Quantum Integrability in 2D sigma-models on supergroups and supercosets

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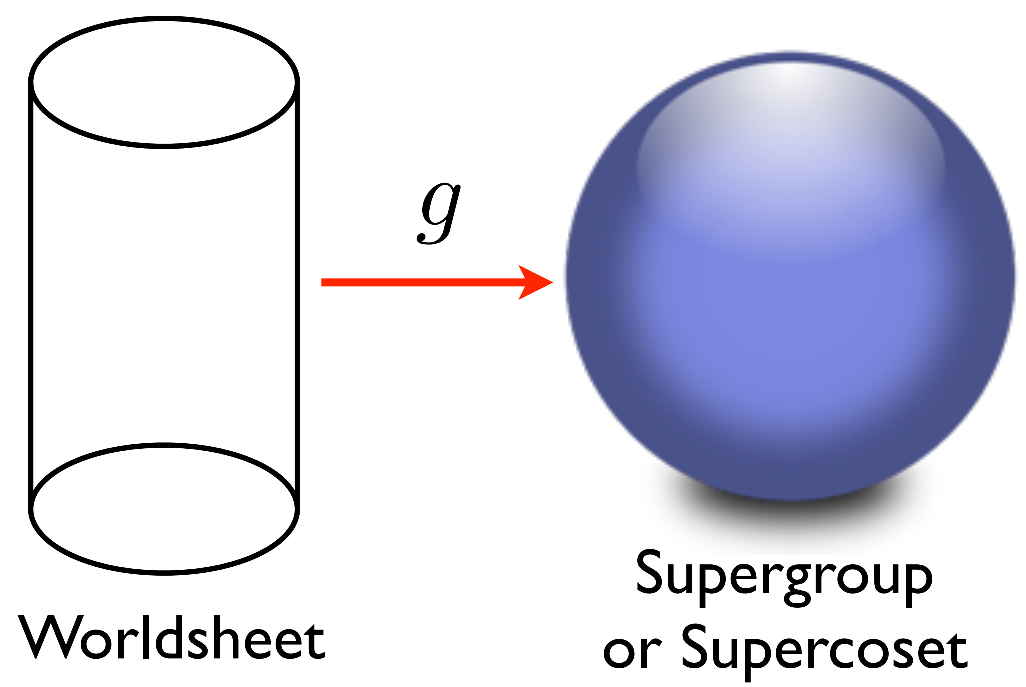
Introduction

In this talk, we consider 2D sigma-models on supergroups and supercosets.



These models are relevant to understand:

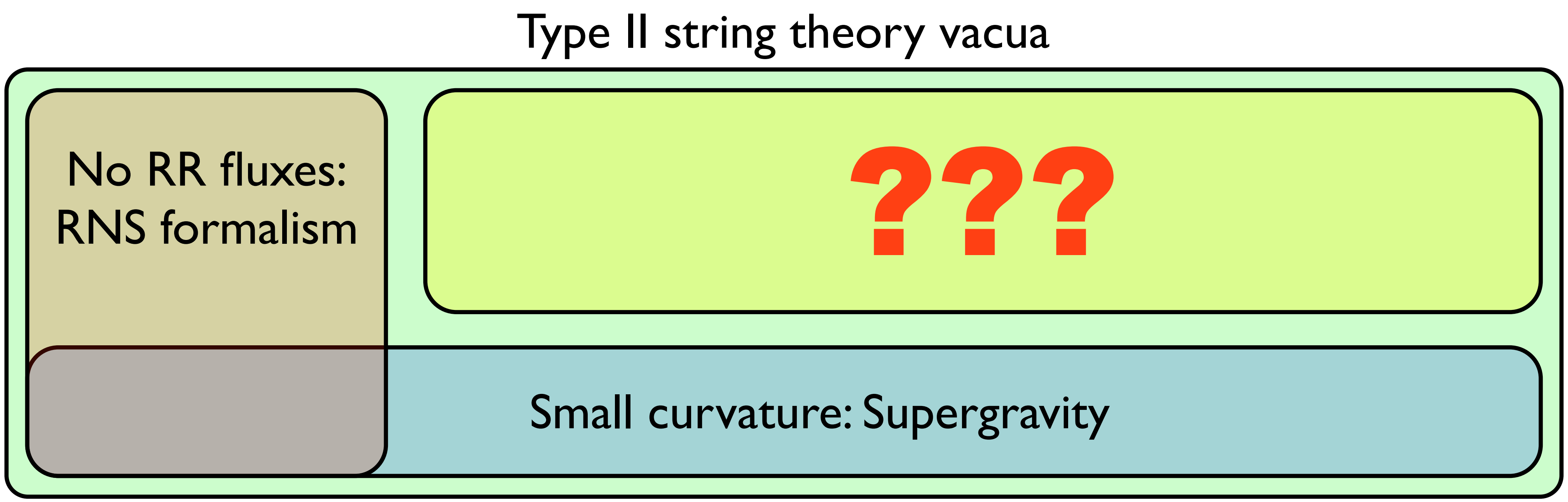
String theory in RR backgrounds

Integrability in AdS/CFT

Superstrings in RR backgrounds

In type II string theory, several fields can take a non-zero expectation value in the vacuum: metric, dilaton... and RR-fluxes.

Quantization of string theory with RR fluxes is not understood.



Superstrings in RR backgrounds

We need to embed spacetime in a superspace.

Green Schwarz formalism	Pure spinor formalism	Hybrid formalism	etc.

Green & Schwarz, 1984

Berkovits et al.

Not a single example is under control.
Sigma models on superspaces need to be understood better.

Sigma models on supergroups and supercosets are natural starting points. In this talk, we mostly discuss the computation of the spectrum in these models.

