Junction type representations of the Temperley-Lieb algebra and associated symmetries

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Motivation and conclusions

- The study of representations of the braid group and its quotients provides solutions to the Yang-Baxter and reflection equations in a systematic way.
- Investigation of the Temperley-Lieb algebra. Generalization of the asymmetric twin representation studied before [Doikou, Martin 03 and 06, Doikou 05] .
- Existence of non-trivial quantum symmetries, in addition to the trivial ones. Emergence of dualities.
- Extension to the case of the boundary Temperley-Lieb algebra.
- Possible applications to physical systems such as spin chains, with various boundary conditions.

Plan of the talk

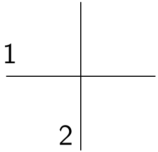
- Introduction to the mathematical concepts and presentation of the building blocks.
- Definition of the junction representation.
- Exposure of the quantum symmetries.
- Extension to the boundary case.
- Discussion of the corresponding quantum spin chain.
- Summary and future directions.

In the heart of quantum integrability lies the Yang-Baxter equation

$$R_{12}(\lambda_1-\lambda_2)\, R_{13}(\lambda_1-\lambda_3)\, R_{23}(\lambda_2-\lambda_3) = R_{23}(\lambda_2-\lambda_3)\, R_{13}(\lambda_1-\lambda_3)\, R_{12}(\lambda_1-\lambda_2),$$

where the R-matrix acts on $V \otimes V$ and indices denote the vector space, i.e. $R_{12} = R \otimes \mathbb{I}$, $R_{23} = \mathbb{I} \otimes R$ and so on. λ is called spectral parameter.

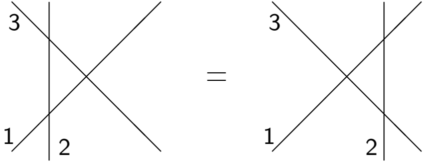
Graphically, one represents R_{12} as



The Yang-Baxter equation (YBE)

$$R_{12}(\lambda_1-\lambda_2)\, R_{13}(\lambda_1-\lambda_3)\, R_{23}(\lambda_2-\lambda_3) = R_{23}(\lambda_2-\lambda_3)\, R_{13}(\lambda_1-\lambda_3)\, R_{12}(\lambda_1-\lambda_2)$$

is then illustrated as below, representing the factorization of scattering between particles



The R-matrix is associated with an integrable Hamiltonian. The machinery of quantum integrability leads to exact solutions of integrable systems. [Faddeev, Korepin, Sklyanin, Takhtajan, et al].

Let $\mathcal{P}$ be the permutation operator, its action on two vectors giving $\mathcal{P}(a \otimes b) = b \otimes a$. Defining $\check{R} = \mathcal{P}R$, YBE is also rewritten as

$$\check{R}_{12}(\lambda_1 - \lambda_2)\ \check{R}_{23}(\lambda_1)\ \check{R}_{12}(\lambda_2) = \check{R}_{23}(\lambda_2)\ \check{R}_{12}(\lambda_1)\ \check{R}_{23}(\lambda_1 - \lambda_2).$$

The A-type Artin braid group is defined by the $N-1$ generators g_i

$$\begin{aligned} &g_i\ g_{i+1}\ g_i = g_{i+1}\ g_i\ g_{i+1} && \text{braid relation} \\ &[g_i,\ g_j] = 0, \qquad |i-j| > 1. \end{aligned}$$

The braid relation is isomorphic to the “modified” YBE and thus provides a systematic way to obtain solutions of the latter one.

The A-type Hecke algebra $H_N(q)$ is defined by the $N-1$ generators g_i satysfying

$$(g_i - q)(g_i + q^{-1}) = 0 \qquad \text{Hecke condition}$$

The Temperley-Lieb algebra

There is an alternative expression of Hecke algebra. Renaming the generators as $\mathbb{U}_i = g_i - q$, it is also written as

$$\begin{aligned} &\mathbb{U}_i\ \mathbb{U}_{i+1}\ \mathbb{U}_i - \mathbb{U}_i = \mathbb{U}_{i+1}\ \mathbb{U}_i\ \mathbb{U}_{i+1} - \mathbb{U}_{i+1} \\ &\mathbb{U}_i^2 = -(q+q^{-1})\mathbb{U}_i \qquad\qquad \text{Hecke condition} \\ &[\mathbb{U}_i,\ \mathbb{U}_j] = 0, \qquad |i-j|>1. \end{aligned}$$

