Spin Geometry 2010

Tutorial Sheet 7

(Harder problems, if any, are adorned with a ✰.)

Problem 7.1. Show that $\mathrm{Spin}(5) \cong \mathrm{Sp}(2)$.

Problem 7.2. Show that in a spin manifold, the Dirac operator D squares to

$$D^2 = \nabla^*\nabla + \frac{1}{4}s\,,$$

where $\nabla^*\nabla$ is the covariant laplacian and s is the Ricci scalar.

Problem 7.3. Show that if a positive-definite riemannian spin manifold (M, g) admits Killing spinor fields, then it is Einstein.

Problem 7.4. Let (M, g) be Einstein. Let $\widetilde{M} = \mathbb{R}^+ \times M$, with metric $\widetilde{g} = dr^2 + \mu^2 r^2 g$. Show that for some value of μ (related to the Ricci scalar of g), $\widetilde{g}$ is Ricci-flat.

Problem 7.5.✰ Let (M, g) be a riemannian manifold and let $(\widetilde{M}, \widetilde{g})$ be its metric cone. Let $\xi = r\frac{\partial}{\partial r}$ be the Euler vector. Show that $\widetilde{\nabla}_X \xi = X$ for all vector fields X. Conversely, suppose that $(\widetilde{M}, \widetilde{g})$ is a riemmanian manifold and that ξ is a vector field such that $\widetilde{\nabla}_X \xi = X$ for every vector field X. Then show that $(\widetilde{M}, \widetilde{g})$ is the metric cone of some (M, g). (Only the converse is difficult.)

Problem 7.6. Let (M, g) be a six-dimensional positive-definite riemannian manifold whose metric cone $(\widetilde{M}, \widetilde{g})$ has $G_2 \subset SO(7)$ holonomy representation. Define $J : TM \to TM$ by $g(J(X), Y) = \phi(\xi, X, Y)$, where ξ the Euler vector and ϕ the G_2-invariant 3-form and everything is evaluated at $r = 1$. Show that J is an orthogonal almost complex structure and show that $(\nabla_X J)(X) = 0$ for all vector fields X. Is it possible for J to be parallel?

Problem 7.7. Let (M, g) be an odd-dimensional positive-definite riemannian manifold whose metric cone $(\widetilde{M}, \widetilde{g})$ is Kähler, with Kähler form ω and complex structure J. Let $\chi = J\xi$, $\theta = \iota_\xi \omega$ and $g(T(X), Y) = \omega(X, Y)$ define the Sasakian structure (M, g, χ, θ, T) on (M, g), obtained by restricting the relevant objects to $r = 1$. Show that χ is a unit-norm Killing vector, that $\theta(X) = g(\chi, X)$ and that

$$(\nabla_X T)(Y) = \theta(Y)X - g(X, Y)\chi\,.$$

Problem 7.8. Let (M, g) be a 7-dimensional positive-definite riemannian manifold whose metric cone $(\widetilde{M}, \widetilde{g})$ has $\mathrm{Spin}(7) \subset SO(8)$ holonomy representation. Let Ω denote the Cayley 4-form on $\widetilde{M}$ and let $\phi = \iota_\xi \Omega$ be its contraction with the Euler vector and pulled back to M via the embedding at $r = 1$. Show that $\nabla\phi = \star\phi$.

Problem 7.9. Let (M, g) be a positive-definite riemannian manifold whose metric cone $(\widetilde{M}, \widetilde{g})$ is hyperkähler with Euler vector ξ. Let $X_1 = -\frac{1}{2}I\xi$, $X_2 = -\frac{1}{2}J\xi$ and $X_3 = -\frac{1}{2}K\xi$, where I, J, K are the hyperkähler structure. Show that X_i are perpendicular to ξ and hence they are lifts to the cone of vector fields on M. Show that they are Killing vectors on M and that they define an infinitesimal action of $\mathfrak{so}(3)$.