Duality and defects in Landau-Ginzburg models

based on work with **Daniel Murfet**

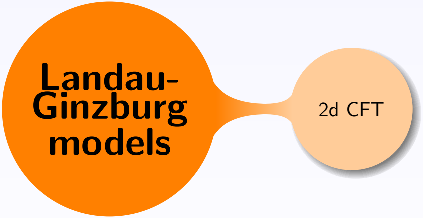
Nils Carqueville

LMU München

Bigger picture

Landau-
Ginzburg
models

Bigger picture



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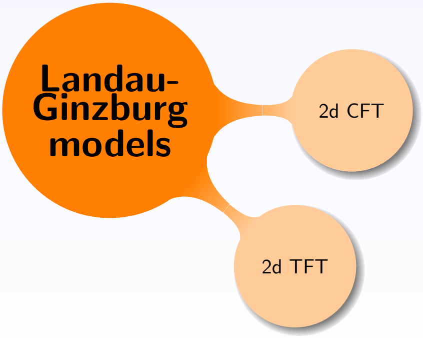

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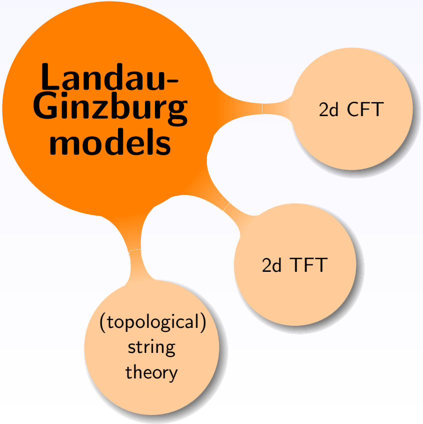

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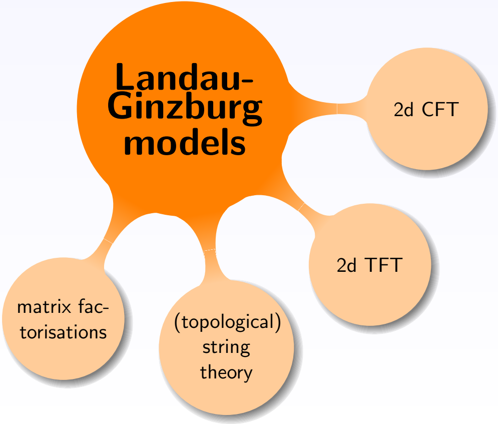

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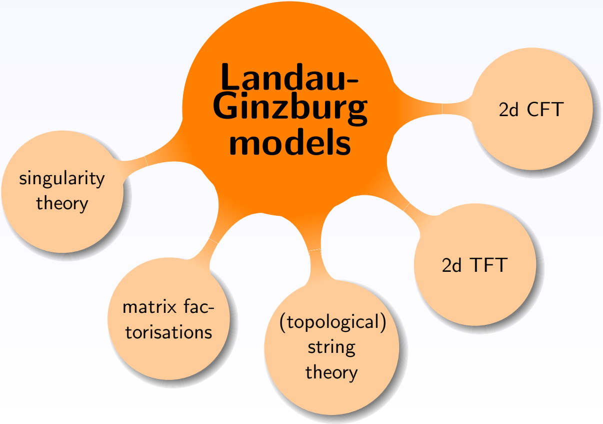

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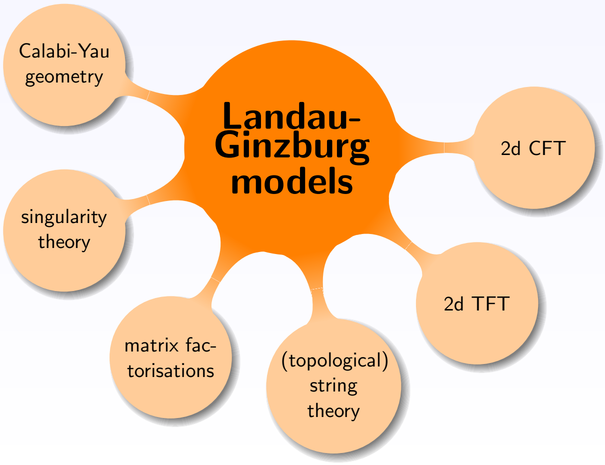

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Point

2d TFTs with defects

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- Why care?