The Temperley-Lieb $T_N(q)$ algebra is defined by an additional requirement

$$\begin{aligned} &\mathbb{U}_i\ \mathbb{U}_{i+1}\ \mathbb{U}_i = \mathbb{U}_i \\ &\mathbb{U}_i^2 = -(q+q^{-1})\mathbb{U}_i \qquad\qquad \text{(Hecke condition)} \\ &[\mathbb{U}_i,\ \mathbb{U}_j] = 0, \qquad |i-j|>1. \end{aligned}$$

The explicit construction

The $\check{R}$-matrix associated with a Temperley-Lieb representations is obtained via [Jimbo '86]

$$\check{R}_{ii+1} = \sinh(\lambda + i\mu)\ \mathbb{I} + \sinh\lambda\ \rho(\mathbb{U}_i)\ ,$$

for any representation ρ of the Temperley-Lieb algebra and with $q = e^{i\mu}$.

The Hamiltonian obtained then has the symmetry of the representation of the T-L algebra.

The XXZ representation

A well-known representation of the $T_N(q)$ algebra is the XXZ one. Consider the matrix

$$U = \sum_{a \neq b = 1}^{2} e_{ab} \otimes e_{ba} - \sum_{a \neq b = 1}^{2} q^{-sgn(a-b)} e_{aa} \otimes e_{bb},$$

where the matrices e_{ab} are 2×2 matrices defined as $(e_{ab})_{cd} = \delta_{ac}\ \delta_{bd}$.

The representation of the Temperley-Lieb algebra $\rho: \ T_N(q) \mapsto \mathrm{End}((\mathbb{C}^2)^{\otimes N})$ is then

$$\rho(\mathbb{U}_i) = \mathbb{I} \otimes \ldots \otimes \mathbb{I} \otimes \underbrace{U}_{i,\ i+1} \otimes \mathbb{I} \otimes \ldots \otimes \mathbb{I}.$$

The XXZ representation (continued)

Using the recipe for the R-matrix

$$\check{R}_{ii+1} = \sinh(\lambda + i\mu)\, \mathbb{I} + \sinh\lambda\ \rho(\mathbb{U}_i)\ ,$$

yields the $\check{R}$-matrix which generates the Hamiltonian of the XXZ spin chain as

$$\frac{d}{d\lambda}\check{R}(\lambda)_{ii+1}\Big|_{\lambda=0} \propto \mathcal{H}_{ii+1},$$

with

$$\mathcal{H}_{ii+1} = -\frac{1}{2}\left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta\ \sigma_i^z\sigma_{i+1}^z\right).$$

It is also important to note that the particular representation has a $U_q(\mathfrak{sl}_2)$ symmetry.

Joining representations together

We construct a fusion type rep. of T-L algebra.

The junction representation involves n copies of the XXZ representation, i.e. $\Theta : \; T_n(q) \mapsto \mathrm{End}(((\mathbb{C}^2)^{\otimes n})^{\otimes N})$

$$\Theta(\mathbb{U}_l) = \prod_{i=1}^{n} \rho_{a_i}(\mathbb{U}_{l^{(i)}}),$$

and is a representation of the Temperley-Lieb algebra provided that:

$$(-1)^n \prod_{i=1}^{n} (a_i + a_i^{-1}) = -(q + q^{-1}).$$

The explicit representation for $n = 2$

The two branches

$$\rho_{a_1}(\mathbb{U}_{l(1)}) = \ldots \mathbb{I} \otimes \left(\sum_{a \neq b} e_{ab} \otimes \mathbb{I} \otimes e_{ba} \otimes \mathbb{I} + \sum_{a \neq b} a_1{}^{-sgn(a-b)} e_{aa} \otimes \mathbb{I} \otimes e_{bb} \otimes \mathbb{I} \right) \otimes \mathbb{I} \ldots$$

$$\rho_{a_2}(\mathbb{U}_{l(2)}) = \ldots \mathbb{I} \otimes \left(\sum_{a \neq b} \mathbb{I} \otimes e_{ab} \otimes \mathbb{I} \otimes e_{ba} + \sum_{a \neq b} a_2{}^{-sgn(a-b)} \mathbb{I} \otimes e_{aa} \otimes \mathbb{I} \otimes e_{bb} \right) \otimes \mathbb{I} \ldots$$

and their fusion

$$\Theta(\mathbb{U}_l) = \rho_{a_1}(\mathbb{U}_{l(1)}) \rho_{a_2}(\mathbb{U}_{l(2)})$$

