Extremal black holes and near-horizon geometry

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Motivation

- Higher dimensional black holes of interest in string theory and the gauge/gravity duality (AdS/CFT) for variety of reasons.
- Classification of $D > 4$ *stationary* black hole solns to Einstein's eqs. Asympt flat, KK and Anti de Sitter (AdS) all of interest.
- *Asymptotically flat vacuum solutions* esp important. Arise as a limit of other cases, e.g. black holes localised in KK dims.

No-hair/uniqueness theorem [Israel, Carter, Robinson, Hawking '70s]

The only asymptotically flat stationary black hole solution to the $D = 4$ vacuum Einstein equations is the Kerr metric (M, J). Spatial horizon topology S^2.

Motivation

- $D > 4$: much richer space of stationary black hole solutions to Einstein's equations. Black hole *non-uniqueness*.
- Asymptotically flat vacuum black holes: Myers-Perry S^{D-2} (analogue of Kerr), black rings $S^1 \times S^2$ [Emparan, Reall '01]
- Key issues: horizon topology? rotational symmetries? Number/moduli space of solutions?
- *Extremal* black holes provide a simplified setting to address some of these questions. May give clues to general problem.

Motivation – extremal black holes

- Extremal black holes ($\kappa = 0$) special in quantum gravity – Hawking temperature $T_H = \frac{\hbar\kappa}{2\pi}$ vanishes.

- Tend to admit simple statistical derivation of Bekenstein Hawking entropy $S_{BH} = \frac{A}{4\hbar}$ (e.g. in String Theory).

- Extremal case often excluded from GR theorems. E.g. no-hair theorem for *extremal* Kerr ($J = M^2$) only recently shown.

- Extremal black holes have a well defined notion of a "near-horizon geometry". Important in string theory.

Higher dimensional black holes

- Uniqueness of asympt flat *static* black holes [Gibbons et al '02]. Schwarzschild-Tangherlini is unique *vacuum* soln: $\mathrm{Ric}(g) = 0$

$$g = -\left(1 - \frac{2M}{r^{D-3}}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r^{D-3}}\right)} + r^2 d\Omega^2_{D-2}$$

- Interesting asymptotically flat solutions must be non-static. Typically this means they *rotate* – complicated!

- *Conserved charges*: mass M, angular momenta J_i where $i = 1, \ldots, \mathrm{rank}\; SO(D-1) = [(D-1)/2]$, Maxwell charges Q.

- Black hole *non-uniqueness*: fixing these conserved charges insufficient to fix black hole solution.

Weyl black holes

- *Weyl solutions*: vacuum black holes with $R \times U(1)^{D-3}$ sym. Compatible with asymptotic flatness only for $D = 4, 5$. [Emparan, Reall '01]
- Einstein eqs reduce to integrable non-linear σ-model on 2d orbit space $\mathcal{M}/(R \times U(1)^{D-3})$.
- This has allowed much progress in $D = 5$ culminating in:

Uniqueness theorem [Hollands, Yazadjiev '07]

There is at most one $D = 5$ asymptotically flat, non-extremal vacuum black hole with $R \times U(1)^2$ symmetry, for given M, J_i and *rod structure* (orbit space data).

Note: method is same as used for 4D black holes with $R \times U(1)$ symmetry. [Carter, Robinson '70s, Mazur '83]

General results/questions

- **Horizon Topology**. Let H be a spatial section of event horizon – closed orientable $(D-2)$-manifold.
  - Einstein eqs $\implies$ Yamabe$(H) > 0$ [Galloway, Schoen '05].
  - Spatial topology at infinity $\implies$ H is *cobordant* to S^{D-2}

  What H are actually realised by black hole solutions?

- **Rigidity Theorem**: any *stationary* rotating black hole solution must have rotational symmetry: isometry $R \times U(1)^s$ for $s \geq 1$.
  [Hawking '72; Hollands, Ishibashi, Wald '06; Isenberg, Moncrief '83 '08]

  Asymptotic flat $\implies$ $s \leq \mathrm{rank}\, SO(D-1) = [(D-1)/2]$.
  Known solns saturate upper bound – e.g. $D = 5$ with $s = 2$

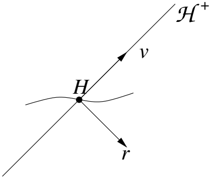
  Are there black hole solutions with $s = 1$?