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(conceptual construction, yet very “computable”)

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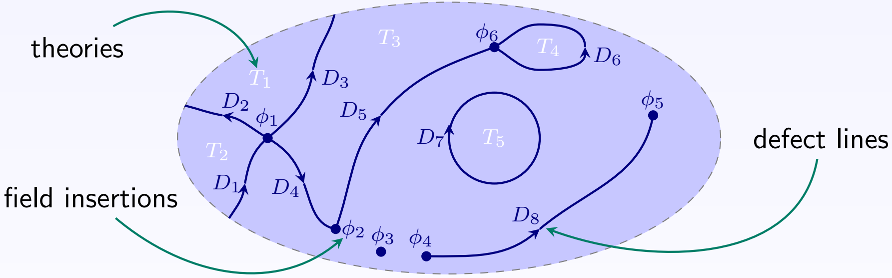
Plan.

- What?
- Why care?
  - see previous slide!
  - "computability" has many applications:
    - **proof of Cardy condition**
    - defect action on bulk fields
    - **generalised orbifolds**
    - ...

2d TFTs with defects

Worldsheet, partitioned into domains:



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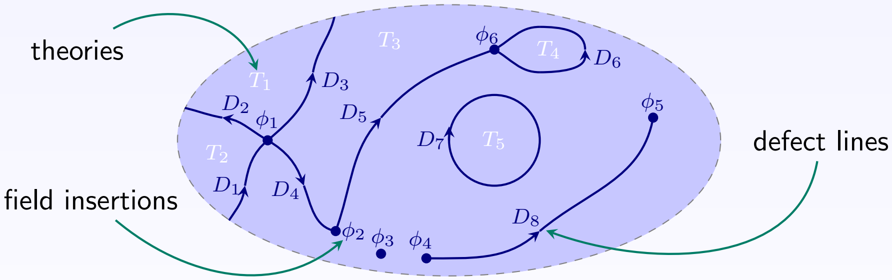

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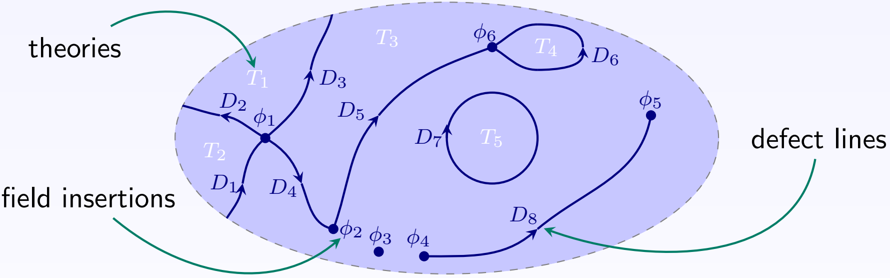

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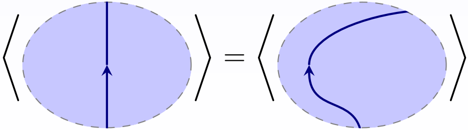

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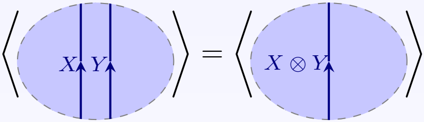
2d TFTs with defects

Defect fusion gives product



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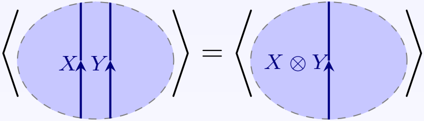

