Spin Geometry 2010

Tutorial Sheet 4

(Harder problems, if any, are adorned with a ✰.)

Problem 4.1. Let T denote a two-dimensional torus thought of as the quotient of the complex plane by a lattice $\mathbb{C}/\Lambda$. Describe the action of the modular group on the four inequivalent spin structures on T.

Problem 4.2.✰ Describe how the mapping class group of a compact Riemann surface of genus $g \geq 2$ acts on the 2^{2g} inequivalent spin structures.

Problem 4.3. Let $\Gamma \subset \mathrm{SO}(n+1)$ be a finite group acting freely on the unit sphere $\mathrm{S}^n \subset \mathbb{R}^{n+1}$. Show that the quotient S^n/Γ admits a spin structure if and only if there is a subgroup $\widehat{\Gamma} \subset \mathrm{Spin}(n+1)$ such that the covering homomorphism $\widetilde{\mathrm{Ad}} : \mathrm{Spin}(n+1) \to \mathrm{SO}(n+1)$ restricts to an isomorphism $\widehat{\Gamma} \cong \Gamma$ (i.e., $-1 \not\in \widehat{\Gamma}$).

Problem 4.4. Let $m, a, b \in \mathbb{N}$ be natural numbers satisfying $1 \leq a, b < m$ with $(a, m) = (b, m) = 1$ and let $\Gamma \subset \mathrm{SO}(6)$ be the cyclic subgroup of order m generated by the following matrix

$$\mathrm{A} = \begin{pmatrix} \mathrm{R}\left(\frac{1}{m}\right) & & \\ & \mathrm{R}\left(\frac{a}{m}\right) & \\ & & \mathrm{R}\left(\frac{b}{m}\right) \end{pmatrix} \qquad \text{with} \qquad \mathrm{R}(\theta) = \begin{pmatrix} \cos 2\pi\theta & -\sin 2\pi\theta \\ \sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}.$$

Show that Γ acts freely on the unit sphere $\mathrm{S}^5 \subset \mathbb{R}^6$. For which m, a, b satisfying the above conditions does the quotient S^5/Γ admit a spin structure? And in those cases, how many inequivalent spin structures does it admit?

Problem 4.5.✰ Do the same for the subgroup $\Gamma \subset \mathrm{SO}(6)$ generated by

$$\mathrm{A} = \begin{pmatrix} \mathrm{R}\left(\frac{1}{m}\right) & & \\ & \mathrm{R}\left(\frac{r}{m}\right) & \\ & & \mathrm{R}\left(\frac{r^2}{m}\right) \end{pmatrix} \qquad \text{and} \qquad \mathrm{B} = \begin{pmatrix} 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \\ \mathrm{R}\left(\frac{3\ell}{n}\right) & 0 & 0 \end{pmatrix},$$

where $m, n, r, \ell \in \mathbb{N}$ and $n \equiv 0 \pmod 3$, $(n, m) = (\ell, n) = 1$ and $r^3 \equiv 1 \pmod m$. (Notice that this means that m has to be odd.)

Problem 4.6.✰ Classify all the finite subgroups $\Gamma \subset \mathrm{SO}(4)$ which act freely on the unit sphere $\mathrm{S}^3 \subset \mathbb{R}^4$ and for which the quotient S^3/Γ admits a spin structure. Classify the number of spin structures in the quotient.

Problem 4.7. Show that $\mathbb{RP}^n$ is orientable if and only if n is odd and show that it is spin if and only if $n = 3 \mod 4$.