Alternating Sign Matrices and Descending Plane Partitions

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Introduction

- Plane Partitions were introduced by Mac Mahon about a century ago. However Descending Plane Partitions (DPPs), as well as other variations on plane partitions (symmetry classes), were considered in the 80s. [Andrews]
- Alternating Sign Matrices (ASMs) also appeared in the 80s, but in a completely different context, namely in Mills, Robbins and Rumsey's study Dodgson's condensation algorithm for the evaluation of determinants.
- One of the possible formulations of the Alternating Sign Matrix conjecture is that these objects are in bijection (for every size n). (proved by Zeilberger in '96 in a slightly different form)

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T. Fonseca and P. Zinn-Justin: proof of the doubly refined Alternating Sign Matrix conjecture ('08).

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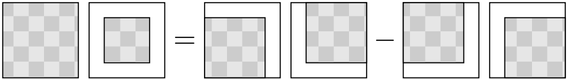
Today's talk is about the proof of another generalization of the ASM conjecture formulated in '83 by Mills, Robbins and Rumsey.

Iterative use of the Desnanot–Jacobi identity:



allows to compute the determinant of a $n \times n$ matrix by computing the determinants of the connected minors of size $1, \ldots, n$.

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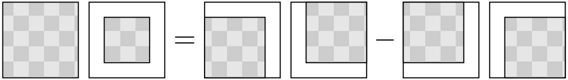

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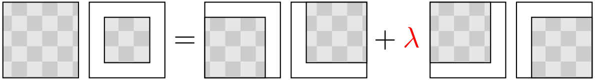

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What happens when we replace the minus sign with an arbitrary parameter?

Theorem (Robbins, Rumsey, '86)

If M is an $n \times n$ matrix, then

$$\det_\lambda M = \sum_{A \in \mathrm{ASM}(n)} \lambda^{\nu'(A)} (1+\lambda)^{\mu(A)} \prod_{i,j=1}^{n} M_{ij}^{A_{ij}}$$

Here $\mathrm{ASM}(n)$ is the set of $n \times n$ Alternating Sign Matrices, that is matrices such that in each row and column, the non-zero entries form an alternation of $+1$s and -1s starting and ending with $+1$.

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Example

For $n = 3$, there are 7 ASMs:

$$\mathrm{ASM}(3) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

$\mu(A)$ is the number of -1s in A.

$\nu'(A)$ is a generalization of the inversion number of A:

$$\nu'(A) = \sum_{\substack{1 \le i < i' \le n \\ 1 \le j' < j \le n}} A_{ij} A_{i'j'}$$

In what follows it is more convenient to consider another generalization of the inversion number, namely

$$\nu(A) = \nu'(A) - \mu(A) = \sum_{\substack{1 \le i \le i' \le n \\ 1 \le j' < j \le n}} A_{ij} A_{i'j'}$$

Finally, for future purposes define $\rho(A)$ to be the number of 0's to the left of the 1 in the first row of A.

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A Descending Plane Partition is an array of positive integers ("parts") of the form

$$\begin{array}{lllll} D_{11} & D_{12} \ldots\ldots\ldots\ldots\ldots\ldots D_{1,\lambda_1} \\ & D_{22} \ldots\ldots\ldots\ldots D_{2,\lambda_2+1} \\ & \quad \ddots \quad \ldots \\ & \qquad\quad D_{tt} \ldots D_{t,\lambda_t+t-1} \end{array}$$

such that

- The parts decrease weakly along rows, i.e., $D_{ij} \geq D_{i,j+1}$.
- The parts decrease strictly down columns, i.e., $D_{ij} > D_{i+1,j}$.
- The first parts of each row and the row lengths satisfy

$$D_{11} > \lambda_1 \geq D_{22} > \lambda_2 \geq \ldots \geq D_{t-1,t-1} > \lambda_{t-1} \geq D_{tt} > \lambda_t$$

Let $\mathrm{DPP}(n)$ be the set of DPPs in which each part is at most n, i.e., such that $D_{ij} \in \{1, \ldots, n\}$.

Example

For $n = 3$, there are 7 DPPs:

$$\mathrm{DPP}(3) = \left\{ \emptyset,\ \begin{matrix} 3 & 3 \\ & 2 \end{matrix},\ 2,\ 3\,3,\ 3,\ 3\,2,\ 3\,1 \right\}$$

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Define statistics for each $D \in \mathrm{DPP}(n)$ as:

$\nu(D) =$ number of parts of D for which $D_{ij} > j - i$,

$\mu(D) =$ number of parts of D for which $D_{ij} \leq j - i$,

$\rho(D) =$ number of parts equal to n in (necessarily the first row of) D.

DPP enumeration

Theorem (Andrews, 79)

The number of DPPs with parts at most n is:

$$|\mathrm{DPP}(n)| = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} = 1, 2, 7, 42, 429 \ldots$$

The Alternating Sign Matrix conjecture

The following result was first conjectured by Mills, Robbins and Rumsey in '82:

Theorem (Zeilberger, '96; Kuperberg, '96)

The number of ASMs of size n is

$$|\mathrm{ASM}(n)| = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!} = 1, 2, 7, 42, 429 \dots$$

NB: a third family is also known to have the same enumeration as ASMs and DPPs: TSSCPPs. In fact, Zeilberger's proof consists of a (non-bijective) proof of equienumeration of ASMs and TSSCPPs.

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Theorem (Behrend, Di Francesco, Zinn-Justin, '11)

The sizes of $\{A \in \mathrm{ASM}(n) \mid \nu(A) = p,\ \mu(A) = m,\ \rho(A) = k\}$ *and* $\{D \in \mathrm{DPP}(n) \mid \nu(D) = p,\ \mu(D) = m,\ \rho(D) = k\}$ *are equal for any* n*,* p*,* m *and* k*.*

Equivalently, if one defines generating series:

$$Z_{\mathrm{ASM}}(n,x,y,z) = \sum_{A \in \mathrm{ASM}(n)} x^{\nu(A)}\, y^{\mu(A)}\, z^{\rho(A)}$$

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$$Z_{\mathrm{ASM/DPP}}(3, x, y, z) = 1 + x^3 z^2 + x + x^2 z^2 + xz + x^2 z + xyz$$

Strategy: write the two generating series as determinants:

modified Izergin

$$Z_{\mathrm{ASM}}(n,x,y,z) = \det M_{\mathrm{ASM}}(n,x,y,z)$$
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LGV

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Let $6\mathrm{VDW}(n)$ be the set of all configurations of the six-vertex model on the $n \times n$ grid with DWBC, i.e., decorations of the grid's edges with arrows such that:

- The arrows on the external edges are fixed, with the horizontal ones all incoming and the vertical ones all outgoing.
- At each internal vertex, there are as many incoming as outgoing arrows.

