# Symplectic structures on moduli spaces of framed sheaves on surfaces

Francesco Sala

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# Gauge Theory

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# Algebraic Geometry

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Instantons are classified by their instanton number n.

## Definition

- A framed SU(r)-instanton on  $S^4$  is a pair  $(A, \phi)$  where
  - $\bullet~A$  is an ASD connection on a principal  $SU(r)\mbox{-bundle}~P$  on  $S^4\mbox{,}$
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We call these objects *framed vector bundles* on  $\mathbb{CP}^2$ .

 $\mathcal{M}_{lf}(r,n)$  is an open subscheme of the moduli space  $\mathcal{M}(r,n)$  of isomorphism classes of *framed sheaves* on  $\mathbb{CP}^2$  of rank r with second Chern class n.

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A framed sheaf on  $\mathbb{CP}^2$  of rank r with second Chern class n is a pair  $(E,\alpha),$  in which

- E is a torsion free sheaf on  $\mathbb{CP}^2$ , a vector bundle in a neighborhood of a fixed line  $l_{\infty}$ ,
- $\alpha$  is an isomorphism  $E|_{l_{\infty}} \xrightarrow{\sim} \mathcal{O}_{l_{\infty}}^{\oplus r}$ .

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# Question

Is there a symplectic structure on  $\mathcal{M}(r, n)$ ?

#### Answer

By using the ADHM data description, Nakajima realized the moduli space  $\mathcal{M}(r,n)$  as a hyper-Kähler quotient.

By fixing a complex structure on  $\mathcal{M}(r, n)$ , one can define a holomorphic symplectic form on  $\mathcal{M}(r, n)$ .

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 Bottacin defined Poisson structures on moduli spaces of framed <u>vector bundles</u> on surfaces.

He constructed a symplectic structure on  $\mathcal{M}_{lf}(r, n)$  and on moduli spaces of framed <u>vector bundles</u> on other rational surfaces.

 By using a modified version of the Atiyah class for a family of framed sheaves, I defined closed two-forms on moduli spaces of framed <u>sheaves</u> on surfaces.

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• Characterize the *Lagrangian subvarieties* of the moduli spaces of framed sheaves.

In the case of  $\mathcal{M}(r, n)$ , the Lagrangian subvarieties parametrize solutions of *vortex equations*. (see Bonelli, Tanzini, Zhao. *Vertices, Vortices and Interacting Surface Operators* (arXiv:1102.0184)).

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# Warning!

In the following I will deal with schemes and coherent sheaves on them.

By using Serre's GAGA principles, one can think:

ALGEBRAIC	$\Leftrightarrow$	COMPLEX
GEOMETRY		DIFFERENTIAL GEOMETRY

 $\begin{array}{rcl} \text{Noetherian schemes} & \Leftrightarrow & \text{Complex analytic spaces} \\ \text{of finite type over } \mathbb{C} & \\ \text{Smooth varieties over } \mathbb{C} & \Leftrightarrow & \text{complex manifolds} \\ & & \text{Coherent sheaves} & \Leftrightarrow & \text{Coherent analytic sheaves} \end{array}$ 

#### Definition

Let D be an effective divisor of X and  $F_D$  a vector bundle on D. We say that a coherent sheaf E on X is  $(D, F_D)$ -framable if

- $\bullet$  E is torsion free,
- E is a vector bundle in a neighborhood of D,
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A  $(D, F_D)$ -framed sheaf is a pair  $\mathcal{E} := (E, \alpha)$  consisting of

- a  $(D, F_D)$ -framable sheaf E,
- a  $(D, F_D)$ -framing  $\alpha$ .

Two  $(D, F_D)$ -framed sheaves  $(E, \alpha)$  and  $(E', \alpha')$  are isomorphic if there is an isomorphism  $f: E \to E'$  such that  $\alpha' \circ f|_D = \alpha$ .

Framed sheaves

# Framed modules

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# Framed modules

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# Theorem (Bruzzo, Markushevich)

There exists a moduli space  $\mathcal{M}(X; F_D, P)$  for  $(D, F_D)$ -framed sheaves on X with Hilbert polynomial P, under the following assumptions:

• D is a big and nef divisor,

•  $F_D$  is a Gieseker semistable vector bundle on D.

