections, according to whether it has determinant -1 or 1, respectively. Show that the same is true for $\mathrm{O}(1,1)$.

Problem 3.2.✰ Prove the Cartan–Dieudonné theorem for euclidean vector spaces.

Problem 3.3. Show that the restriction to $\mathrm{Pin}(V)$ of an irreducible representation of $C\ell(V)$ remains irreducible. Similarly, show that the restriction to $\mathrm{Spin}(V)$ of an irreducible representation of $C\ell(V)_0$ remains irreducible.

Problem 3.4. Prove the identities satisfied by the functors $c, e_{\mathbb{R}}^{\mathbb{C}}, e_{\mathbb{C}}^{\mathbb{H}}, r_{\mathbb{R}}^{\mathbb{C}}, r_{\mathbb{C}}^{\mathbb{H}}$ between the categories of real, complex and quaternionic representations of a Lie group.

Problem 3.5. Verify the properties of the Clifford involutions $\tilde{}$, $\check{}$ and $\hat{}$ stated in the lecture.

Problem 3.6.✰ Convince yourself of the low-dimensional isomorphisms given at the end of the third lecture or, at the very least, those of the corresponding Lie algebras. This involves understanding the inner products for the relevant spinor representations.

Problem 3.7. Let W be a real vector space and consider $V = W \oplus W^*$ with the inner product given by the dual pairing between W and W^*. Show that the unique pinor representation of $C\ell(V)$ is isomorphic to ΛW^*, with $\theta \in W^*$ acting by the wedge product and $w \in W$ acting by the interior product. Show that the spinor representations are naturally isomorphic to $\Lambda^{\mathrm{even}} W^*$ and $\Lambda^{\mathrm{odd}} W^*$. What is the $\mathrm{Spin}(V)$-invariant inner product? (It goes by the name of the *Mukai pairing* of forms.)