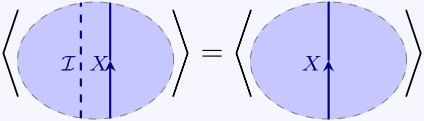

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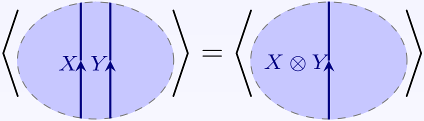
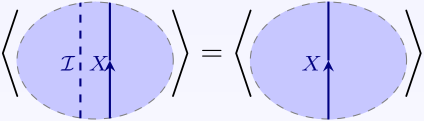
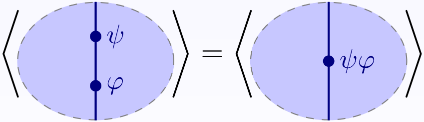




operator product of fields



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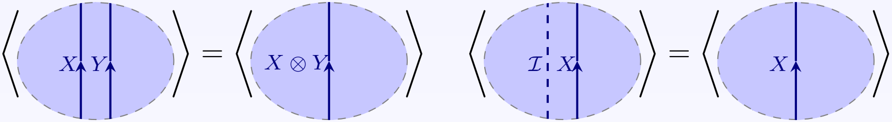

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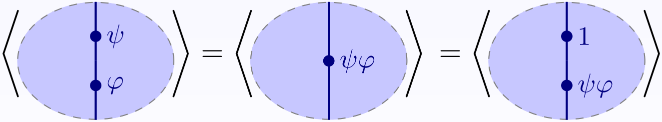

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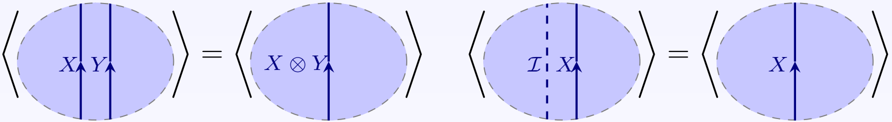

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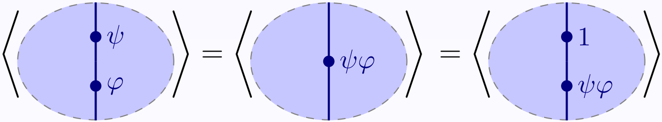

Claim. 2d TFTs with defects give **bicategory**:

- objects (domains) = theories
- 1-morphisms (lines) = defects
- 2-morphisms (points) = fields

Diagrammatics in bicategories

$= 1_X$

Joyal/Street 1991

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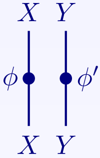

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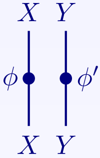

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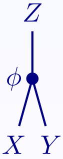

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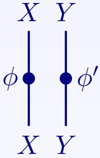

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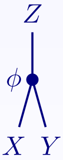

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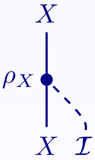

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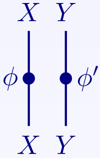

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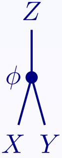

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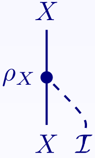

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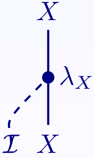

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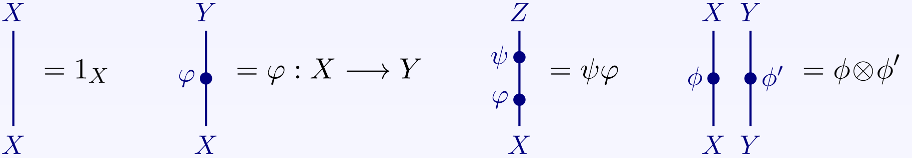
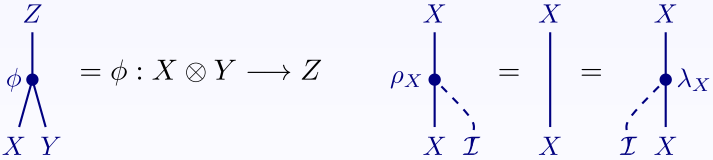
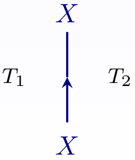




