T-duality and bundle gerbes

- T-duality

$$\boxed{\begin{array}{c}\text{str. th. on}\\ M\times S^1\end{array}} \xleftrightarrow{\;T\;} \boxed{\begin{array}{c}\text{str. th. on}\\ M\times S^1\end{array}}$$

$\downarrow$ radius R $\qquad\qquad$ $\downarrow$ radius $\frac{1}{R}$

$$\begin{tikzcd}
p^*H=\hat{p}^*\hat{H} \arrow[r] & Q\times_M \hat{Q} \arrow[dl, "p"'] \arrow[dr, "\hat{p}"] & \{(q,\hat{q}) \mid \pi(q)=\hat{\pi}(\hat{q})\} \arrow[l] \\
Q \arrow[dr, "\pi"'] & & \hat{Q} \arrow[dl, "\hat{\pi}"] \\
H\in H^3(Q) \arrow[u] & M & \hat{H}\in H^3(Q) \arrow[u]
\end{tikzcd}$$

T-duality changes the topology of the bundle

Q = circle bundle

Gysin sequence

$$\longrightarrow H^k(M) \xrightarrow{\pi^*} H^k(Q) \xrightarrow{\pi_*} H^{k-1}(M) \xrightarrow{F\wedge} H^{k+2}(M)$$

so

$$\begin{array}{ccccccc}
\longrightarrow H^3(M) & \xrightarrow{\pi^*} & H^3(Q) & \xrightarrow{\pi_*} & H^2(M) & \xrightarrow{F\wedge} & H^4(M) \\
 & & \Psi & & \Psi & & \\
 & & H & \longmapsto & \pi_*H & \longmapsto & F\wedge\pi_*H=0
\end{array}$$

$$\begin{array}{ccccccc}
H^3(M) & \xrightarrow{\hat{\pi}^*} & H^3(\hat{Q}) & \xrightarrow{\hat{\pi}_*} & H^2(M) & \xrightarrow{\hat{F}\wedge} & H^4(M) \longrightarrow \\
 & & \Psi & & & & \\
 & & \hat{H} & \longleftarrow & F & &
\end{array}$$

$\exists\, \hat{H}\in H^3(\hat{Q})$ so that $\hat{\pi}_*\hat{H}=F$

- Bundle gerbes

p-gerbe on M $\longleftrightarrow$ $H^{p+2}(M,\mathbb{Z})$

Ex.

$p=-1$: something that represents $H^1(M,\mathbb{Z})$

$$M \longrightarrow U(1) \rightsquigarrow H^1(M,\mathbb{Z})$$

$p=0$: $U(1)$ bundle

$$\begin{array}{c} P \\ \big\downarrow U(1) \\ M \end{array} \rightsquigarrow H^2(M,\mathbb{Z})$$

Equivalent description for a circle bundle

$$\begin{tikzcd} & U(1) & \\ P\times_M P \arrow[ur, "\mu"] \arrow[r, shift left] \arrow[r, shift right] & P \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

so that

$$\mu(p_1,p_2)\,\mu(p_2,p_3)=\mu(p_1,p_3)$$

$$P\times_M P = \{(p,p') \mid \pi(p)=\pi(p')\}$$

So $p=1$ gerbe:

$$\begin{tikzcd} P \arrow[d] \arrow[r] & \text{s.t. } P_{y_1y_2} \otimes P_{y_2y_3} \simeq P_{y_1y_3} \\ Y \times_M Y \arrow[r, shift left] \arrow[r, shift right] & Y \arrow[d] \\ & M \end{tikzcd}$$

Clarification:

$$\begin{tikzcd} & P \arrow[d] & \\ Y \times_M Y \times_M Y \arrow[r, shift left=2, "\pi_1"] \arrow[r, "\pi_2"] \arrow[r, shift right=2, "\pi_3"'] & Y \times_M Y \arrow[r, shift left] \arrow[r, shift right] & Y \end{tikzcd}$$

$$\pi_3^* P \otimes \pi_1^* P \xrightarrow{\sim} \pi_2^* P$$

or

$\delta P = \pi_1^* P \otimes (\pi_2^* P)^* \otimes \pi_3^* P$ has to be trivial

- Put geometry on the bundle gerbe

α a connection

F the curvature from α

$$\begin{tikzcd} P \arrow[d] & & \\ Y \times_M Y \arrow[r, shift left] \arrow[r, shift right] & Y \arrow[d] & B \in \Omega^2(Y) \arrow[l] \\ & M & \omega \arrow[l] \end{tikzcd}$$

"curving" or B-field for str. th.