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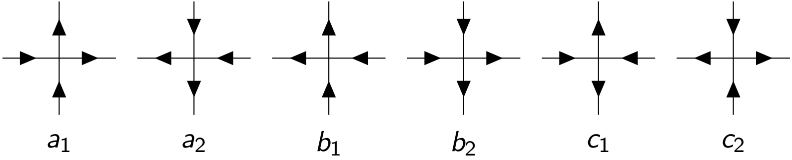

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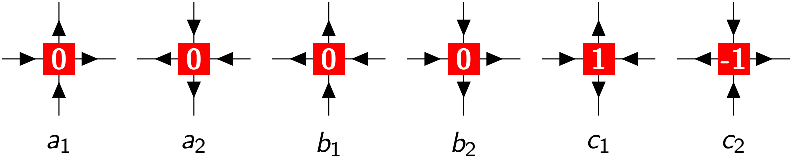
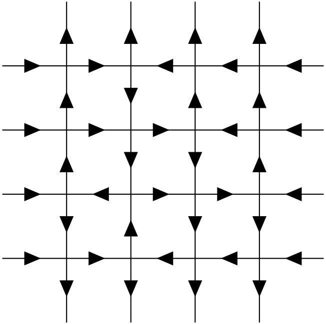


The bijection from $6\mathrm{VDW}(n)$ to $\mathrm{ASM}(n)$



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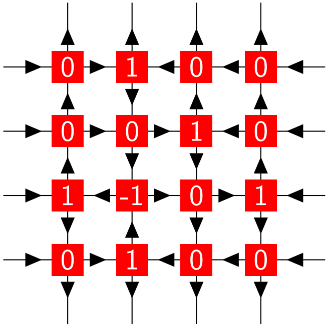

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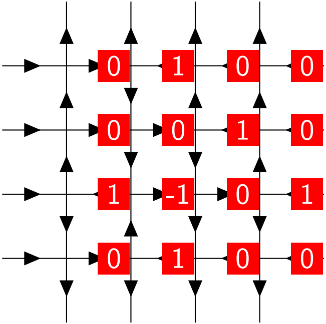

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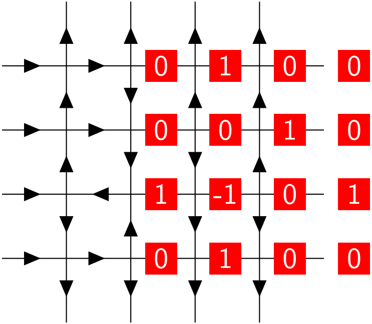

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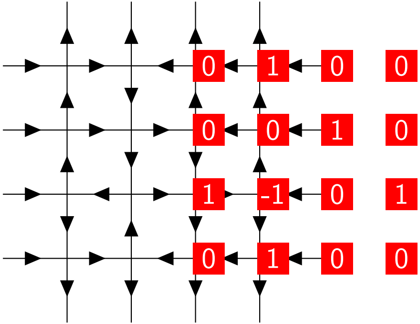

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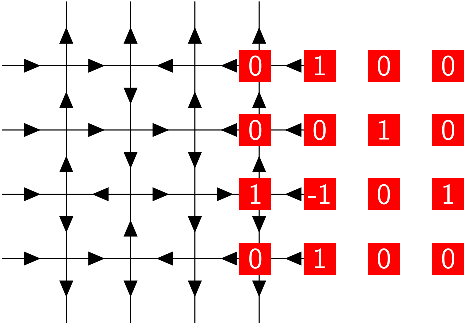

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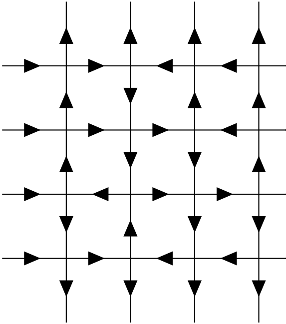

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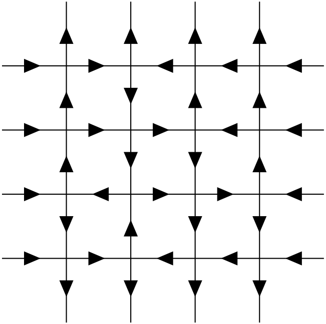

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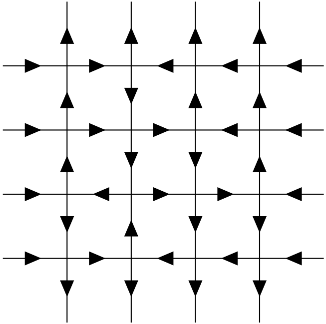

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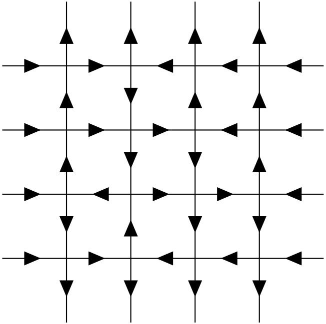

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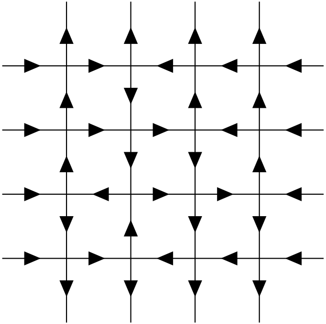