Orientation matters:



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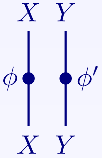

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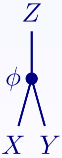

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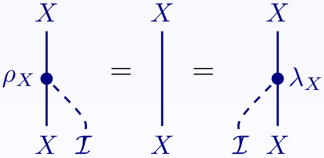

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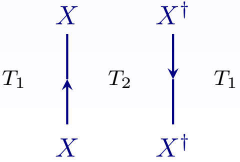

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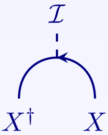
Orientation and adjoints



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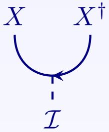

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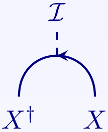

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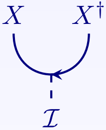

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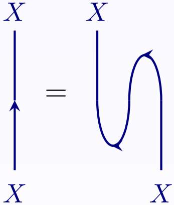
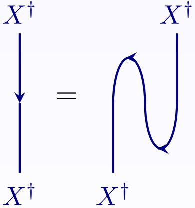
Defects are **topological**:





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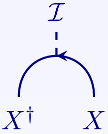

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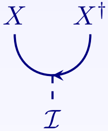

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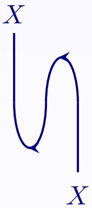

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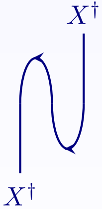

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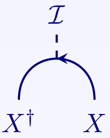

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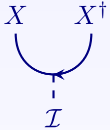

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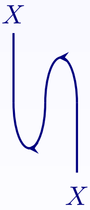

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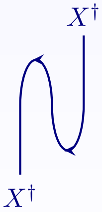

Definition. A bicategory **has adjoints** if for each 1-morphism X there is a 1-morphism $X^\dagger$ with 2-morphisms $\mathrm{ev}_X, \mathrm{coev}_X$ such that the above *Zorro moves* hold.

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$$d_X = \begin{pmatrix} 0 & d_X^1 \\ d_X^0 & 0 \end{pmatrix} \in \mathrm{End}^1_{R \otimes S}(X) \, , \qquad d_X^2 = (V - W) \cdot 1_X$$

Kontsevich, Kapustin/Li 2002, Lazaroiu 2003, Brunner/Roggenkamp 2007

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$$d_X = \begin{pmatrix} 0 & x^n \\ x^{d-n} & 0 \end{pmatrix} \, , \qquad \psi = \begin{pmatrix} x^i & 0 \\ 0 & x^i \end{pmatrix}$$

Kontsevich, Kapustin/Li 2002, Lazaroiu 2003, Brunner/Roggenkamp 2007

Landau-Ginzburg models

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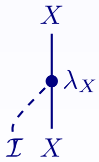

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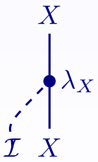

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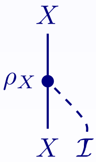

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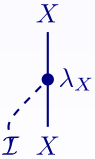

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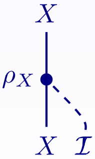

$: X\otimes\mathcal{I} \longrightarrow X$

- operator product: matrix multiplication

Main result

Theorem. Landau-Ginzburg models give a bicategory, called $\mathcal{LG}$.