The H-field in st. theory

- Examples

$$\begin{tikzcd} \delta Q \arrow[d] & Q \arrow[d] \\ Y \times_M Y \arrow[r, shift left] \arrow[r, shift right] & Y \arrow[d] \\ & M \end{tikzcd}$$

Trivial bundle gerbe

Consider the central extension of a group G

$$U(1) \longrightarrow \hat{G} \longrightarrow G$$

Can we lift the transition functions to a $\hat{G}$ bundle ?.

$$\begin{tikzcd}
\mu^*\hat{G} \arrow[dd, "U(1)"'] & \hat{G} \arrow[d, "U(1)"] \\
 & G \\
P \times_M P \arrow[ur, "\mu"] \arrow[r, shift left] \arrow[r, shift right] & P \arrow[d] \\
 & M
\end{tikzcd}$$

"Lifting" bundle gerbe

Comment:

" Everything we can do with line bundles we can do with bundle gerbes (bg) (ex. tensor product 2 bg's , dualise etc) "

- How to dualise a bg (Topological)

bg:

$$\begin{pmatrix} \hat{P}(y_1,y_2,[\tau]) = \tau_{y_1}^{*}\tau_{y_2} \\ \text{for any } \tau\in[\tau] \end{pmatrix} \qquad \begin{pmatrix} R(y,\tau)=\tau_y \\ \delta R(y_1,y_2,\tau) = \tau_{y_1}^{*}\otimes\tau_{y_2} \end{pmatrix}$$

$$\begin{tikzcd}
P \arrow[d] & & & \hat{P} \arrow[d] & R \arrow[dd] & \delta R \arrow[d] \\
Y\times_Q Y \arrow[dr, shift left] \arrow[dr, shift right] & & & (Y\times_M Y)\times_M \hat{Q} \arrow[dl, shift left] \arrow[dl, shift right] & & (Y\times_M Y)\times_M T \arrow[dl, shift left] \arrow[dl, shift right] \\
& Y \arrow[d] & Y\times_M \hat{Q} \arrow[d] & & Y\times_M T \arrow[d] & \\
& Q \arrow[dr] & \hat{Q} = T/\sim \arrow[dl] & & T \arrow[ll] & \\
& & M & & &
\end{tikzcd}$$

$(Y\times_M Y)\times_M \hat{Q} \ni (y_1,y_2,[\tau])$, $Y\times_M \hat{Q} \ni (y,[\tau])$, $Y\times_M T \ni (y,\tau)$, $(Y\times_M Y)\times_M T \ni (y_1\, y_2\, \tau)$, $\hat{Q} \ni [\tau]$ "classes of τ", $T \ni \tau$, $M \ni x$

bg

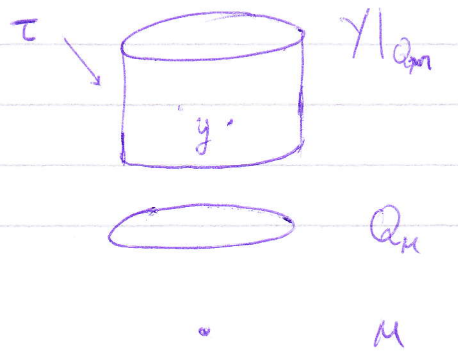
U(1)-circle bundle

$T_M = \{$ all trivilisations of $P|_{Q_M}$ with connections $\} =$

$= \{ \tau \longrightarrow Y|_{Q_M}$ with connections $| \; \delta\tau \simeq P|_{Q_M} \}$

so $\hat{Q}_M = \{ \tau \longrightarrow Y|_{Q_M} \ldots \; | \; hol(\tau_1^{*}\otimes\tau_2) = 1 \}$

↳ the equivalence relation



• Geometic T-duality

$$
\begin{tikzcd}
\alpha \arrow[r] & P \arrow[d] & & & & \hat{P} \arrow[d] \\
& Y \times_M Y \arrow[r, shift left] \arrow[r, shift right] & Y \arrow[d] & & Y \times_M \hat{Q} \arrow[d] & (Y \times_M Y) \times_M \hat{Q} \arrow[l, shift left] \arrow[l, shift right] \\
B \arrow[urr] & & Q \arrow[dr] & & \hat{Q} = T/\sim \arrow[dl] & \\
A \arrow[urr] & & & M & &
\end{tikzcd}
$$

$$\begin{cases} P \text{ has a connection and curving } B \\ Q \text{ has a connection } A \end{cases}$$

$$\Longrightarrow \begin{cases} \hat{Q} \text{ has a connection } \cancel{\text{with}} \text{ and} \\ \hat{P} \text{ has a connection with curving } \hat{B} = B + A \wedge \hat{A} \end{cases}$$

$$\left(\text{if } H^2(Y \times_M \hat{Q}) = 0 \right)$$