Explicit construction for n copies

The junction rep. is obtained through

$$\Theta(\mathbb{U}_l) = \prod_{i=1}^{n} \rho_{a_i}(\mathbb{U}_{l^{(i)}}),$$

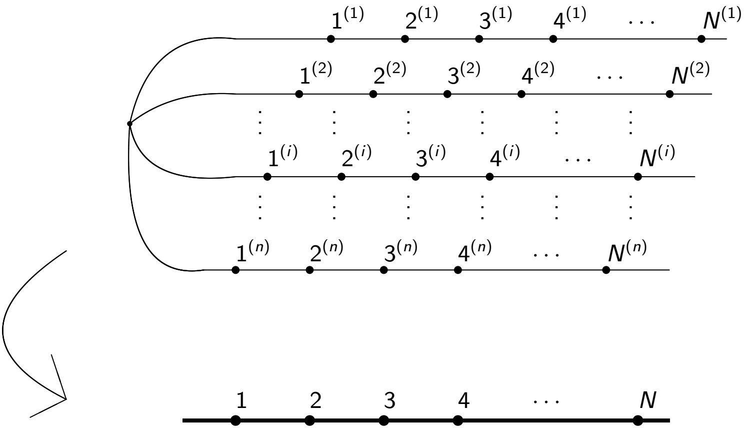
Each one of the terms in the product can be written explicitly

$$\rho_{a_m}(\mathbb{U}_{l^{(m)}}) = \ldots \mathbb{I} \otimes \Bigg(\sum_{a \neq b} \underbrace{\mathbb{I} \otimes \ldots \otimes \mathbb{I}}_{m-1} \otimes \underbrace{e_{ab}}_{l^{(m)}} \otimes \underbrace{\mathbb{I} \otimes \ldots \otimes \mathbb{I}}_{n-1} \otimes \underbrace{e_{ba}}_{(l+1)^{(m)}} \otimes \underbrace{\mathbb{I} \otimes \ldots \otimes \mathbb{I}}_{n-m}$$

$$+ \sum_{a \neq b} a_m{}^{-sgn(a-b)} \underbrace{\mathbb{I} \otimes \ldots \otimes \mathbb{I}}_{m-1} \otimes \underbrace{e_{aa}}_{l^{(m)}} \otimes \underbrace{\mathbb{I} \otimes \ldots \otimes \mathbb{I}}_{n-1} \otimes \underbrace{e_{bb}}_{(l+1)^{(m)}} \otimes \underbrace{\mathbb{I} \otimes \ldots \otimes \mathbb{I}}_{n-m} \Bigg) \otimes \mathbb{I} \ldots$$

with $m \in \{1, \ldots, n\}$.

Graphical illustration of the junction representation



The q-deformed $\mathfrak{sl}_2$ algebra

- Recall the $U_q(\mathfrak{sl}_2)$ algebra [Jimbo 85 and 86] defined by the algebraic relations

$$[h,\ e] = 2e, \quad [h,\ f] = -2f\ , \quad [e,\ f] = \frac{q^h - q^{-h}}{q - q^{-1}},$$

- It is equipped with a non-trivial co-product

$$\Delta :\ U_q(\mathfrak{sl}_2) \mapsto U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$$

$$\Delta(q^h) = q^h \otimes q^h, \qquad \Delta(x) = x \otimes q^{\frac{h}{2}} + q^{-\frac{h}{2}} \otimes x, \qquad x \in \{e,\ f\}.$$

We may also define N co-products $\Delta^{(N)} : U_q(\mathfrak{sl}_2) \mapsto U_q(\mathfrak{sl}_2)^{\otimes N}$ which are given by iteration

$$\Delta^{(N)} = (\mathrm{id} \otimes \Delta^{(N-1)})\Delta$$

- The limit $q \to 1$ gives the usual $\mathfrak{sl}_2$ Lie algebra.