Carqueville/Runkel 2009

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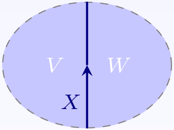

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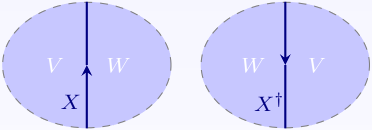

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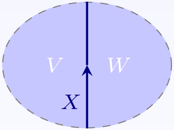

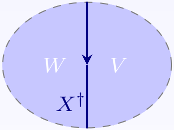

$$X^\dagger = X^\vee[n] \quad d_{X^\dagger} = \begin{pmatrix} 0 & (d_X^0)^{\mathrm{T}} \\ -(d_X^1)^{\mathrm{T}} & 0 \end{pmatrix}[n]$$

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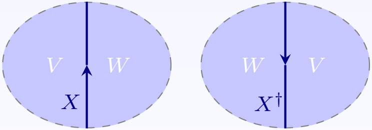

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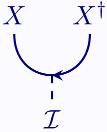

$$= \mathrm{coev}_X : 1 \longmapsto \partial_{[1]} d_X \dots \partial_{[n]} d_X \in X \otimes X^\vee$$

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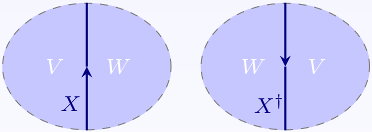

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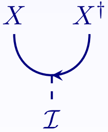

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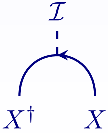

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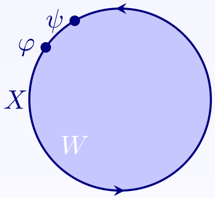
Applications

(1) **Boundary topological metric** / 2-point disk correlator



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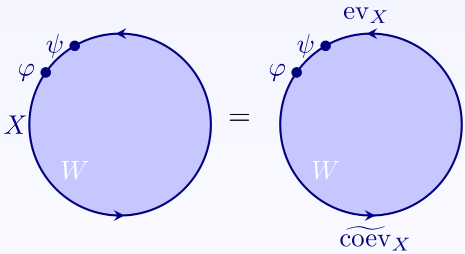

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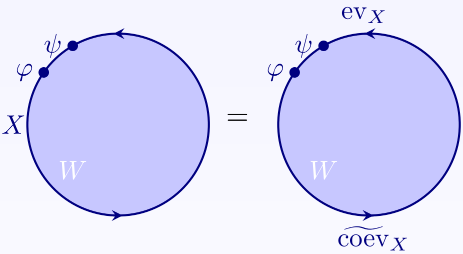

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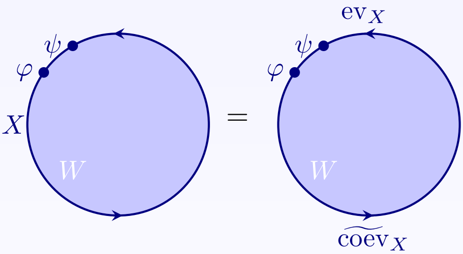

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Recover **Kapustin-Li pairing** as a 2-morphism!

Kapustin/Li 2003, Herbst/Lazaroiu 2004

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(2) **Defect action on bulk fields**

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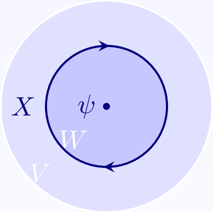
$\mathcal{D}_X^r(\psi) =$ 

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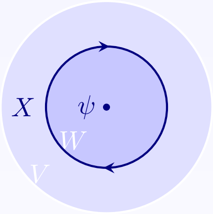

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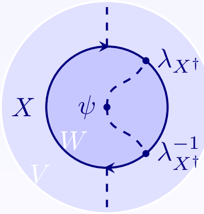

$$= \operatorname{Res}\left[\frac{\psi \operatorname{str}\left(\Lambda_X^{(x)}\Lambda_X^{(z)}\right)\underline{\mathrm{d}x}}{\partial_{x_1}W \ldots \partial_{x_n}W}\right]$$