 $\mathcal{M}(X; F_D, P)$  is a quasi-projective scheme over  $\mathbb{C}$ .

If the surface X is rational and D is a smooth connected curve such that  $D \cong \mathbb{CP}^1$  and  $D^2 > 0$ ,  $\mathcal{M}(X; \mathcal{O}_D^{\oplus r}, P)$  is a smooth quasi-projective variety.

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# The Atiyah class

# Let Y be a Noetherian scheme of finite type over $\mathbb{C}$ .

#### Definition

Let E be a coherent sheaf on Y. We call *sheaf of first jets*  $J^1(E)$  of E the coherent sheaf on Y defined as follows:

- as a sheaf of  $\mathbb{C}$ -vector spaces, we set  $\mathrm{J}^1(E):=(\Omega^1_Y\otimes E)\oplus E$ ,
- for any  $y \in Y$ ,  $a \in \mathcal{O}_{Y,y}$  and  $(z \otimes e, f) \in \mathrm{J}^1(E)_y$ , we define

 $a(z \otimes e, f) := (az \otimes e + d(a) \otimes f, af).$ 

The sheaf  $J^1(E)$  fits into an exact sequence of coherent sheaves

$$0 \longrightarrow \Omega^1_Y \otimes E \longrightarrow \mathcal{J}^1(E) \longrightarrow E \longrightarrow 0.$$
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Let E be a coherent sheaf on Y. We call *Atiyah class* of E the class at(E) in  $Ext^1(E, \Omega^1_Y \otimes E)$  associated to the extension (1).

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Let E be a coherent sheaf on Y. An algebraic connection  $\nabla$  on E is a  $\mathbb{C}\text{-linear morphism}$ 

$$\nabla \colon E \longrightarrow \Omega^1_Y \otimes E$$

such that (locally)  $\nabla(f\cdot e)=f\cdot\nabla(e)+d(f)\otimes e.$ 

#### Proposition

The Atiyah class at(E) is the obstruction to the existence of an algebraic connection on E, i.e., at(E) = 0 iff there exists an algebraic connection on E.

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Let S be a Noetherian scheme of finite type over  $\mathbb{C}$ .

#### Definition

A flat family of  $(D, F_D)$ -framed sheaves parametrized by S is a pair  $\mathcal{E} = (E, \alpha)$  where

- E is a coherent sheaf on  $S \times X$ , flat over S,
- $\alpha \colon E \to p_X^*(F_D)$  is a morphism,

such that for any  $s \in S$  the pair  $(E|_{\{s\}\times X}, \alpha|_{\{s\}\times X})$  is a  $(\{s\}\times D, p_X^*(F_D)|_{\{s\}\times D})$ -framed sheaf on  $\{s\}\times X$ .

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To the coherent sheaf E, we associate its sheaf of first jets

 $J^1(E) \cong (\Omega^1_{S \times X} \otimes E) \oplus E$  as sheaf of  $\mathbb{C}$ -vector spaces.

Since  $\Omega^1_{S \times X} \cong p^*_S(\Omega^1_S) \oplus p^*_X(\Omega^1_X)$ , we have

$$\begin{split} \mathrm{J}^1(E) &\cong & ((p_S^*(\Omega_S^1) \oplus p_X^*(\Omega_X^1)) \otimes E) \oplus E \\ &\cong & (p_S^*(\Omega_S^1) \otimes E) \oplus (p_X^*(\Omega_X^1) \otimes E) \oplus E \\ & \text{ as sheaf of } \mathbb{C}\text{-vector spaces.} \end{split}$$

The *framed sheaf of first jets*  $J_{fr}^1(\mathcal{E})$  is the subsheaf of the sheaf of first jets  $J^1(E)$  consisting of those sections whose  $p_S^*(\Omega_S^1)$ -part vanishes along  $S \times D$ .