Superstrings in RR backgrounds

Two families of supergroups are particularly attractive:

$$\boxed{PSl(n|n)} \qquad \boxed{OSp(2n+2|2n)}$$

They have vanishing dual Coxeter number: sigma-models on these supergroups are conformal. Some of their cosets inherit this property.

$$\boxed{PSU(1,1|2) \leftrightarrow AdS_3 \times S^3}$$

$$\boxed{\boxed{\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)} \leftrightarrow AdS_5 \times S^5}} \qquad \boxed{\frac{OSp(6|4)}{SO(3,1) \times U(3)} \leftrightarrow AdS_4 \times CP^5}$$

Integrability in AdS/CFT

Type IIB string theory in $AdS_5 \times S^5$ $\xleftrightarrow{\text{AdS/CFT}}$ $\mathcal{N}=4 \;\; SU(N)$ SYM

Large N limit: Integrable structures appear.

Beisert *et al.*, 2011

In this talk we focus on the spectrum problem.

Energy of string states $\longleftrightarrow$ Conformal dimensions

The spectrum problem: history

- The dilatation operator of N=4 SYM can be related to the Hamiltonian of an integrable spin chain.

Minahan & Zarembo, 2002

- The string worldsheet theory is integrable, at least classically.

Bena, Polchinski & Roiban, 2003

- The dimension of long operators is given by the Asymptotic Bethe Ansatz.

Beisert, Eden & Staudacher, 2006

- A solution has been proposed for the spectrum of all operators: the Y-system.

Gromov, Kazakov & Vieira, 2009

The Y- and T-systems

- It is an infinite system of equations for the so-called Y-functions, that can be solved numerically.

$$\boxed{\text{Y-system}}$$

$$\Updownarrow$$

$$\boxed{\mathcal{T}_{a,s}(u+1)\mathcal{T}_{a,s}(u-1) = \mathcal{T}_{a+1,s}(u+1)\mathcal{T}_{a-1,s}(u-1) + \mathcal{T}_{a,s+1}(u-1)\mathcal{T}_{a,s-1}(u+1)}$$

T-system, or Hirota equation

- Each string state corresponds to a solution of the Hirota equation with specific analytic properties.

- The energy of a string state can be computed easily from the T-functions.

Good reasons to appreciate the Y-system

- It is compatible with the Asymptotic Bethe Ansatz.

Gromov, Kazakov & Vieira, 2009a

- It reproduces the spectrum of the quasi-classical string a large ‘t Hooft coupling.

Gromov, 2009

Gromov, Kazakov & Tsuboi, 2010

- It gave correct predictions for the dimension of the Konishi operator at large and small ‘t Hooft coupling.

Gromov, Kazakov & Vieira, 2009c

Arutyunov, Frolov & Suzuki, 2010

Now it would be nice to prove the validity of the Y-system.

Derivations of the Y-system

- The Y-system can be derived using the Thermodynamic Bethe Ansatz.

Gromov, Kazakov, Kozak & Vieira, 2009

Bombardelli, Fioravanti & Tateo, 2009

Arutyunov & Frolov, 2009

- This derivation relies on some crucial assumptions:
  - Quantum integrability
  - String hypothesis
  - Analytic continuation for the excited states

In this talk we present another approach:

☺ First-principles

☹ Perturbative

closer in spirit to the work of

Bazhanov, Lukyanov & Zamolodchikov, 1994

Plan

1. Introduction
2. Line operators and integrability
3. Derivation of the Hirota equation
4. Conclusions

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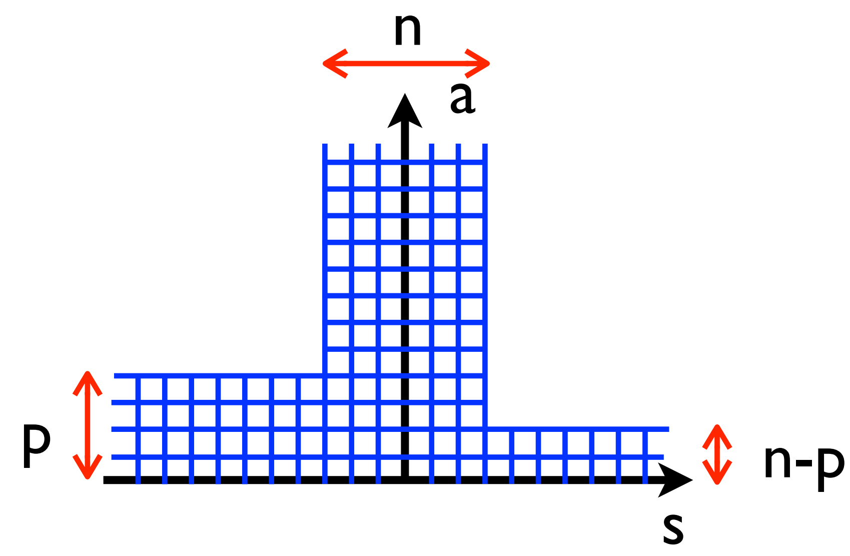
The Hirota equation: generalities

$$\mathcal{T}_{a,s}(u+1)\mathcal{T}_{a,s}(u-1) = \mathcal{T}_{a+1,s}(u+1)\mathcal{T}_{a-1,s}(u-1) + \mathcal{T}_{a,s+1}(u-1)\mathcal{T}_{a,s-1}(u+1)$$

- The integer indices (a,s) label representations of PSl(n|n). They take value in a T-shaped lattice.

The precise shape of the lattice depends on the real form of the supergroup. For PSU(p,n-p|n):



- The T-functions are presumably related to the transfer matrices of the underlying theory.

see e.g. Gromov, Kazakov & Tsuboi, 2010

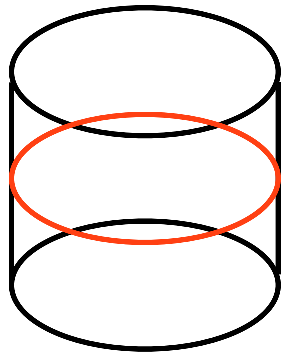
Classical integrability

A two-dimensional field theory is classically integrable if one can find a one-parameter family of flat connections:

$$\forall u, \quad dA(u) + A(u) \wedge A(u) = 0$$

From the flat connection, one can construct the transfer matrix:

$$\boxed{\mathcal{T}_R(u) = STr\ P\ \exp\left(-\oint A_R(u)\right)}$$



Flatness of the connection implies that the transfer matrix is independant of the integration contour. Thus it encodes an infinite number of conserved charges.