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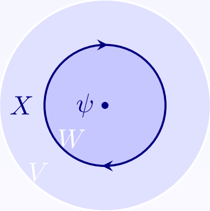

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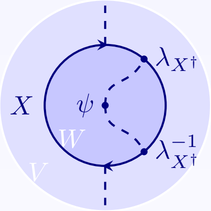

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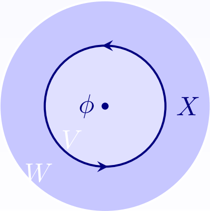

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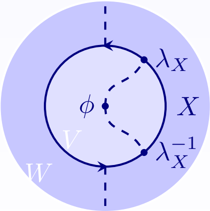

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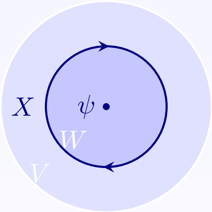

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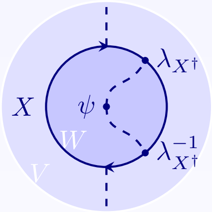

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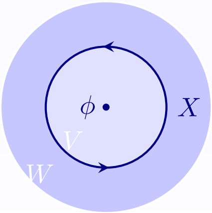

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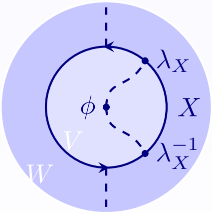

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Special cases $\mathcal{D}^l_X(1)$ and $\mathcal{D}^r_X(1)$ are left and right **quantum dimensions**.

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Left and right defect actions are adjoint with respect to the bulk topological metric:

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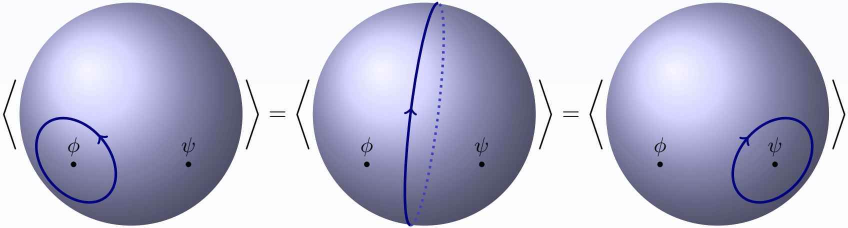

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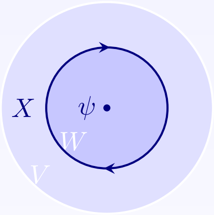

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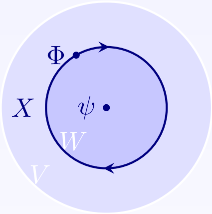

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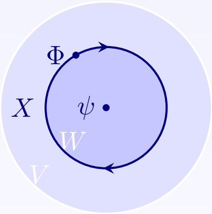

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- $V = 0$ gives Kapustin-Li **disk correlator**

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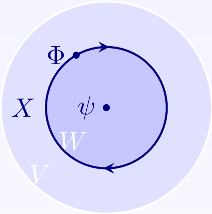

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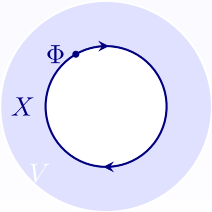

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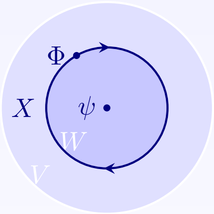

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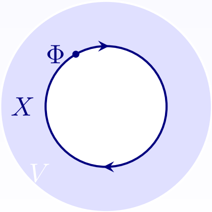

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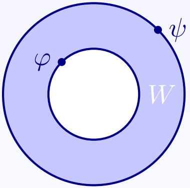
$\operatorname{ch}(X) := \beta^X(1) = \operatorname{str}\left(\partial_{z_1} d_X \ldots \partial_{z_m} d_X\right)$ is the **Chern character**