Extremal (Killing) horizons

- Rigidity theorem $\implies$ event horizon is a *Killing horizon*: i.e. a null hypersurface with a normal Killing field $V = \frac{\partial}{\partial t} + \Omega_i \frac{\partial}{\partial \phi^i}$

- Near Killing horizon use Gaussian null coords. $V = \frac{\partial}{\partial v}$, horizon $r = 0$, x^a coords on *compact* cross-section H.

$$g = r\, f(r,x) dv^2 + 2dvdr + 2r\, h_a(r,x) dvdx^a + \gamma_{ab}(r,x) dx^a dx^b$$



- Surface gravity κ defined by $d(V^2) = -2\kappa V$ on horizon. Extremal horizon $\iff$ $\kappa = 0$ $\iff$ $f(r,x) = r\, F(r,x)$.

Near-horizon geometry

- Metric near an extremal horizon in Gaussian null coordinates:

$$g = r^2 F(r,x)dv^2 + 2dvdr + 2r\, h_a(r,x)dvdx^a + \gamma_{ab}(r,x)dx^a dx^b$$

- Near-horizon limit [Reall '02]: $v \to v/\epsilon$, $r \to \epsilon r$ and $\epsilon \to 0$. Limit is *near-horizon geometry* (NHG).

$$g_{NH} = r^2 F(x)dv^2 + 2dvdr + 2r\, h_a(x)dvdx^a + \gamma_{ab}(x)dx^a dx^b$$

- Near-horizon data (γ_{ab}, h_a, F) all defined on H, r-dependence fixed.

- New symmetry: $v \to v/\lambda$, $r \to \lambda r$. Together with $v \to v + c$ these form 2d non-abelian group.

General strategy

- NHG of extremal black hole *solution* to some theory of gravity, must also be a solution. Classify *NHG solutions*!
- Einstein eqs for NHG equivalent to Einstein-like eqs on H. Problem of compact Riemannian geometry in $D-2$ dims.
- Gives potential horizon *topologies* and *geometries* of full extremal black hole solns. Can rule out black hole topologies.
- Data outside black hole horizon lost. Existence of NHG soln does not guarantee existence of corresponding black hole.

Application to black hole classification

- Aim: use NHG classification to derive corresponding extremal black hole classification.

- Has been achieved in some cases where extra structures constrain exterior of black hole. Otherwise very difficult!

- *Supersymmetric* black holes in minimal supergravities. 4D: multi Reissner-Nordström. 5D & $H = S^3$: BMPV black hole.
  [Reall '02, Chrusciel, Reall, Tod '05]

- $D = 4, 5$ asymptotically flat *extremal vacuum* black holes with $R \times U(1)^{D-3}$-symmetry (Weyl solutions) [Figueras, JL '09].

  4D: uniqueness of extremal Kerr. Fills a gap in no-hair thrm!
  [Meinel et al '08; Amsel et al '09; Figueras, JL '09; Chrusciel, Nguyen '10]

Examples

- Einstein-Maxwell: extremal Reissner-Nordstrom $M = Q$, near-horizon limit is $AdS_2 \times S^2$ (homogeneous, static)

$$ds^2 = Q^2[-r^2 dv^2 + 2dvdr + d\Omega_2^2]$$

- Vacuum: extremal Kerr $J = M^2$, NH limit is S^2-bundle over AdS_2. Isometry $SO(2,1) \times U(1)$ (inhomogeneous, non-static)

$$\begin{aligned} ds^2 &= \frac{(1+\cos^2\theta)}{2}\left[-\frac{r^2dv^2}{2a^2} + 2dvdr + a^2 d\theta^2\right] \\ &\quad + \frac{2a^2\sin^2\theta}{1+\cos^2\theta}\left(d\phi + \frac{rdv}{2a^2}\right)^2 \end{aligned}$$

- $D > 4$ many *extremal* examples... all have $SO(2,1)$ isometry!

Near-horizon symmetries

- $SO(2,1)$ NHG symmetry not obvious! In general only have 2d symmetry in (v,r) plane.