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Statistics

Statistics also have a nice interpretation in terms of the six-vertex model: if $A \in \mathrm{ASM}(n) \mapsto C \in 6\mathrm{VDW}(n)$,

$$\mu(A) = \frac{1}{2}\left((\text{number of vertices of type } c \text{ in } C) - n\right)$$
$$\nu(A) = \frac{1}{2}(\text{number of vertices of type } a \text{ in } C)$$

Define the six-vertex partition function of the six-vertex model with DWBC to be:

$$Z_{6\mathrm{VDW}}(u_1,\ldots,u_n;v_1,\ldots,v_n) = \sum_{C\in 6\mathrm{VDW}(n)} \prod_{i,j=1}^{n} C_{ij}(u_i,v_j)$$

where the u_i (resp. the v_j) are parameters attached to each row (resp. a column), and C_{ij} is the type of configuration at vertex (i,j).

$$a(u,v) = uq - \frac{1}{vq}, \qquad b(u,v) = \frac{u}{q} - \frac{q}{v}, \qquad c(u,v) = \left(q^2 - \frac{1}{q^2}\right)\sqrt{\frac{u}{v}}$$

Based on Korepin's recurrence relations for Z_{6VDW}, Izergin found the following determinant formula:

Theorem (Izergin, '87)

$$Z_{\mathrm{6VDW}}(u_1,\ldots,u_n;v_1,\ldots,v_n) \propto \frac{\det\limits_{1\le i,j\le n}\left(\dfrac{1}{a(u_i,v_j)b(u_i,v_j)}\right)}{\prod\limits_{1\le i<j\le n}(u_j-u_i)(v_j-v_i)}$$

Problem: what happens in the homogeneous limit $u_1,\ldots,u_n,v_1,\ldots,v_n \longrightarrow r$?

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Theorem (Izergin, '87)

$$Z_{\mathrm{6VDW}}(u_1, \ldots, u_n; v_1, \ldots, v_n) \propto \frac{\det\limits_{1\le i,j\le n} \left(\dfrac{1}{a(u_i, v_j)b(u_i, v_j)} \right)}{\prod\limits_{1\le i<j\le n} (u_j - u_i)(v_j - v_i)}$$

Problem: what happens in the homogeneous limit $u_1, \ldots, u_n, v_1, \ldots, v_n \to r$?

The "naive" homogeneous limit:

$$Z_{\mathrm{6VDW}}(r,\ldots,r;r,\ldots,r) \propto \det_{0\le i,j\le n-1} \frac{\partial^{i+j}}{\partial u^i \partial v^j}\left(\frac{1}{a(u,v)b(u,v)}\right)_{|u,v=r}$$

$$\propto \det_{0\le i,j\le n-1} \frac{\partial^{i+j}}{\partial u^i \partial v^j}\left(\frac{1}{uv-q^2}-\frac{1}{uv-q^{-2}}\right)_{|u,v=r}$$

Define L_{ij} to be the $n \times n$ lower-triangular matrix with entires $\binom{i}{j}$, and D to be the diagonal matrix with entries $\left(\frac{qr-q^{-1}r^{-1}}{q^{-1}r-qr^{-1}}\right)^i$, $i = 0, \dots, n-1$.

Proposition (Behrend, Di Francesco, Zinn-Justin, '11)

$$Z_{\mathrm{6VDW}}(r, \dots, r; r, \dots, r) \propto \det\left(I - \frac{r^2 - q^{-2}}{r^2 - q^2} DLDL^T\right)$$

Proof: write the determinant as $\det(A_+ - A_-)$, note that $A_\pm$ is up to a diagonal conjugation $\frac{1}{r^2-q^{\pm 2}} D_\pm L D_\pm L^T$, pull out $\det A_+$ and conjugate $I - A_- A_+^{-1} \dots$

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Proposition (Behrend, Di Francesco, Zinn-Justin, '11)

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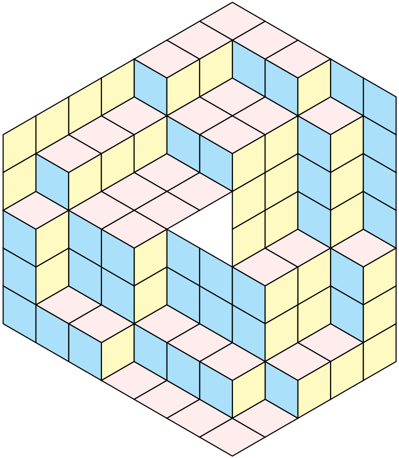
Proof: write the determinant as $\det(A_+ - A_-)$, note that $A_\pm$ is up to a diagonal conjugation $\frac{1}{r^2 - q^{\pm 2}} D_\pm L D_\pm L^T$, pull out $\det A_+$ and conjugate $I - A_- A_+^{-1} \ldots$

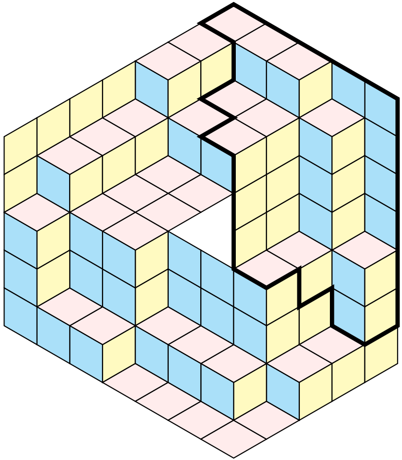
Rewriting the previous proposition in terms of Boltzmann weights a, b, c, and then switching to $x = (a/b)^2$, $y = (c/b)^2$, we finally find $Z_{\mathrm{ASM}}(n, x, y, 1) = \det M_{\mathrm{ASM}}(n, x, y, 1)$ with

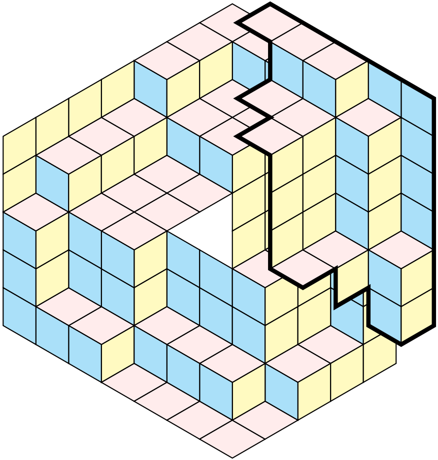
$$M_{\mathrm{ASM}}(n, x, y, 1)_{ij} = (1-\omega)\delta_{ij} + \omega \sum_{k=0}^{\min(i,j)} \binom{i}{k}\binom{j}{k} x^k\, y^{i-k}$$