Applications

(3) **Cardy condition**



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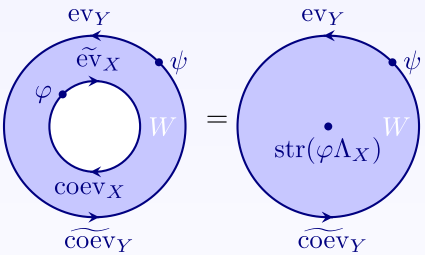

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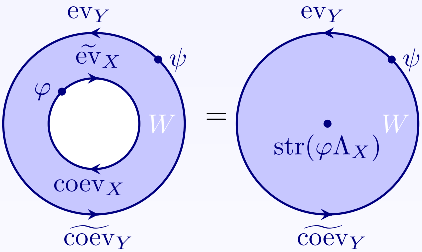

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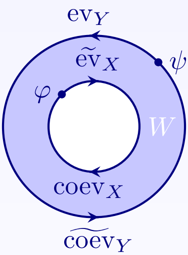

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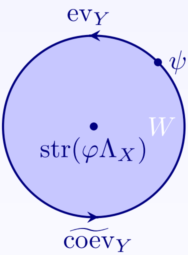

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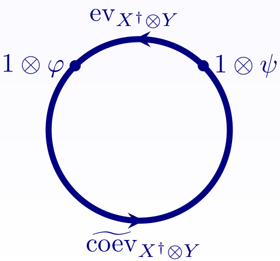

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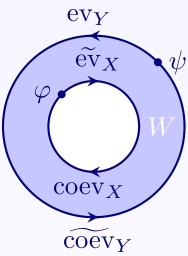

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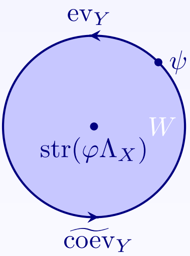

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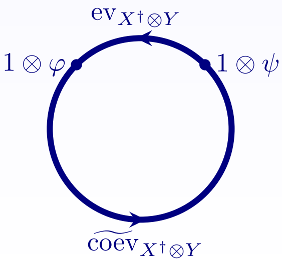

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Applications

Theorem. The **Cardy condition** holds in $\mathcal{LG}$

Polishchuck/Vaintrob 2010, Carqueville/Murfet 2012

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Theorem. The **Cardy condition** holds in $\mathcal{LG}$: for matrix factorisations X, Y of W and maps $\varphi : X \longrightarrow X$, $\psi : Y \longrightarrow Y$ we have

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Special case $\varphi = 1_X$, $\psi = 1_Y$ gives the Landau-Ginzburg version of the **Hirzebruch-Riemann-Roch theorem**:

$$\chi(\operatorname{Hom}(\mathcal{E}, \mathcal{F})) = \int \operatorname{ch}(\mathcal{E}^*)\,\operatorname{ch}(\mathcal{F})\,\operatorname{Td}(X)$$

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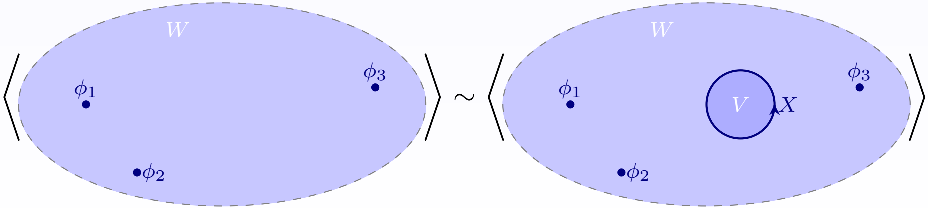
Idea. Introducing X-bubbles in W-correlator is scaling by $\mathrm{qdim}(X)$. Blowing up all X-bubbles produces V-correlator with defect network.



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Task. Classify all defects with invertible quantum dimensions (and find new equivalences this way)!

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Also allows to find new structure: **generalised orbifolds**