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The framed sheaf of first jets  $J_{fr}^1(\mathcal{E})$  of  $\mathcal{E}$  fits into an exact sequence:

$$0 \longrightarrow \left(p_{S}^{*}(\Omega_{S}^{1})(-S \times D) \oplus p_{X}^{*}(\Omega_{X}^{1})\right) \otimes E \longrightarrow \mathcal{J}_{fr}^{1}(\mathcal{E}) \longrightarrow E \longrightarrow 0,$$
(2)
where  $p_{S}^{*}(\Omega_{S}^{1})(-S \times D) := p_{S}^{*}(\Omega_{S}^{1}) \otimes \mathcal{O}_{S \times X}(-S \times D).$ 

#### Definition

Let  $\mathcal{E} = (E, \alpha)$  be a flat family of framed sheaves parametrized by a scheme S. We call *framed Atiyah class* of the family  $\mathcal{E}$  the class  $at(\mathcal{E})$  in Ext<sup>1</sup> $(E, (\alpha^*(\Omega^1)), S \times D) \oplus \alpha^*(\Omega^1)) \otimes E$ )

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# The Kodaira-Spencer map for framed sheaves

The framed Atiyah class  $at(\mathcal{E})$  of  $\mathcal{E}$  induces a morphism  $\mathcal{A}t_S(\mathcal{E})$  $\mathcal{O}_S \to \mathcal{E}xt^1_{n_S}(E, p^*_S(\Omega^1_S) \otimes p^*_X(\mathcal{O}_X(-D)) \otimes E).$ 

#### Definition

The *framed Kodaira-Spencer map* associated to the family  $\mathcal{E}$  is the composition

$$\begin{split} KS_{fr} \colon (\Omega_{S}^{1})^{\vee} & \stackrel{\mathrm{id} \otimes \mathcal{A}t_{S}(\mathcal{E})}{\longrightarrow} \\ & \longrightarrow (\Omega_{S}^{1})^{\vee} \otimes \mathcal{E}xt_{p_{S}}^{1}(E, p_{S}^{*}(\Omega_{S}^{1}) \otimes p_{X}^{*}(\mathcal{O}_{X}(-D)) \otimes E) \to \\ & \longrightarrow \mathcal{E}xt_{p_{S}}^{1}(E, p_{S}^{*}((\Omega_{S}^{1})^{\vee} \otimes \Omega_{S}^{1}) \otimes p_{X}^{*}(\mathcal{O}_{X}(-D)) \otimes E) \to \\ & \longrightarrow \mathcal{E}xt_{p_{S}}^{1}(E, p_{X}^{*}(\mathcal{O}_{X}(-D)) \otimes E). \end{split}$$

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$$\begin{split} KS_{fr} \colon (\Omega_{S}^{1})^{\vee} & \stackrel{\operatorname{id} \otimes \mathcal{A}t_{S}(\mathcal{E})}{\longrightarrow} \\ & \longrightarrow (\Omega_{S}^{1})^{\vee} \otimes \mathcal{E}xt_{p_{S}}^{1}(E, p_{S}^{*}(\Omega_{S}^{1}) \otimes p_{X}^{*}(\mathcal{O}_{X}(-D)) \otimes E) \to \\ & \longrightarrow \mathcal{E}xt_{p_{S}}^{1}(E, p_{S}^{*}((\Omega_{S}^{1})^{\vee} \otimes \Omega_{S}^{1}) \otimes p_{X}^{*}(\mathcal{O}_{X}(-D)) \otimes E) \to \\ & \longrightarrow \mathcal{E}xt_{p_{S}}^{1}(E, p_{X}^{*}(\mathcal{O}_{X}(-D)) \otimes E). \end{split}$$

## Remark

Let S be a smooth projective variety over  $\mathbb C$  and s a point on it. Then the framed Kodaira-Spencer map at the point s is

$$KS_{fr}: T_{S,s} \rightarrow (\mathcal{E}xt^{1}_{p_{S}}(E, p^{*}_{X}(\mathcal{O}_{X}(-D)) \otimes E))_{s}$$
$$\cong \operatorname{Ext}^{1}(E|_{\{s\} \times X}, E|_{\{s\} \times X}(-D)).$$

Let  $\mathcal{E} = (E, \alpha)$  be a flat family of framed sheaves parametrized by a smooth affine Noetherian scheme S of finite type over  $\mathbb{C}$ . From the Atiyah class  $at(\mathcal{E})$  of  $\mathcal{E}$ , we can define a class  $\gamma$  in

 $\mathrm{H}^{0}(S, \Omega_{S}^{2}) \otimes \mathrm{H}^{2}(X, \mathcal{O}_{X}(-2D)).$ 

 $\gamma$  is the (0,2)-part of the *Newton polynomial* of  $at(\mathcal{E})$ .