Theorem [Kunduri, JL, Reall '07]

Consider extremal black hole soln to Einstein-Maxwell-CS-scalar (+higher derivatives) with $R \times U(1)^{D-3}$ symmetry. NHG has $SO(2,1) \times U(1)^{D-3}$ symmetry:

$$g_{NH} = \Gamma(\rho)[-r^2 dv^2 + 2dvdr] + d\rho^2 + \gamma_{ij}(\rho)(d\phi^i + k^i rdv)(d\phi^j + k^j rdv)$$

$$F_{NH} = d[e\, rdv + b_i(\rho)\,(d\phi^i + k^i rdv)]$$

- Applicable to asympt flat/AdS cases only in $D = 4, 5$. Note if $D > 5$ then $D - 3 > [(D-1)/2]$.

Near-horizon symmetries

- $D > 5$ known examples outside validity of theorem. NHG Myers-Perry and some new examples later!
- NHG of extremal Myers-Perry has $SO(2,1) \times U(1)^n$ symmetry with $n = [(D-1)/2]$. If $J_i = J$ then $U(1)^n \to U(n)$.
- $D > 5$ vacuum: can also prove $SO(2,1)$ for cohomogeneity-1 *non-abelian* rotational sym G, such that $U(1)^{[(D-1)/2]} \subset G$ [Figueras, Kunduri, JL, Rangamani '08]

Classifying (vacuum) near-horizon geometries

- Focus on vacuum Einstein eqs $R_{\mu\nu} = \Lambda g_{\mu\nu}$ with $\Lambda \leq 0$. For NHG *equivalent* to solving eqs on H (recall dim $H = D - 2$):

$$Ric(\gamma)_{ab} = \frac{1}{2} h_a h_b - \nabla_{(a} h_{b)} + \Lambda \gamma_{ab}$$

- Difficult to solve in general. 4D: general *axisymmetric* solution gives NHG of extremal Kerr/Kerr-AdS, $H = S^2$.
  [Hajicek '73; Lewandowski, Pawlowski '03; Kunduri, JL '08]

- Topology $\Lambda = 0$ (& dom energy). 4D: $\chi(H) = \int_H \frac{R_\gamma}{4\pi} > 0$.
  $D > 4$: can show Yamabe$(H) > 0$ directly [JL unpublished].

$D = 5$ vacuum near-horizon geometries

[Kunduri, JL '08]

- Classification of vacuum $U(1)^2$-NHG. Horizon eqs reduce to ODEs on $H/U(1)^2 \cong$ interval. Can be solved for $\Lambda = 0$!
- $\Lambda = 0$ results: all such vacuum NHG solns arise from *known* extremal black holes (either asympt flat or KK)!
  - S^3: Myers-Perry; slow/fast KK black holes [Rasheed '95]
  - $L(p, q)$ (Lens spaces): all quotients of S^3 case above
  - $S^1 \times S^2$: black ring [Pomeransky, Senkov '06]; boosted Kerr string.
- $\Lambda < 0$ still open! Example of NHG of black ring in AdS_5?

Electro-vacuum near-horizon geometries

- $D = 4$: results generalise to Einstein-Maxwell-Λ [Kunduri, JL '08]. NHG extremal Kerr-Newman-AdS in only axisymmetric soln.

- $D = 5$: adding Maxwell field complicates classification. No electro-magnetic duality; local dipole charges...

  Restrict to *minimal supergravity* (Einstein-Maxwell-CS-Λ):

  - *supersymmetric* NHG classified for $\Lambda \leq 0$ (assuming $U(1)^2$). $\Lambda < 0 \implies$ no supersymmetric AdS_5 black rings! [Reall '01; Kunduri, JL, Reall '06]

  - Non-supersymmetric $U(1)^2$-NHG with $\Lambda = 0$. Can exploit hidden symmetry of supergravity to solve classification. Reduces to complicated algebraic problem. [Kunduri, JL to appear]

$D > 5$ near-horizon geometries

- Can determine all vacuum $U(1)^{D-3}$–NHG [Hollands, Ishibashi '09]. $H = S^3 \times T^{D-5}$, $L(p,q) \times T^{D-5}$, $S^2 \times T^{D-4}$ (c.f. $D = 4, 5$)

- Classification of NHG with asymptotically flat rotational symmetry $U(1)^{[(D-1)/2]} < U(1)^{D-3}$ not yet possible.