The classical limit of the Hirota equation

- The classical transfer matrix is a super-character:

$$\mathcal{T}_R(u) = STr \boxed{P \ \exp\left(-\oint A_R(u)\right)} \longrightarrow \boxed{\text{Supergroup element}}$$

- Characters of PSl(n|n) satisfy:

$$\boxed{\chi_{(a,s)}\ \chi_{(a,s)} \ = \ \chi_{(a+1,s)}\ \chi_{(a+1,s)} \ + \ \chi_{(a,s+1)}\ \chi_{(a,s-1)}}$$

$u \gg 1$ ~ Classical limit

$$\boxed{\mathcal{T}_{a,s}(u+1)\mathcal{T}_{a,s}(u-1) = \mathcal{T}_{a+1,s}(u+1)\mathcal{T}_{a-1,s}(u-1) + \mathcal{T}_{a,s+1}(u-1)\mathcal{T}_{a,s-1}(u+1)}$$

- The shifts of the spectral parameter presumably come from some kind of quantum effects.

The strategy of the derivation

$\mathcal{T}$'s = Transfer matrices

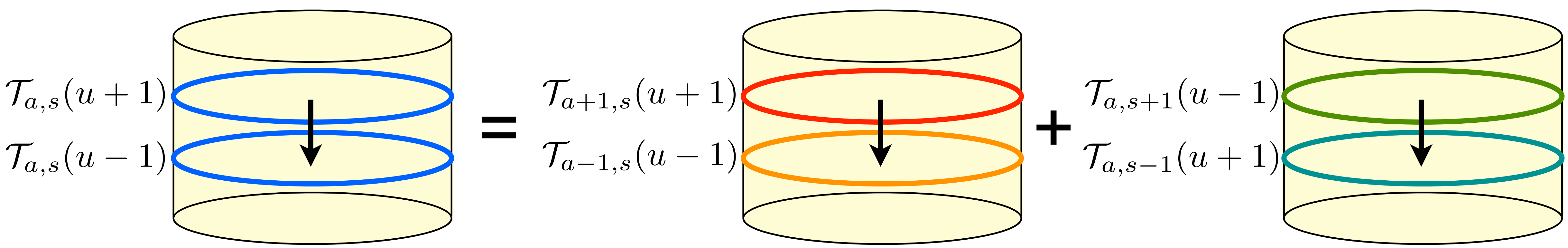
→ The Hirota equation is promoted to an operator identity.

Product of $\mathcal{T}$'s = Fusion of line operators

→ The shifts come from quantum effects associated with fusion.

$$\mathcal{T}_{a,s}(u+1) \triangleright \mathcal{T}_{a,s}(u-1) = \mathcal{T}_{a+1,s}(u+1) \triangleright \mathcal{T}_{a-1,s}(u-1) + \mathcal{T}_{a,s+1}(u-1) \triangleright \mathcal{T}_{a,s-1}(u+1)$$

$$\Updownarrow$$



We will demonstrate that this picture is correct at first order in perturbation theory.

Quantum currents

Sigma-models on supergroups admit a one-parameter family of flat connections:

$$\boxed{A(u) = f(u)\, J\, dz + \bar{f}(u)\, \bar{J}\, d\bar{z}}$$

Noether currents

The structure of the current-current OPEs is the following:

$$\boxed{J(z)J(0) = (2\mathrm{nd}-\mathrm{order\ pole})Id + (1\mathrm{st}-\mathrm{order\ pole})J(0) + \ldots}$$

Known to all orders

Ashok, R.B. & Troost, 2009

Konechny & Quella, 2010

Perturbation theory is easily implemented:

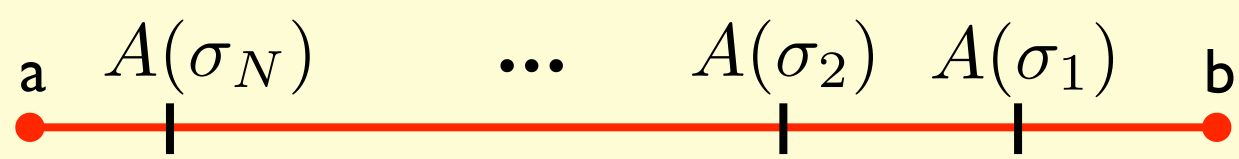
The coefficients of all poles are of order R^{-2} → Computation at order p $\Leftrightarrow$ Perform p OPEs.

UV divergences in line operators

We expand the line operators:

$$W^{b,a} = P\exp\left(-\int_a^b A\right) = \sum_{N=0}^{\infty} W_N^{b,a}$$

with: $W_N^{b,a}$:

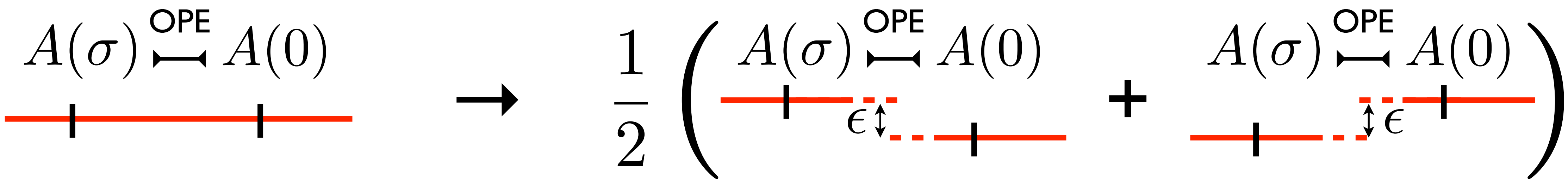


Collisions of integrated operators lead to divergences.