The manifest symmetries

- We focus on the case $n = 3$, where the junction representation is

$$\Theta(\mathbb{U}_I) = \rho_{a_1}(\mathbb{U}_{I(1)})\ \rho_{a_2}(\mathbb{U}_{I(2)})\ \rho_{a_3}(\mathbb{U}_{I(3)}).$$

  Analytics results for $n = 3$ will be presented and a plausible conjecture will be made for generic n.

- First, recall that the XXZ representation of $T_N(q)$ has a $U_q(\mathfrak{sl}_2)$ symmetry.

- Second, recall the spin 1/2 representation of $U_q(\mathfrak{sl}_2)$ is given by $\pi_q : U_q(\mathfrak{sl}_2) \mapsto \mathrm{End}(\mathbb{C}^2)$

$$\pi_q(h) = \sigma^z, \qquad \pi_q(e) = \sigma^+, \qquad \pi_q(f) = \sigma^-.$$

The manifest ones [continuing]

- Now define

$$\pi_1(x) = \pi_{a_1}(x) \otimes \mathbb{I} \otimes \mathbb{I}, \qquad \pi_2(x) = \mathbb{I} \otimes \pi_{a_2}(x) \otimes \mathbb{I},$$
$$\pi_3(x) = \mathbb{I} \otimes \mathbb{I} \otimes \pi_{a_3}(x), \qquad x \in U_{a_i}(\mathfrak{sl}_2).$$

- It is straightforward to check that the junction representation commutes with the following actions

$$\left[\Theta(\mathbb{U}_I),\ \pi_i^{\otimes N}(\Delta^{(N)}(x))\right] = 0, \qquad x \in U_{a_i}(\mathfrak{sl}_2), \quad i = 1,\ 2,\ 3.$$

- Hence, the junction representation enjoys a manifest

$$\mathcal{G}_0 \equiv U_{a_1}(\mathfrak{sl}_2) \otimes U_{a_2}(\mathfrak{sl}_2) \otimes U_{a_3}(\mathfrak{sl}_2)$$

symmetry, as is trivially expected by its construction.

The non-trivial ones

Consider the following family of representations

$$
\begin{aligned}
\mathfrak{f}_0(h) &= e_{11} \otimes e_{11} \otimes e_{11} - e_{22} \otimes e_{22} \otimes e_{22} \equiv h_1^{(0)} - h_2^{(0)} \\
\mathfrak{f}_0(e) &= e_{12} \otimes e_{12} \otimes e_{12}, \quad \mathfrak{f}_0(f) = e_{21} \otimes e_{21} \otimes e_{21} \\
\mathfrak{f}_0(q_0^h) &= \mathbb{I} + (q_0 - 1)h_1^{(0)} + (q_0^{-1} - 1)h_2^{(0)}
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{f}_1(h) &= e_{22} \otimes e_{11} \otimes e_{11} - e_{11} \otimes e_{22} \otimes e_{22} \equiv h_1^{(1)} - h_2^{(1)} \\
\mathfrak{f}_1(e) &= e_{21} \otimes e_{12} \otimes e_{12}, \quad \mathfrak{f}_1(f) = e_{12} \otimes e_{21} \otimes e_{21} \\
\mathfrak{f}_1(q_1^h) &= \mathbb{I} + (q_1 - 1)h_1^{(1)} + (q_1^{-1} - 1)h_2^{(1)}
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{f}_2(h) &= e_{11} \otimes e_{22} \otimes e_{11} - e_{22} \otimes e_{11} \otimes e_{22} \equiv h_1^{(2)} - h_2^{(2)} \\
\mathfrak{f}_2(e) &= e_{12} \otimes e_{21} \otimes e_{12}, \quad \mathfrak{f}_2(f) = e_{21} \otimes e_{12} \otimes e_{21} \\
\mathfrak{f}_2(q_2^h) &= \mathbb{I} + (q_2 - 1)h_1^{(2)} + (q_2^{-1} - 1)h_2^{(2)}
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{f}_3(h) &= e_{11} \otimes e_{11} \otimes e_{22} - e_{22} \otimes e_{22} \otimes e_{11} \equiv h_1^{(3)} - h_2^{(3)} \\
\mathfrak{f}_3(e) &= e_{12} \otimes e_{12} \otimes e_{21}, \quad \mathfrak{f}_3(f) = e_{21} \otimes e_{21} \otimes e_{12} \\
\mathfrak{f}_3(q_3^h) &= \mathbb{I} + (q_3 - 1)h_1^{(3)} + (q_3^{-1} - 1)h_2^{(3)}.
\end{aligned}
$$

They all form representations of the $U_{q_i}(\mathfrak{sl}_2)$ quantum algebra.