- Myers-Perry black holes & MP-black strings give examples of vacuum NHG with $H = S^{D-2}$ & $S^1 \times S^{2n}$ and $U(1)^{[(D-1)/2]}$ [Figueras, Kunduri, JL, Rangamani '08]

- Can we find new examples with appropriate symmetry? Focus on non-static and vacuum near-horizon geometries.

New infinite class of $D > 5$ near-horizon geometries

[Kunduri, JL'10]

- $D = 2n + 2$: have found new NHG solutions to $R_{\mu\nu} = \Lambda g_{\mu\nu}$ with $\leq n = [(D-1)/2]$ commuting rotational KVF.

- $S^2 \to H \to K$: H is inhomogeneous S^2-bundle over any compact positive Kähler-Einstein base manifold K.

- For fixed base, specified by one continuous param L (spin) and an integer $m > p > 0$ ($p =$ Fano index of K).

- All H cobordant to S^{2n} and positive Yamabe type. *Candidates for NHG of new black holes!*

New infinite class of $D > 5$ near-horizon geometries

[Kunduri, JL'10]

- "Calabi Ansatz": (g_K, J) is Kähler-Einstein structure on base K, where $J = \frac{1}{2} d\sigma$, $Ric(g_K) = 2n g_K$ and

$$\gamma_{ab} dx^a dx^b = d\rho^2 + B(\rho)^2 (d\phi + \sigma)^2 + A(\rho)^2 g_K$$
$$h_a dx^a = C(\rho)(d\phi + \sigma) + \lambda'(\rho) d\rho$$

New solns are of this form (not necessarily most general)

- Local form of solns, with $K = CP^{n-1}$, include NHG of Myers-Perry $a_i = a$. $H = S^{2n}$ with $SU(n) \times U(1)$ sym.

$D = 6$ near-horizon geometries

- $D = 6$: H is S^2 bundle over $\mathbb{CP}^1 \cong S^2$. Topology classified by $\pi_1(SO(3)) = Z_2$. One non-trivial bundle.

- $m > 2$: m even $S^2 \times S^2$; m odd $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ (i.e. 1-pt blow-up of $\mathbb{CP}^2$). Metrics cohomogeneity-1 with $SU(2) \times U(1)$ sym.

- Remark 1: analogous to Page instanton, which is an *Einstein* metric on $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ with $m = 1$.

- Remark 2: s.c. closed 4-manifolds, $U(1)^2$-action & cobordant to S^4 must be connected sums of S^4, $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, $S^2 \times S^2$.
  [Orlik, Raymond '70]

$D > 6$ near-horizon geometries

- $D > 6$: different m gives different topology. Infinite number of topologies for fixed KE base. Many choices for KE base...

- If KE base has no (continuous) isometries get NHG with exactly $U(1)$ rotational symmetry!

  E.g. $KE = \mathbb{CP}^2 \# k\, \overline{\mathbb{CP}^2}$ for $4 \leq k \leq 8$. Further $k \geq 5$ have moduli space: extra continuous parameters.

- If there are corresponding black holes must have $R \times U(1)$ symmetry. Saturate lower bound of rigidity theorem!

Open problems

- Complete classification of 5D vacuum $R \times U(1)^2$-black holes . Open for both extremal and non-extremal – NHG cannot help!
- 5D BH/NHG with exactly $U(1)$ rotational symmetry. Applications: KK black holes, brane-world BH, AdS/CFT...
- Uniqueness/classification theorems for *Anti de Sitter black holes*. Even $D = 4$? Classification of 5D NHG?
- $D > 5$. Black holes with non-spherical horizons? Classification of NHG with appropriate symmetries?

Summary

- Near-horizon geometries can be used to learn about geometry and topology of horizons of extremal black holes.

- Much progress in 4D/5D: classification of NHG, uniqueness theorems for extremal black holes.

- $D > 5$ black holes poorly understood. Examples of possible black hole NHG with new horizon topology.

- Many open problems remain in higher dimensions...