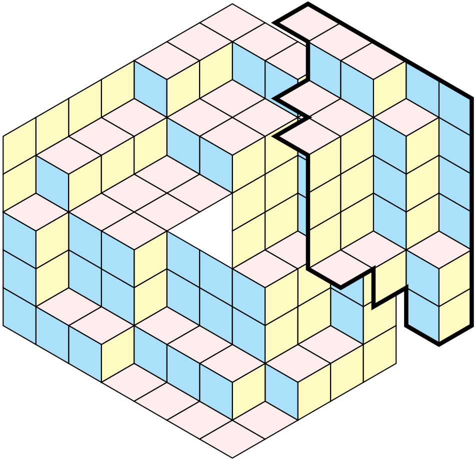
with $i, j = 0, \ldots, n-1$ and ω a solution of

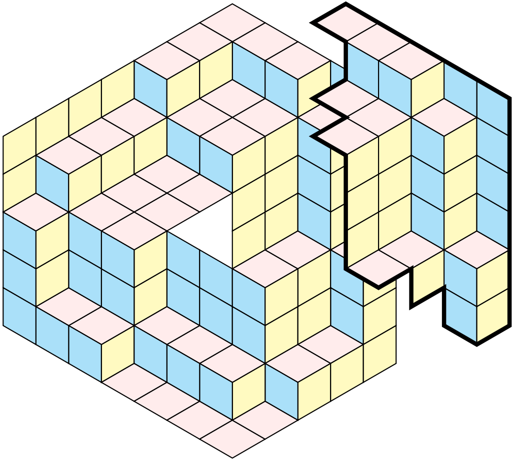
$$y\omega^2 + (1 - x - y)\omega + x = 0$$