The non-trivial ones [continuing]

- The junction representation is found to commute with this family of representations

$$[\Theta(\mathbb{U}_I),\ \mathfrak{f}_i^{\otimes N}(\Delta^{(N)}(x))] = 0, \qquad x \in U_{q_i}(\mathfrak{sl}_2), \qquad i = 0, 1, 2, 3,$$

with q_i being determined by the symmetry requirements. For the particular representations, the respective q_i's are

$$\begin{aligned} q_0 &= a_1 a_2 a_3 \\ q_1 &= a_1^{-1} a_2 a_3 \\ q_2 &= a_1 a_2^{-1} a_3 \\ q_3 &= a_1 a_2 a_3^{-1}, \end{aligned}$$

- Hence there exists a non-trivial quantum symmetry of the representation, namely a

$$\mathcal{G} \equiv U_{q_0}(\mathfrak{sl}_2) \otimes U_{q_1}(\mathfrak{sl}_2) \otimes U_{q_2}(\mathfrak{sl}_2) \otimes U_{q_3}(\mathfrak{sl}_2)$$

symmetry.

A generic family of representations of the quantum algebra

Consider the representation $\mathfrak{f}_0 : U_{q_0}(\mathfrak{sl}_2) \mapsto \mathrm{End}((\mathbb{C}^{(2)})^{\otimes n})$ as the starting point

$$\begin{aligned}
&\mathfrak{f}_0(h) = (e_{11})^{\otimes n} - (e_{22})^{\otimes n} \equiv h_1^{(0)} - h_2^{(0)}, \\
&\mathfrak{f}_0(e) = (e_{12})^{\otimes n}, \qquad \mathfrak{f}_0(f) = (e_{21})^{\otimes n}, \\
&\mathfrak{f}_0(q_0^h) = \mathbb{I} + (q_0 - 1)h_1^{(0)} + (q_0^{-1} - 1)h_2^{(0)},
\end{aligned}$$

with $q_0 = a_1 a_2 \cdots a_n$. Deform the parameter q then as $q_{i_1 i_2 \dots i_m} = a_1 \ a_2 \dots a_{i_1}^{-1} \ a_{i_1+1} \dots a_{i_2}^{-1} \dots a_{i_m}^{-1} \dots a_n$.

The structure of the representations changes along with the deformation of the parameter.

The generic representation is then defined as
$\mathfrak{f}_{i_1 i_2 \dots i_m} : \; U_{q_{i_1 i_2 \dots i_m}}(\mathfrak{sl}_2) \to \mathrm{End}((\mathbb{C}^2)^{\otimes n})$

$$
\begin{aligned}
\mathfrak{f}_{i_1 i_2 \dots i_m}(h) &= e_{11} \otimes \dots e_{11} \otimes \dots \otimes \underbrace{e_{22}}_{i_1} \otimes e_{11} \dots \otimes \underbrace{e_{22}}_{i_2} \otimes \dots \otimes \underbrace{e_{22}}_{i_m} \otimes \dots \otimes e_{11} \\
&\quad - e_{22} \otimes \dots e_{22} \otimes \dots \otimes \underbrace{e_{11}}_{i_1} \otimes e_{22} \dots \otimes \underbrace{e_{11}}_{i_2} \otimes \dots \otimes \underbrace{e_{11}}_{i_m} \otimes \dots \otimes e_{22} \\
&\equiv h_1^{(i_1 \dots i_m)} - h_2^{(i_1 \dots i_m)} \\
\mathfrak{f}_{i_1 i_2 \dots i_m}(e) &= e_{12} \otimes \dots e_{12} \otimes \dots \otimes \underbrace{e_{21}}_{i_1} \otimes e_{12} \dots \otimes \underbrace{e_{21}}_{i_2} \otimes \dots \otimes \underbrace{e_{21}}_{i_m} \otimes \dots \otimes e_{12} \\
\mathfrak{f}_{i_1 i_2 \dots i_m}(f) &= e_{21} \otimes \dots e_{21} \otimes \dots \otimes \underbrace{e_{12}}_{i_1} \otimes e_{21} \dots \otimes \underbrace{e_{12}}_{i_2} \otimes \dots \otimes \underbrace{e_{12}}_{i_m} \otimes \dots \otimes e_{21} \\
\mathfrak{f}_{q_{i_1 \dots i_m}}(q^h_{i_1 \dots i_m}) &= \mathbb{I} + (q_{i_1 \dots i_m} - 1) h_1^{(i_1 \dots i_m)} + (q^{-1}_{i_1 \dots i_m} - 1) h_2^{(i_1 \dots i_m)}.
\end{aligned}
$$