$\Rightarrow$ $\boxed{\text{We need to regularize and potentially renormalize the line operators.}}$

Regularization of divergences

We use a "principal value" regularization scheme:



For instance for a simple pole:

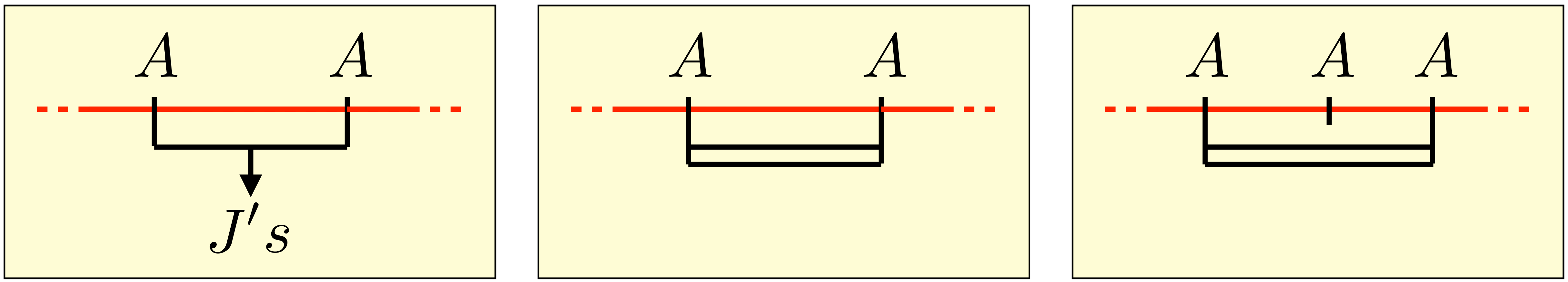
$$\frac{1}{\sigma} \longrightarrow \frac{1}{2}\left(\frac{1}{\sigma + i\epsilon} + \frac{1}{\sigma - i\epsilon} \right)$$

$$= \frac{\sigma}{\sigma^2 + \epsilon^2} \equiv P.V.\frac{1}{\sigma}$$

Line operator: Divergences at first order

There are three sources of divergences:

1st-order poles: 2nd-order poles:



When the dual Coxeter number is zero, the sum of these three terms cancels, but there are less of these.

We end up with a logarithmic divergence:

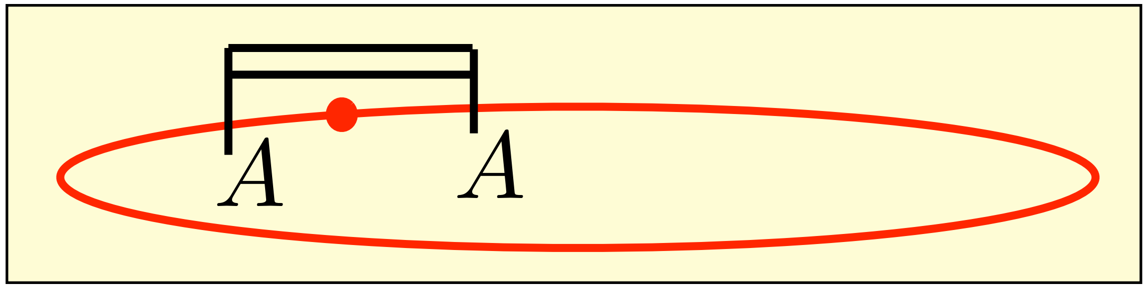
$$log\,\epsilon\,(W t^a t_a + t^a t_a W)$$

Line operator

Generators of the algebra.

Divergences in the loop operators

There is a new source of divergences in loop operators:



It contributes to the logarithmic divergences:

$$log\,\epsilon\,(\Omega t^a t_a + t^a t_a \Omega - 2t^a \Omega t_a)$$

Loop operator

We deduce that:

The transfer matrix is free of divergences
up to first order in perturbation theory.

The vanishing of the dual Coxeter number is crucial.

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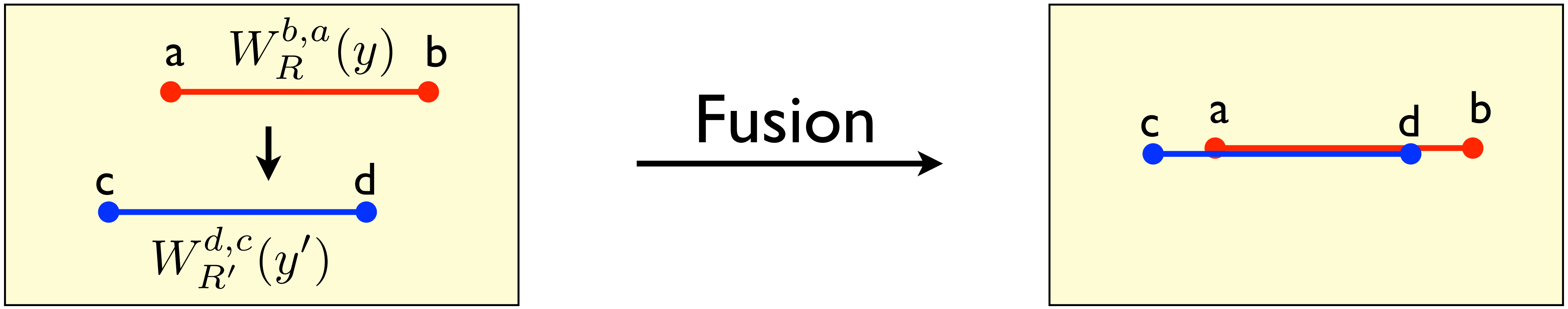
Fusion of line operators