Definition

Let  $\tau_S$  be the homomorphism given by

 $\tau_S \colon \mathrm{H}^0(X, \omega_X(2D)) \cong \mathrm{H}^2(X, \mathcal{O}_X(-2D))^{\vee} \xrightarrow{\cdot \gamma} \mathrm{H}^0(S, \Omega_S^2),$ 

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Fix  $\omega \in \mathrm{H}^0(X, \omega_X(2D))$ . The two-form  $\tau_S(\omega)$  at a point  $s_0 \in S$  coincides with the following composition of maps:

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For any  $\omega \in \mathrm{H}^0(X, \omega_X(2D))$ , the associated two-form  $\tau_S(\omega)$  on S is closed.

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# Tangent bundle of moduli spaces of framed sheaves

Let D be big and nef effective divisor,  $F_D$  a Gieseker semistable vector bundle on D and P a numerical polynomial of degree two.

 $\mathcal{M}(X; F_D, P) =$ moduli space of  $(D, F_D)$ -framed sheaves on X with Hilbert polynomial P.

 $\mathcal{M}(X; F_D, P)^{sm}$  = the smooth locus of  $\mathcal{M}(X; F_D, P)$ .

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The framed Kodaira-Spencer map defined by  ${\mathcal E}$  induces a canonical isomorphism

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*Proof.* It suffices to prove that given a smooth affine scheme S, for any S-flat family  $\mathcal{E} = (E, \alpha)$  of  $(D, F_D)$ -framed sheaves on X defining a classifying morphism

$$\psi \colon S \longrightarrow \mathcal{M}(X; F_D, P)^{sm},$$
  
$$s \longmapsto [\mathcal{E}|_{\{s\} \times X}],$$

the pullback  $\psi^*( au(\omega))\in \mathrm{H}^0(S,\Omega^2_S)$  is closed.

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Let  $\omega \in \mathrm{H}^0(X, \omega_X(2D))$  and  $[(E, \alpha)]$  a point in  $\mathcal{M}(X; F_D, P)^{sm}$ .

#### Proposition

The closed two-form  $\tau(\omega)$  is non-degenerate at the point  $[(E, \alpha)]$  if and only if the multiplication by  $\omega$  induces an isomorphism

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If  $\omega_X(2D)$  is trivial, then  $1 \in \mathrm{H}^0(X, \omega_X(2D)) \cong \mathbb{C}$  defines a non-degenerate closed two-form.

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- $\mathbb{F}_p$  is the projective closure of the total space of the line bundle  $\mathcal{O}_{\mathbb{CP}^1}(-p)$  on  $\mathbb{CP}^1$ .
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#### Fact

 $l_\infty$  is a smooth connected big and nef curve of genus zero.

The Picard group of  $\mathbb{F}_p$  is generated by  $l_\infty$  and the fibre F of the projection  $\mathbb{F}_p \to \mathbb{CP}^1$ . One has

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In particular, the canonical divisor  $K_p$  can be expressed as

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• D is a smooth connected curve such that  $D \cong \mathbb{CP}^1$  and  $D^2 > 0$ .

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$$K_{\mathbb{F}_1} = -2 l_{\infty} - F \Rightarrow \omega_{\mathbb{F}_1}(2D) \cong \mathcal{O}_{\mathbb{F}_1}(F).$$

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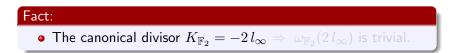
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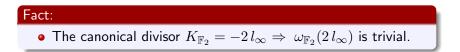
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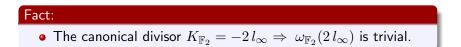
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The two-form  $\tau(1)$  defines a symplectic structure on  $\mathcal{M}(\mathbb{F}_2; F_D, r, a \, l_\infty + b \, F, n)^{sm}$ .

If  $F_D \cong \mathcal{O}_D^{\oplus r}$ , we have b = -2a.

Let us define  $C = l_{\infty} - 2F$ . This is the only irreducible curve in  $\mathbb{F}_2$  with negative self intersection. We can normalize the value a in the range  $0 \le a \le r - 1$  upon twisting by  $\mathcal{O}_{\mathbb{F}_2}(C)$ .

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