Turns out to be a combinatorial problem with 2^{n-1} number of representations.

The symmetry and an emerging duality

- A plausible conjecture: the representation constructed enjoys these non-trivial quantum symmetries, i.e.

$$[\Theta(\mathbb{U}_I),\ \mathfrak{f}^{\otimes N}_{i_1\dots i_m}(\Delta^{(N)}(x))] = 0, \qquad x \in U_{q_{i_1\dots i_m}}(\mathfrak{sl}_2).$$

- Verified explicitly for $n = 3$ and for $n = 4, 5$ by numerical means.
- Some kind of duality arises: the quantum algebra with parameter $q_{i_1,\dots,i_m}$ is equivalent to the algebra with parameter $q^{-1}_{i_1,\dots,i_m}$, after interchanging $e \leftrightarrow f$ and $h \leftrightarrow -h$.

The boundary T-L algebra (blob)

The boundary Temperley-Lieb algebra $B_N(q, Q)$ is defined by generators $\mathbb{U}_i \in T_N(q)$

$$\begin{aligned}
&\mathbb{U}_i \ \mathbb{U}_{i+1} \ \mathbb{U}_i = \mathbb{U}_i \\
&\mathbb{U}_i^2 = -(q + q^{-1})\mathbb{U}_i \\
&[\mathbb{U}_i, \ \mathbb{U}_j] = 0, \qquad |i-j| > 1,
\end{aligned}$$

and an additional generator $\mathbb{U}_0$ satisfying

$$\begin{aligned}
&\mathbb{U}_1 \ \mathbb{U}_0 \ \mathbb{U}_1 = \kappa \mathbb{U}_1 \\
&\mathbb{U}_0^2 = \delta_0 \mathbb{U}_0 \\
&[\mathbb{U}_0, \ \mathbb{U}_i] = 0, \qquad i > 1,
\end{aligned}$$

with $\kappa = qQ^{-1} + q^{-1}Q$ and $\delta_0 = -(Q + Q^{-1})$.

The XXZ representation of $B_N(q,Q)$ is defined as

$$\rho_{q,Q}:\ B_N(q,Q) \mapsto \mathrm{End}\big((\mathbb{C}^2)^{\otimes N}\big),$$
$$\rho_{q,Q}(\mathbb{U}_i) = \mathbb{I}\otimes\cdots\otimes\mathbb{I}\otimes\underbrace{U}_{i,\,i+1}\otimes\mathbb{I}\otimes\cdots\otimes\mathbb{I}$$
$$\rho_{q,Q}(\mathbb{U}_0) = U_0\otimes\mathbb{I}\otimes\cdots\otimes\mathbb{I},$$

where

$$U_0 = -Q^{-1}e_{11} - Qe_{22} + e_{12} + e_{21}.$$

The junction type representation of the boundary element is then

$$\Theta(\mathbb{U}_0) = \prod_{i=1}^{n}\rho_{a_i,Q_i}(\mathbb{U}_0),$$

Alongside with $\Theta(\mathbb{U}_i)$ they satisfy the algebraic relations of $B_N(q,Q)$.