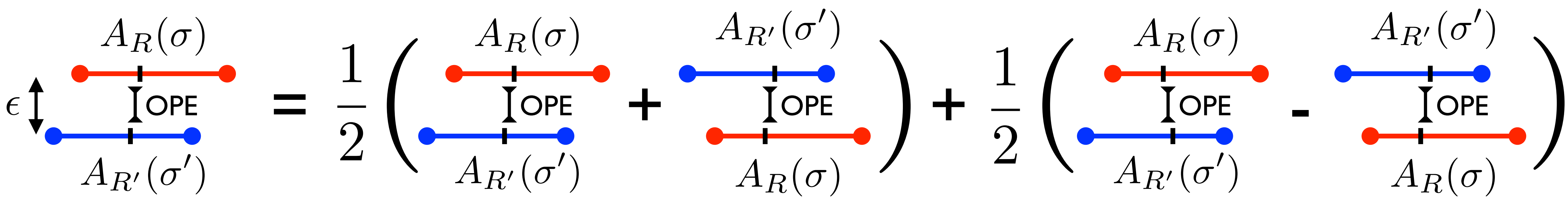
We denote the fusion as: $W_R^{b,a}(y) \triangleright W_{R'}^{d,c}(y')$

- The classical process is simple.
- Collisions of integrated connections induce quantum corrections that we are going to compute.

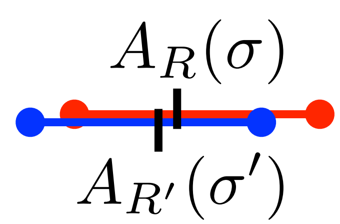
Disentangling the OPEs

Mikhailov & Schafer-Nameki, 2007b

We write the OPE between two connections as:



Regularized OPE in the double-line operator



Quantum correction associated with fusion.

For instance for a simple pole:

$$\frac{1}{\sigma + i\epsilon - \sigma'} = \underbrace{\frac{1}{2}\left(\frac{1}{\sigma + i\epsilon - \sigma'} + \frac{1}{\sigma - i\epsilon - \sigma'}\right)}_{\boxed{P.V.\ \frac{1}{\sigma - \sigma'}}} + \underbrace{\frac{1}{2}\left(\frac{1}{\sigma + i\epsilon - \sigma'} - \frac{1}{\sigma - i\epsilon - \sigma'}\right)}_{\boxed{-i\pi\delta_\epsilon(\sigma - \sigma')}}$$

Commutator of connections

- To compute the quantum corrections in the process of fusion, the relevant OPE is:

$$\lim_{\epsilon \to 0^+} (1 - P.V.) A_R(y; \sigma + i\epsilon) A_{R'}(y'; \sigma') \;=\; \frac{1}{2}[A_R(y;\sigma), A_{R'}(y';\sigma')]$$

- From the current-current OPEs, we obtain:

$$\boxed{[A_R(y;\sigma), A_{R'}(y';\sigma')] = 2s\delta'(\sigma - \sigma') + [A_R(y;\sigma), r + s]\,\delta(\sigma - \sigma') + [A_{R'}(y';\sigma'), r - s]\,\delta(\sigma - \sigma')}$$

We recognize a (r,s) system with:

$$r, s \;\sim\; t^{a,R} \otimes t_a^{R'}$$

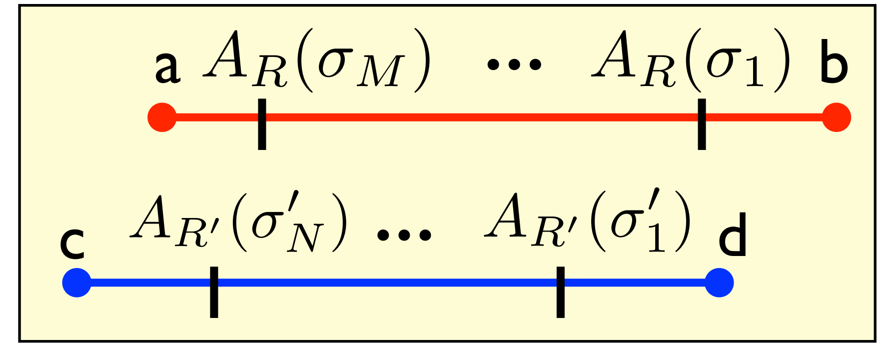
Maillet, 1985

Maillet, 1986

Fusion at first order

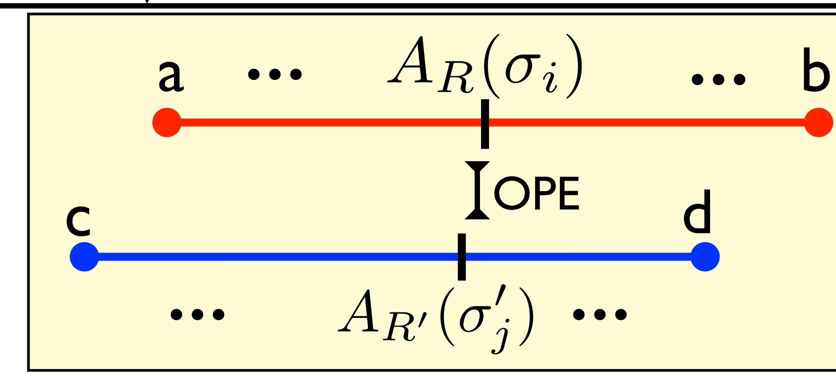
We consider the line operators:

$$\sum_{M,N=0}^{\infty}$$

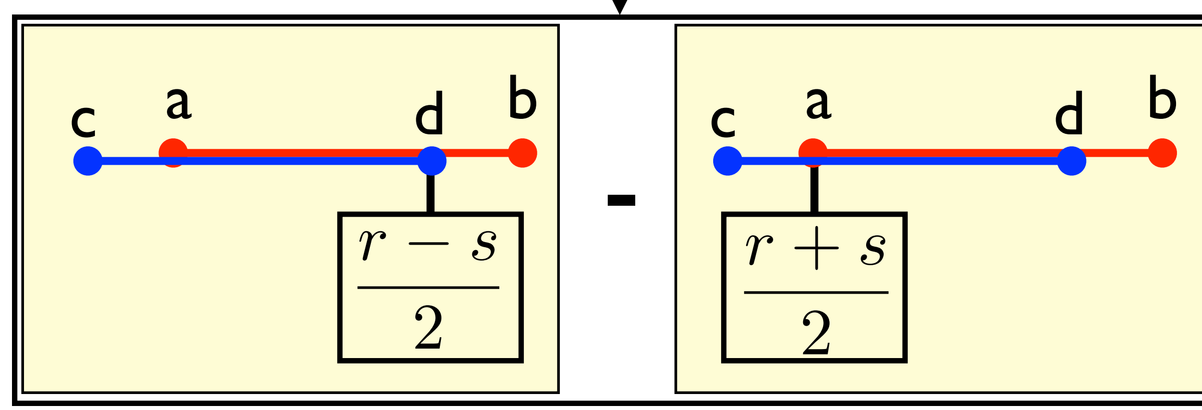


We perform one OPE between two connections sitting on different contours:

$$\sum_{M,N=0}^{\infty}\sum_{i=1}^{M}\sum_{j=1}^{N}$$



With some efforts we can sum all terms to get:

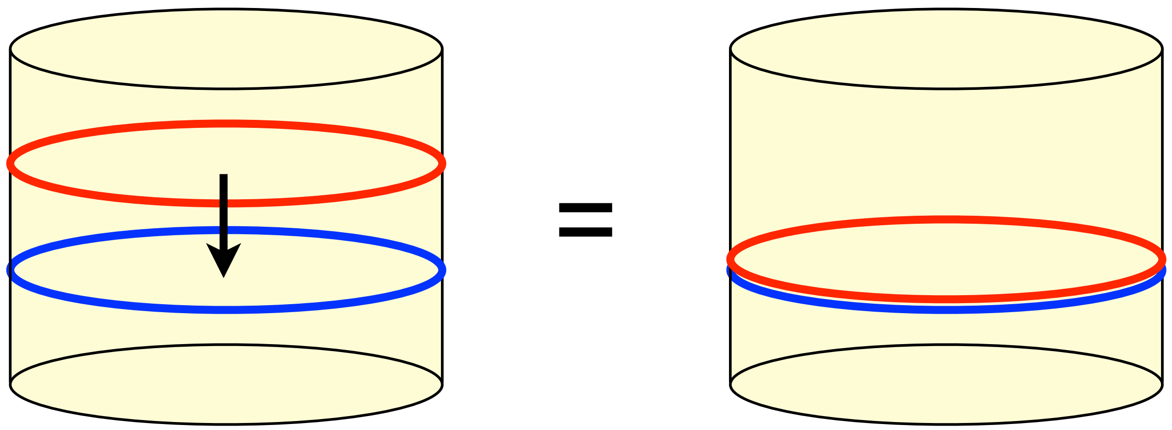


This agrees with the commutator of transition matrices derived in the Hamiltonian formalism.

Maillet, 1986

Fusion of transfer matrices at first order

The fusion of transfer matrices is trivial at first order:



$$+\,\mathcal{O}(R^{-4})$$

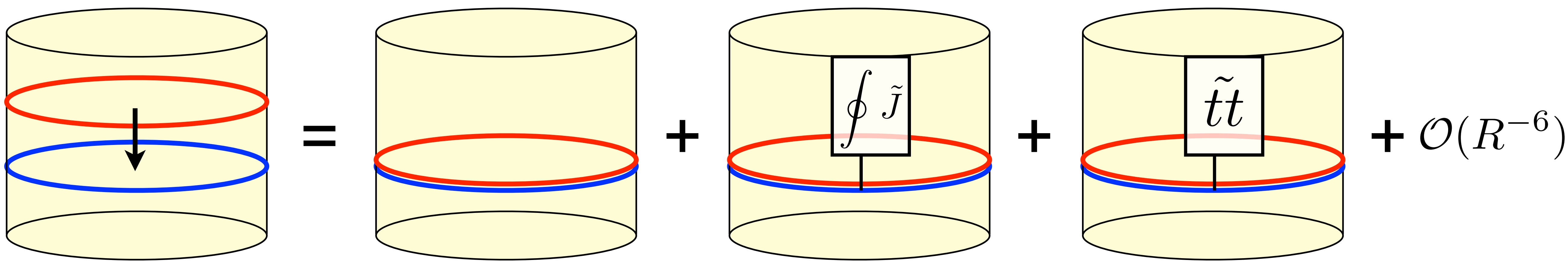
In particular the transfer matrices commute:

$$[\mathcal{T}_R(x), \mathcal{T}_{R'}(x')] = 0 + \mathcal{O}(R^{-4})$$

→ To get the leading quantum correction to the fusion of transfer matrices, we have to go to second order.

Symmetric fusion of transfer matrices

We obtain:



Additional operator integrated on the contour

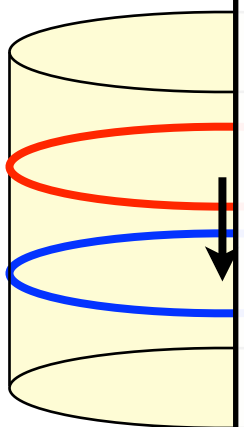
$$\tilde{J} \sim \tilde{J}^a \times f_a{}^{bc} f_c{}^{de} \times t_e t_d t_b$$