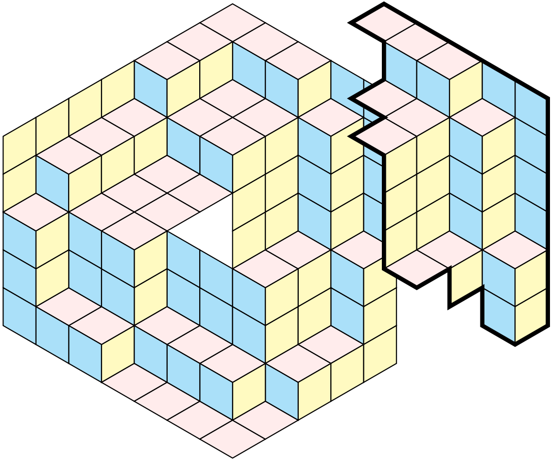


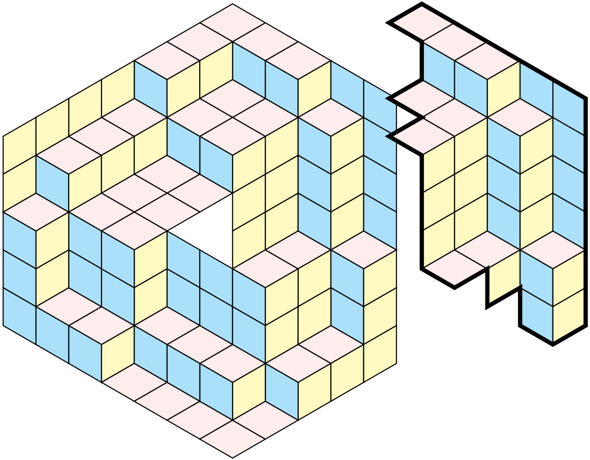


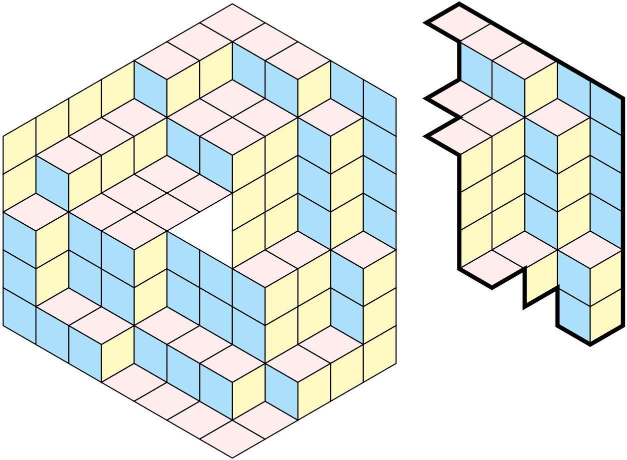


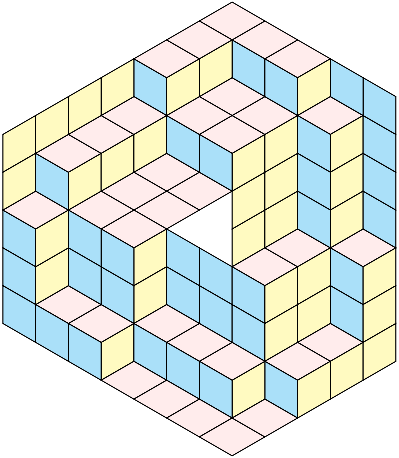


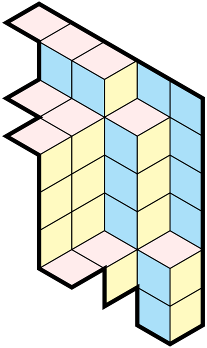


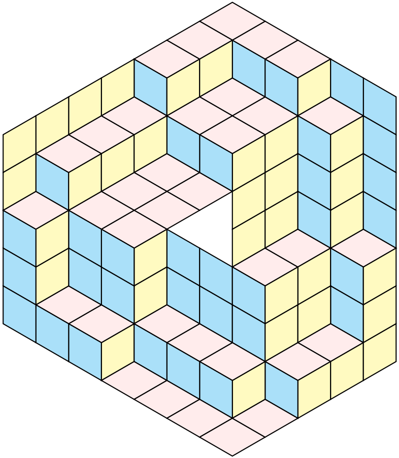


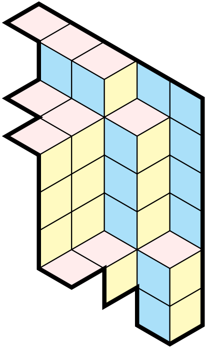


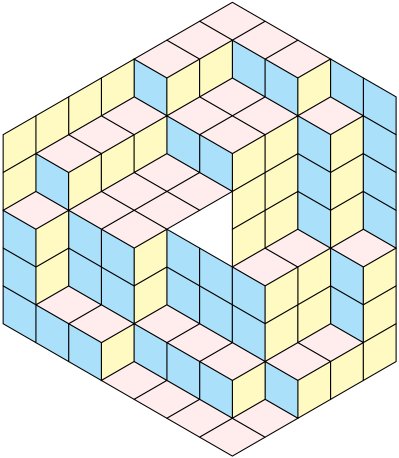


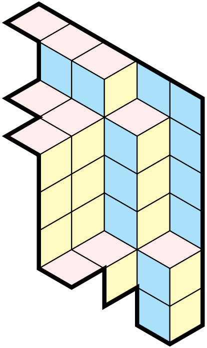


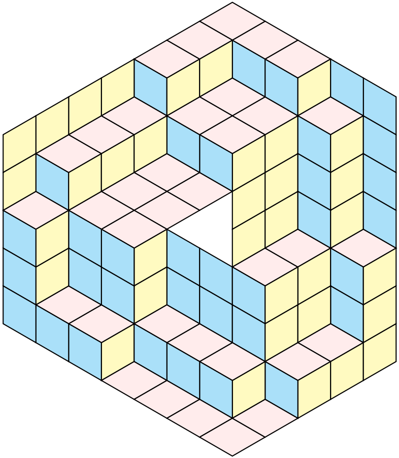


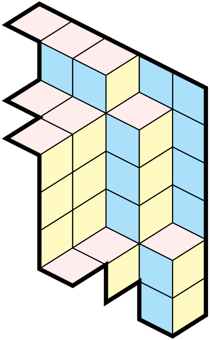


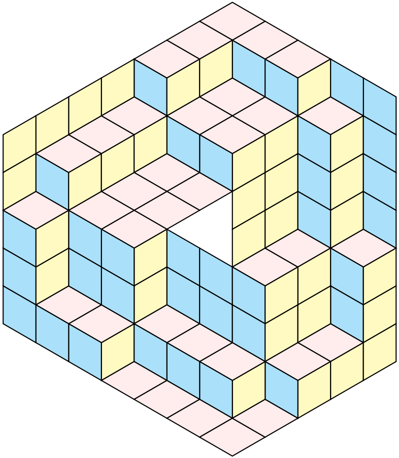


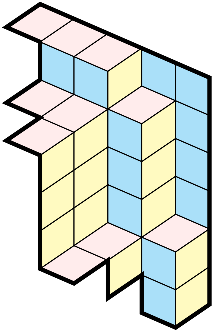


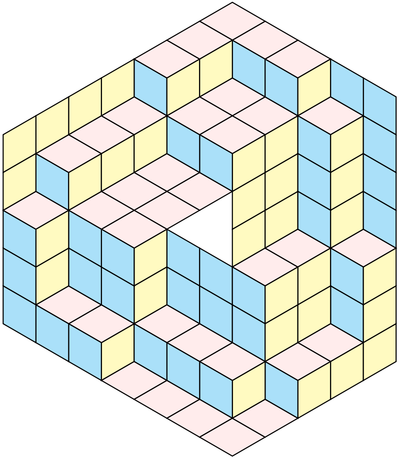


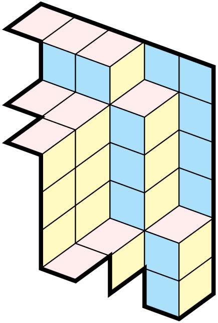


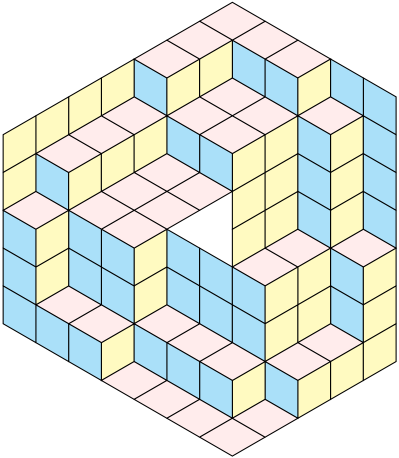


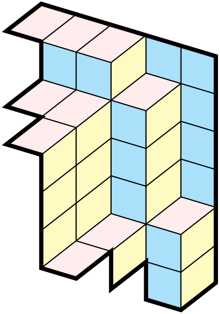


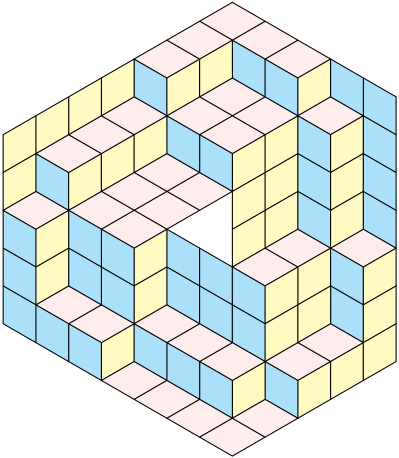