However, nontrivial boundary elements may be constructed with the general form

$$M_i = -Q^{-1} h_1^i - Q h_2^i + \mathfrak{f}_i(e) + \mathfrak{f}_i(f).$$

Let also $\mathcal{M}_i = M_i \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}$. For $n = 3$ then

$$\begin{aligned}
M_0 &= -Q^{-1} e_{11} \otimes e_{11} \otimes e_{11} - Q e_{22} \otimes e_{22} \otimes e_{22} + e_{12} \otimes e_{12} \otimes e_{12} + e_{21} \otimes e_{21} \otimes e_{21}, \\
M_1 &= -Q^{-1} e_{22} \otimes e_{11} \otimes e_{11} - Q e_{11} \otimes e_{22} \otimes e_{22} + e_{21} \otimes e_{12} \otimes e_{12} + e_{12} \otimes e_{21} \otimes e_{21}, \\
M_2 &= -Q^{-1} e_{11} \otimes e_{22} \otimes e_{11} - Q e_{22} \otimes e_{11} \otimes e_{22} + e_{12} \otimes e_{21} \otimes e_{12} + e_{21} \otimes e_{12} \otimes e_{21}, \\
M_3 &= -Q^{-1} e_{11} \otimes e_{11} \otimes e_{22} - Q e_{22} \otimes e_{22} \otimes e_{11} + e_{12} \otimes e_{12} \otimes e_{21} + e_{21} \otimes e_{21} \otimes e_{12}.
\end{aligned}$$

it is straightforward to show that

$$\Theta(\mathbb{U}_1) \mathcal{M}_i \Theta(\mathbb{U}_1) = \kappa \Theta(\mathbb{U}_1),$$

for $i = 0, 1, 2, 3$. Hence the above is representation of the blob algebra.

Residual symmetries

The boundary elements introduced before brake part of the symmetry. In particular,

$$[f_i(x), M_j] = 0, \qquad x \in U_q(\mathfrak{sl}_2), \qquad i \neq j.$$

However, consider the following combination of $U_q(\mathfrak{sl}_2)$ generators

$$\mathcal{Q}_i = q_i^{-\frac{1}{2}} q_i^{\frac{h}{2}} e + q_i^{\frac{1}{2}} q_i^{-\frac{h}{2}} f + x_i \; q_i^h - x_i \; \mathbb{I}.$$

Applied for $n = 3$ it is shown that

$$[f_i(\mathcal{Q}_i), M_j] = 0,$$

not only for $i \neq j$ but for $i = j$ also, provided that the constants satisfy

$$x_i = \frac{Q - Q^{-1}}{q_i - q_i^{-1}}.$$

The corresponding spin chain

- Recall the Yang-Baxter equation

$$\check{R}_{12}(\lambda_1-\lambda_2)\ \check{R}_{23}(\lambda_1)\ \check{R}_{12}(\lambda_2)=\check{R}_{23}(\lambda_2)\ \check{R}_{12}(\lambda_1)\ \check{R}_{23}(\lambda_1-\lambda_2).$$

- The $\check{R}$-matrix associated to representations of the Temperley-Lieb algebra may be expressed as [Jimbo 86].

$$\check{R}_{ii+1}(\lambda)=\sinh(\lambda+i\mu)\ \mathbb{I}+\sinh\lambda\ \rho(\mathbb{U}_i),$$

  for any representation ρ of the TL algebra, and where $q=e^{i\mu}$.
- In particular, we may use $\rho=\Theta$ to construct an R-matrix associated with the junction representation.

- Since (for $n = 3$)

$$\left[\Theta(\mathbb{U}_l),\ f_i^{\otimes N}(\Delta^{(N)}(x))\right] = 0, \qquad x \in U_{a_i}(\mathfrak{sl}_2), \qquad i = 1,\ 2,\ 3.$$

  it immediately follows that these are symmetries of the spin chain

$$\left[\check{R},\ f_i^{\otimes N}(\Delta^{(N)}(x))\right] = 0, \qquad x \in U_{a_i}(\mathfrak{sl}_2), \qquad i = 1,\ 2,\ 3.$$

- In the same spirit, these are symmetries of the transfer matrix

$$\left[t(\lambda),\ f_i^{\otimes N}(\Delta^{(N)}(x))\right] = 0, \qquad x \in U_{a_i}(\mathfrak{sl}_2), \qquad i = 1,\ 2,\ 3.$$

- The same holds for the non-trivial quantum symmetries associated with parameters q_i.
- Insertion of integrable boundaries leads to symmetry breaking and different physical behaviors.

Summary and future directions

- A novel family of representations is introduced.
- The number of exact symmetries is drastically increased with the number n.
- Construction of the relevant physical system, i.e. the respective quantum spin chain. Deserves further research.
- What about the duality observed at the level of the algebra? Is it reflected in the physical picture?
- Consideration of similar constructions for other quantum algebras or representations?
- Possible application in other physical systems?.

Thank you!