Constant matrix inserted between the integrated connections

$$\tilde{t}t \sim f^{abc} f_{cb}{}^{d} \times t_d t_a$$

Symmetric fusion of transfer matrices

We obtain:



$$
\begin{aligned}
\tilde{J} = (i\pi R^{-2})^2 \sum_{m,n,p,q,r} & f_{C_p}{}^{B_n A_m} f_{E_r}{}^{C_p D_q} \{t^R_{D_q}, t^R_{A_m}] t^{R'}_{B_n} \\
& \times \Big(J_r^{E_r}\big(\tilde{D}'^{p}_{mn} F_q C^r_{pq} - \tilde{D}'^{p}_{mn} \bar{F}_q C^r_{p\bar{q}} - \tilde{D}'^{\bar{p}}_{mn} F_q C^r_{\bar{p}q} - \tilde{D}'^{\bar{p}}_{mn} \bar{F}_q C^r_{\bar{p}\bar{q}} \\
& \qquad + \frac{1}{2} F_r(\tilde{D}'^{s}_{mn} F_p C_{sp} + \tilde{D}'^{s}_{pn} F_m C_{sm} - \tilde{D}'^{\bar{s}}_{mn} \bar{F}_p C_{\bar{s}\bar{p}} - \tilde{D}'^{\bar{s}}_{pn} \bar{F}_m C_{\bar{s}\bar{m}})\big) \\
& \quad + \bar{J}_r^{E_r}\big(\tilde{D}'^{p}_{mn} F_q C^{\bar{r}}_{pq} + \tilde{D}'^{p}_{mn} \bar{F}_q C^{\bar{r}}_{p\bar{q}} + \tilde{D}'^{\bar{p}}_{mn} F_q C^{\bar{r}}_{\bar{p}q} - \tilde{D}'^{\bar{p}}_{mn} \bar{F}_q C^{\bar{r}}_{\bar{p}\bar{q}} \\
& \qquad + \frac{1}{2} \bar{F}_r(\tilde{D}'^{s}_{mn} F_p C_{sp} + \tilde{D}'^{s}_{pn} F_m C_{sm} - \tilde{D}'^{\bar{s}}_{mn} \bar{F}_p C_{\bar{s}\bar{p}} - \tilde{D}'^{\bar{s}}_{pn} \bar{F}_m C_{\bar{s}\bar{m}})\big)\Big) \\
& + f_{C_p}{}^{B_n A_m} f_{E_r}{}^{D_q C_p} t^R_{A_m} \{t^{R'}_{B_n}, t^{R'}_{D_q}] \\
& \times \Big(J_r^{E_r}\big(-\tilde{D}^{p}_{mn} F_q C^r_{pq} + \tilde{D}^{p}_{mn} \bar{F}_q C^r_{p\bar{q}} + \tilde{D}^{\bar{p}}_{mn} F_q C^r_{\bar{p}q} + \tilde{D}^{\bar{p}}_{mn} \bar{F}_q C^r_{\bar{p}\bar{q}} \\
& \qquad - \frac{1}{2} F_r(\tilde{D}^{s}_{mn} F_p C_{sp} + \tilde{D}^{s}_{pn} F_m C_{sm} - \tilde{D}^{\bar{s}}_{mn} \bar{F}_p C_{\bar{s}\bar{p}} - \tilde{D}^{\bar{s}}_{pn} \bar{F}_m C_{\bar{s}\bar{m}})\big) \\
& \quad + \bar{J}_r^{E_r}\big(-\tilde{D}^{p}_{mn} F_q C^{\bar{r}}_{pq} - \tilde{D}^{p}_{mn} \bar{F}_q C^{\bar{r}}_{p\bar{q}} - \tilde{D}^{\bar{p}}_{mn} F_q C^{\bar{r}}_{\bar{p}q} + \tilde{D}^{\bar{p}}_{mn} \bar{F}_q C^{\bar{r}}_{\bar{p}\bar{q}}\big) \\
& \qquad - \frac{1}{2} \bar{F}_r(\tilde{D}^{s}_{mn} F_p C_{sp} + \tilde{D}^{s}_{pn} F_m C_{sm} - \tilde{D}^{\bar{s}}_{mn} \bar{F}_p C_{\bar{s}\bar{p}} - \tilde{D}^{\bar{s}}_{pn} \bar{F}_m C_{\bar{s}\bar{m}})\big)\Big)
\end{aligned}
$$

$+\, \mathcal{O}(R^{-6})$

Derivation of the T-system I

The goal is to show that:

$$\boxed{\mathcal{T}_{a,s}(u+1) \triangleright \mathcal{T}_{a,s}(u-1) = \mathcal{T}_{a+1,s}(u+1) \triangleright \mathcal{T}_{a-1,s}(u-1) + \mathcal{T}_{a,s+1}(u-1) \triangleright \mathcal{T}_{a,s-1}(u+1)}$$

- We perform a semi-classical expansion:

$$\sum_{R,R'} \mathcal{T}_R(u+1) \triangleright \mathcal{T}_{R'}(u-1)$$

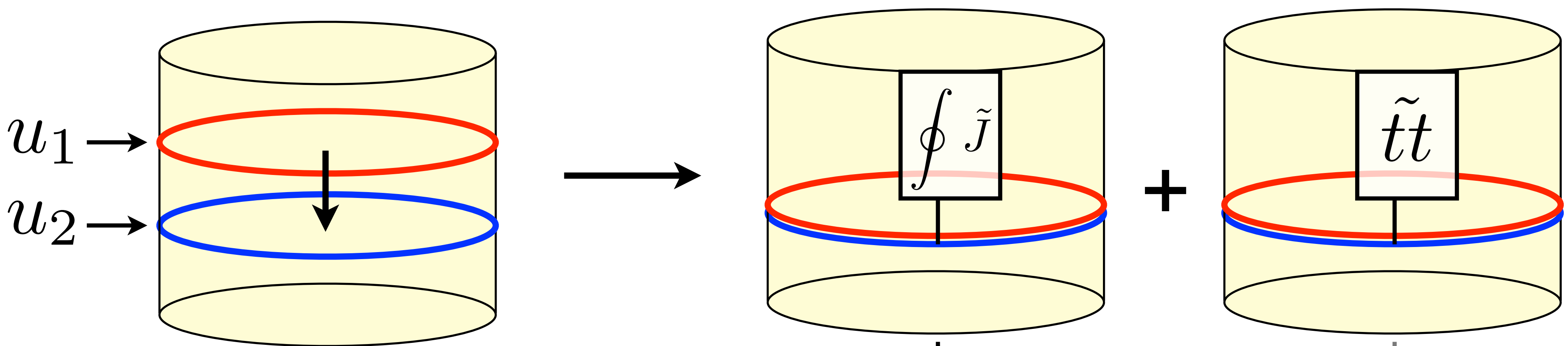
$$= \boxed{\sum_{R,R'} \mathcal{T}_R(u)\mathcal{T}_{R'}(u)} + \underline{\sum_{R,R'} \left(\partial_u \mathcal{T}_R(u)\mathcal{T}_{R'}(u) - \mathcal{T}_R(u)\partial_u \mathcal{T}_{R'}(u)\right)} + \boxed{\text{Leading quantum corrections from fusion}} + \dots$$