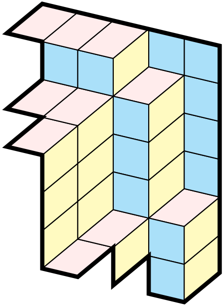


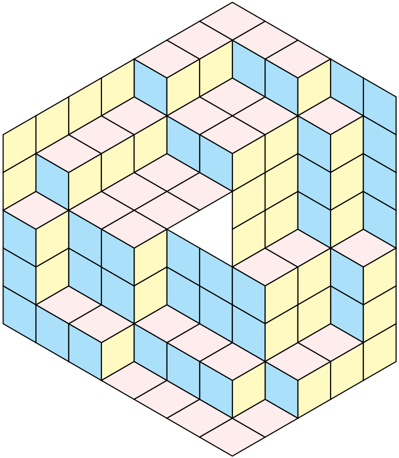


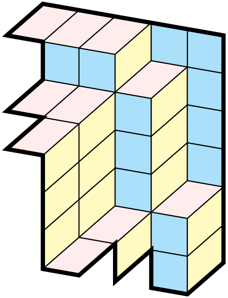


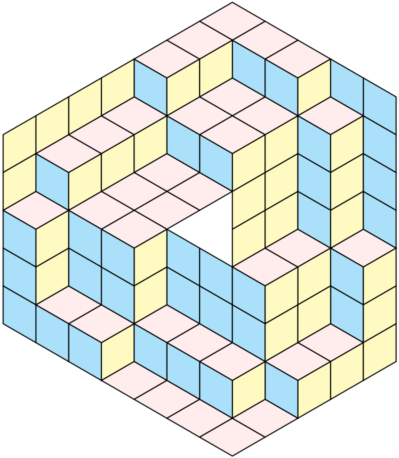


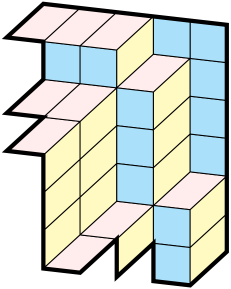


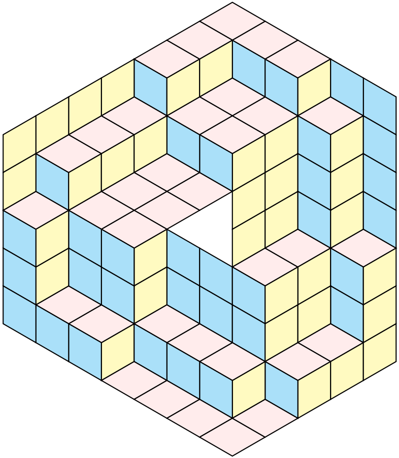


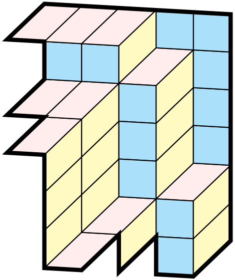


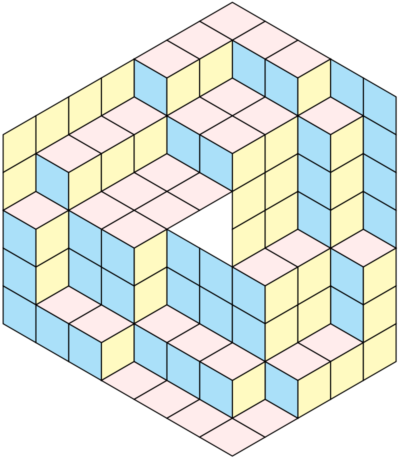


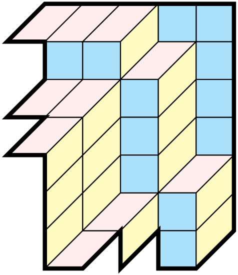


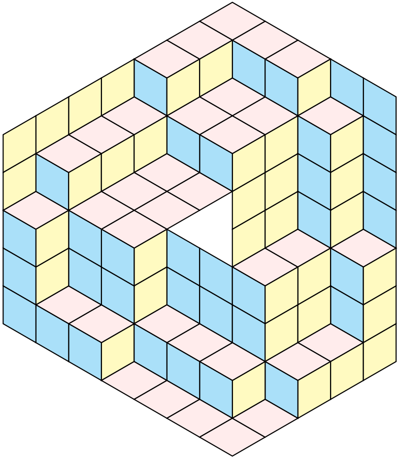


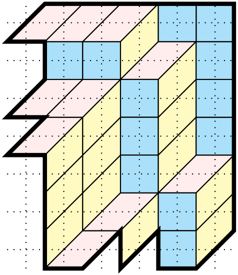


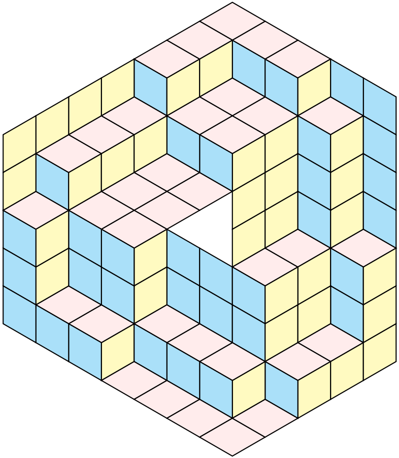


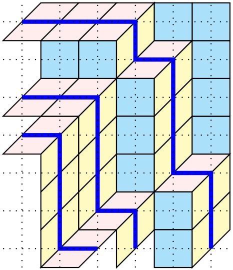


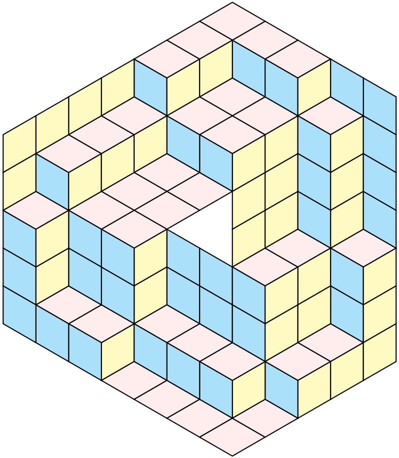


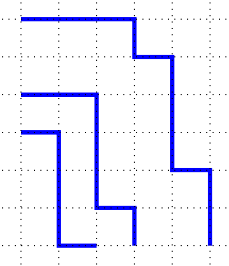


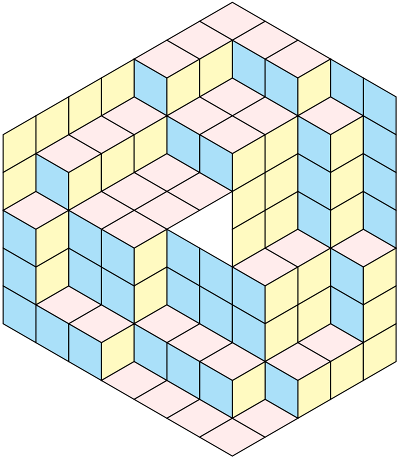


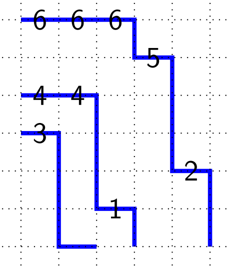








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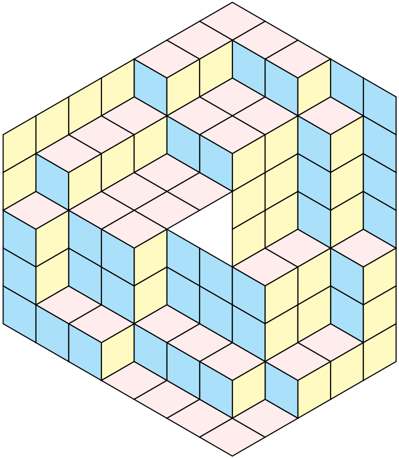

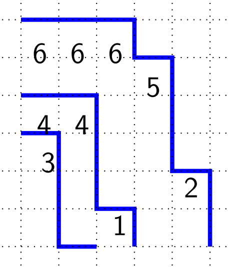
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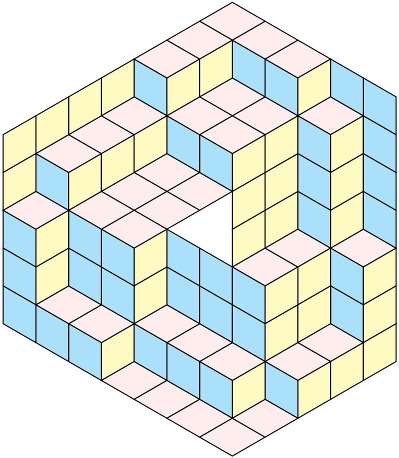

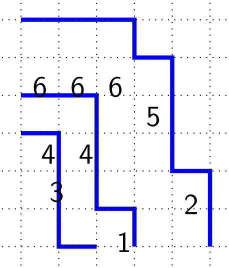
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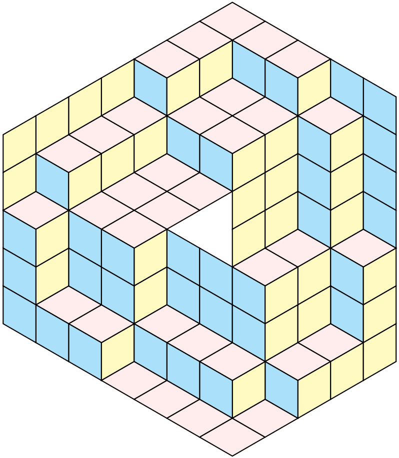

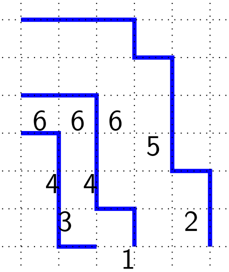
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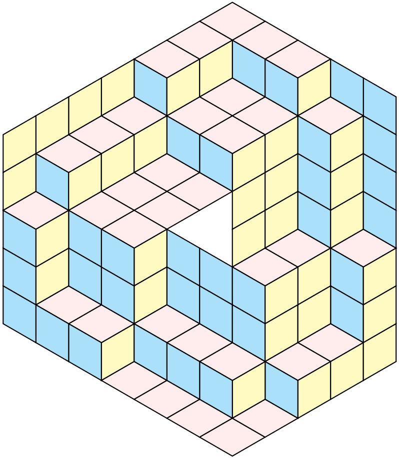

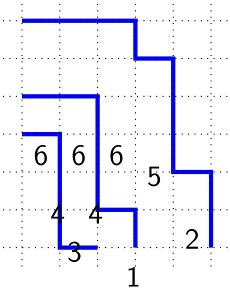
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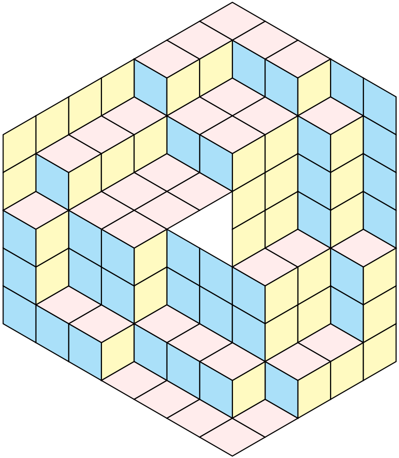

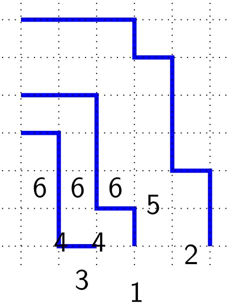
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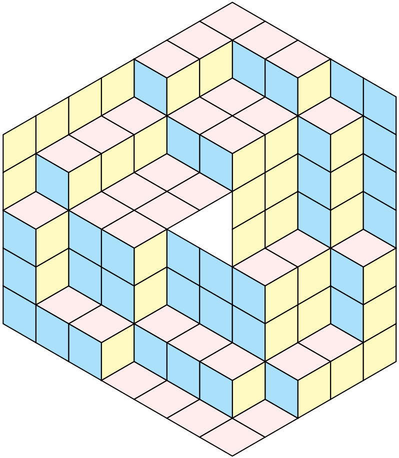