Character identity
$\Rightarrow \varnothing$

$\varnothing$??

Derivation of the T-system II

- Previously we computed the leading quantum correction:



Subleading

$$\tilde{J} \sim \boxed{\tilde{J}^a} \times \boxed{f_a{}^{bc} f_c{}^{de} \times t_e t_d t_b}$$

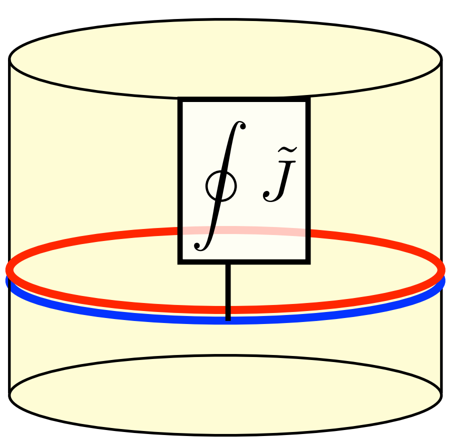
$u_1 - u_2 \ll u_1$

$\tilde{J}^a = \partial_u A^a(u) + \text{subleading}$

Character identities from Kazakov & Vieira, 2007

t_a

$$\sum_{R,R'}$$



$$= -\sum_{R,R'} \left(\partial_u \mathcal{T}_R(u) \mathcal{T}_{R'}(u) - \mathcal{T}_R(u) \partial_u \mathcal{T}_{R'}(u) \right) + \ldots$$

Derivation of the T-system III

We obtain eventually:

$$\sum_{R,R'} \mathcal{T}_R(u+1) \triangleright \mathcal{T}_{R'}(u-1) = \boxed{\sum_{R,R'} \mathcal{T}_R(u)\mathcal{T}_{R'}(u)}$$

Character identity $\Rightarrow \varnothing$

$$+ (\boxed{1} - \boxed{1}) \sum_{R,R'} (\partial_u \mathcal{T}_R(u)\mathcal{T}_{R'}(u) - \mathcal{T}_R(u)\partial_u \mathcal{T}_{R'}(u)) \; + \ldots$$

From the derivative expansion

From the quantum effects in fusion

We have derived from first principles the T-system up to first order in perturbation theory.

" The shifts come from fusion "

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Summary of the technical results

We studied quantum integrability of conformal sigma models on supergroups and supercosets:

- Starting from the current-current OPEs, we computed the fusion of line operators up to second order.

- We deduced a perturbative proof of the Hirota equation as an operator identity.

Summary of the conceptual results

In the case of string theory on $AdS_5 \times S^5$: we obtained a first-principles, perturbative derivation of the AdS/CFT Y-system.

The same integrability techniques can be used to solve the spectrum of generic conformal sigma-models on supergroups and supercosets.

This applies to string theory on:

$AdS_4 \times CP^3$ $AdS_2 \times S^2$ $AdS_3 \times S^3 \times S^3$ ... Zarembo, 2010

Fusion vs TBA

☺ No hypothesis	☺ All states	← Fusion wins
? Energy(T's)	? Analytic properties	← More work is needed
☹ Perturbative		← TBA wins

At that point, the two approaches are complementary.

Thank you.

Superstrings in $AdS_3 \times S^3$

- Strings in $AdS_3 \times S^3$ with RR and/or NS fluxes can be described in the hybrid formalism.

Hybrid string on $AdS_3 x S^3$ $\Leftrightarrow$ Sigma model on $PSU(1,1|2)$ + ghosts

Can be treated pertubatively.

Berkovits, Vafa & Witten, 1999

$\Rightarrow$ String theory in $AdS_3 \times S^3$ realizes the T-system

- Up to first order in the large radius expansion.
- At zeroth-order in the ghosts expansion.

The pure spinor string on $AdS_5 \times S^5$

The worldsheet theory is a sigma-model on $\frac{PSU(2,2|4)}{SO(5) \times SO(4,1)}$ coupled to ghosts.

Berkovits, 2000

The action is:

$$S = \frac{R^2}{4\pi} STr \int d^2w \left(J_2 \bar{J}_2 + \frac{3}{2} J_3 \bar{J}_1 + \frac{1}{2} \bar{J}_3 J_1 \right)$$
$$+ \frac{R^2}{2\pi} STr \int d^2w \left(N \bar{J}_0 + \hat{N} J_0 - N \hat{N} + w \bar{\partial} \lambda + \hat{w} \partial \hat{\lambda} \right)$$

The J_i's are the $\mathbb{Z}_4$ components of the Maurer-Cartan current:

$$g \in PSU(2,2|4): \ g^{-1} dg = J_0 + J_1 + J_2 + J_3$$

Pure spinor ghosts and their conjugate momenta (λ, w) $(\hat{\lambda}, \hat{w})$ $\rightarrow$ $N = -\{w, \lambda\}$ $N = -\{\hat{w}, \hat{\lambda}\}$ Pure spinor Lorentz currents