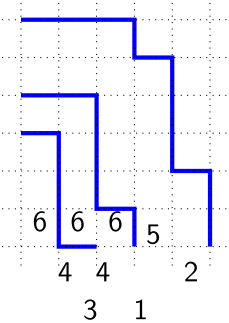
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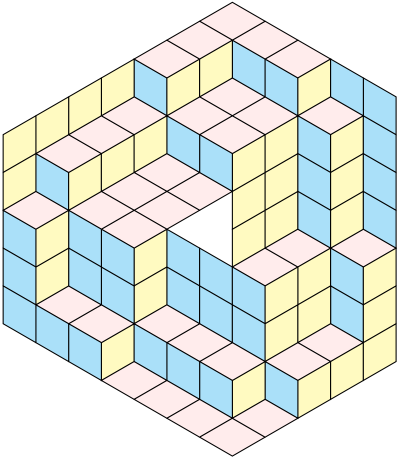

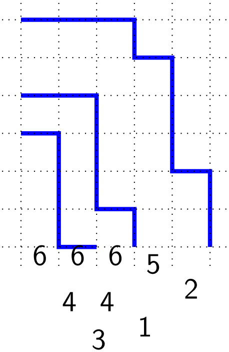
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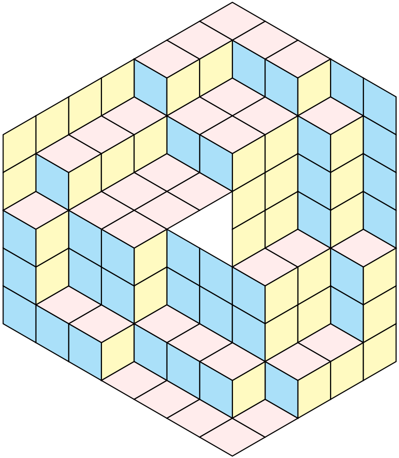

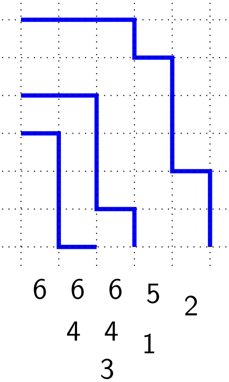
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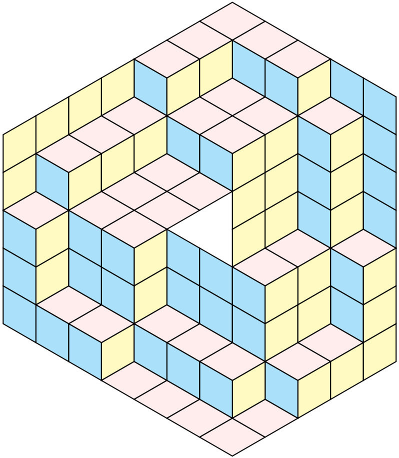

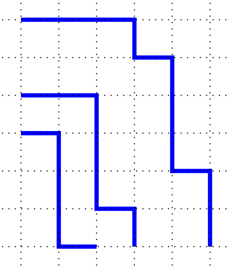

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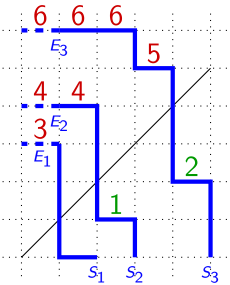
Statistics

Statistics also have a nice interpretation in terms of Nonintersecting lattice paths (NILPs):

$D =$ 

$\nu(D) = 7$

$\mu(D) = 2$

LGV formula / free fermions

NILPS are (lattice) free fermions:

Number of NILPs from S_i to E_i, $i = 1, \ldots, n$

$$= \det_{i,j=1,\ldots,n} (\text{Number of (single) paths from } S_i \text{ to } E_j)$$

and similarly with weighted sums.

LGV formula / free fermions

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Number of NILPs from S_i to E_i, $i = 1, \ldots, n$

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and similarly with weighted sums.

Here we are also summing over endpoints and the number of paths ("grand canonical partition function"):
$Z_{\mathrm{DPP}}(n, x, y, 1) = \det M_{\mathrm{DPP}}(n, x, y, 1)$ with

$$M_{\mathrm{DPP}}(n, x, y, 1) = \delta_{ij} + \sum_{k=0}^{i-1} \sum_{\ell=0}^{\min(j,k)} \binom{j}{\ell} \binom{k}{\ell} x^{\ell+1} y^{k-\ell}$$

Note that the second term is a product of two discrete transfer matrices...

We have

$$(I-S)M_{\mathrm{DPP}}(n,x,y,1)(I+(\omega-1)S^T) \\ = (I+(x-\omega y-1)S)M_{\mathrm{ASM}}(n,x,y,1)(I-S^T)$$

where $I_{ij}=\delta_{i,j}$ and $S_{ij}=\delta_{i,j+1}$.

Therefore,

$$Z_{\mathrm{DPP}}(n,x,y,1)=Z_{\mathrm{ASM}}(n,x,y,1)$$

We have

$$(I - S)M_{\mathrm{DPP}}(n, x, y, 1)(I + (\omega - 1)S^T) \\ = (I + (x - \omega y - 1)S)M_{\mathrm{ASM}}(n, x, y, 1)(I - S^T)$$

where $I_{ij} = \delta_{i,j}$ and $S_{ij} = \delta_{i,j+1}$.

Therefore,

$$Z_{\mathrm{DPP}}(n, x, y, 1) = Z_{\mathrm{ASM}}(n, x, y, 1)$$

We are working on various generalizations:

- At least one more statistic can be introduced: the double refinement. For ASMs this consists in recording the positions of the 1's on both the first row and last row.
- There are symmetry operations on ASMs and DPPs. For example, there is an operation * which for ASMs is symmetry wrt a vertical axis, and for DPPs viewed as rhombus tilings is refelection in any of the three lines bisecting the central triangular hole.
  De Gier, Pyatov and Zinn-Justin have proved in '09 a conjecture of Mills, Robbins and Rumsey concerning these.
  The proof can probably be simplified and the result generalized.
  Other symmetries?

We are working on various generalizations:

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- There are symmetry operations on ASMs and DPPs. For example, there is an operation * which for ASMs is symmetry wrt a vertical axis, and for DPPs viewed as rhombus tilings is refelection in any of the three lines bisecting the central triangular hole.

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