

Duality and defects in Landau-Ginzburg models

based on work with Daniel Murfet

Nils Carqueville

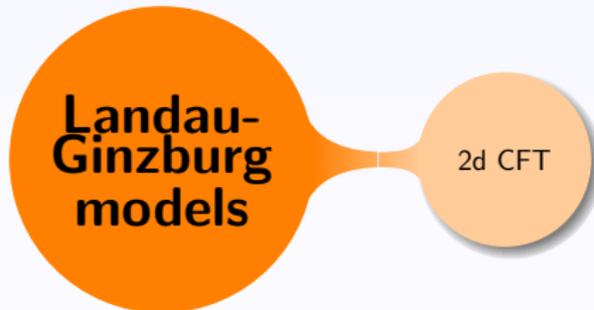
LMU München

Bigger picture

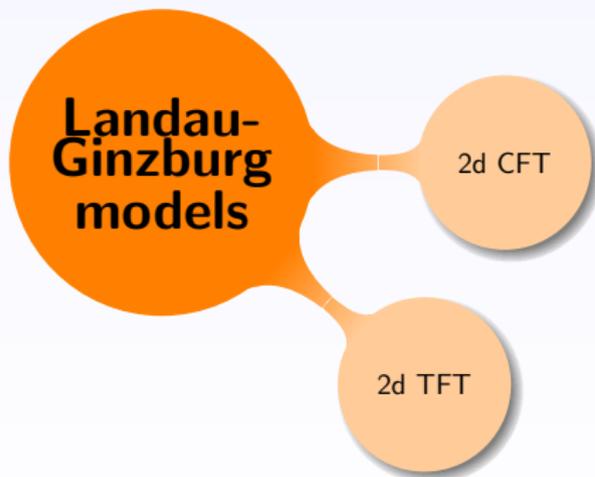
A large orange circle is centered on the page. Inside the circle, the text "Landau-Ginzburg models" is written in a bold, black, sans-serif font, arranged in three lines.

**Landau-
Ginzburg
models**

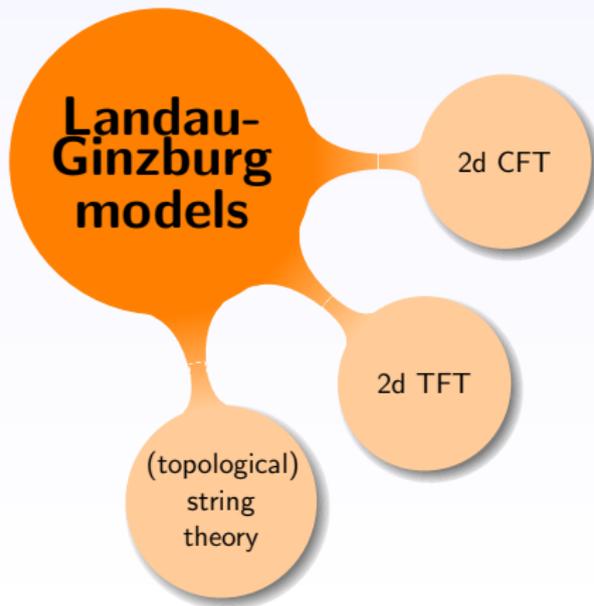
Bigger picture



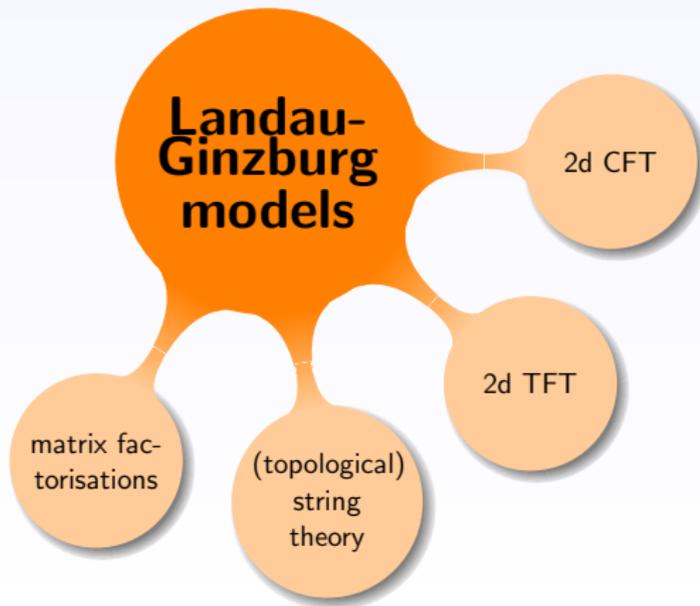
Bigger picture



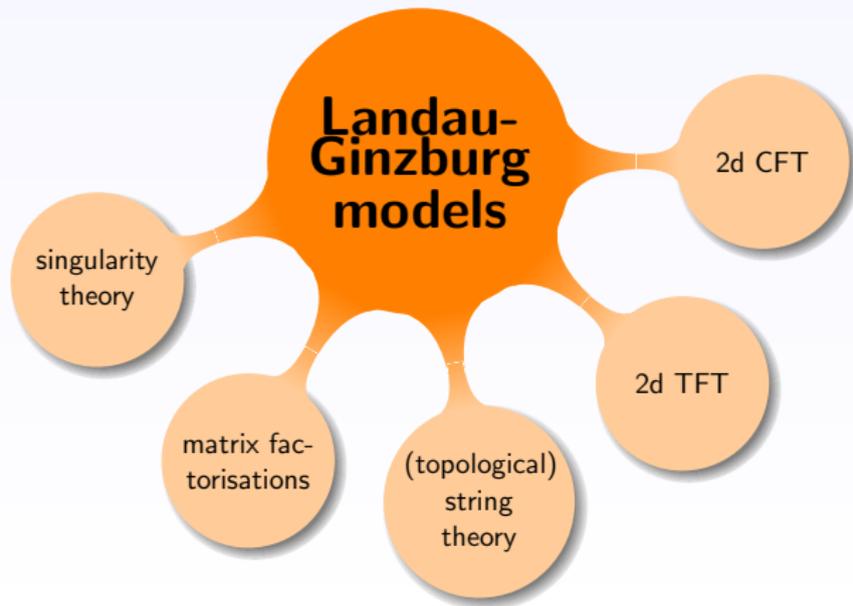
Bigger picture



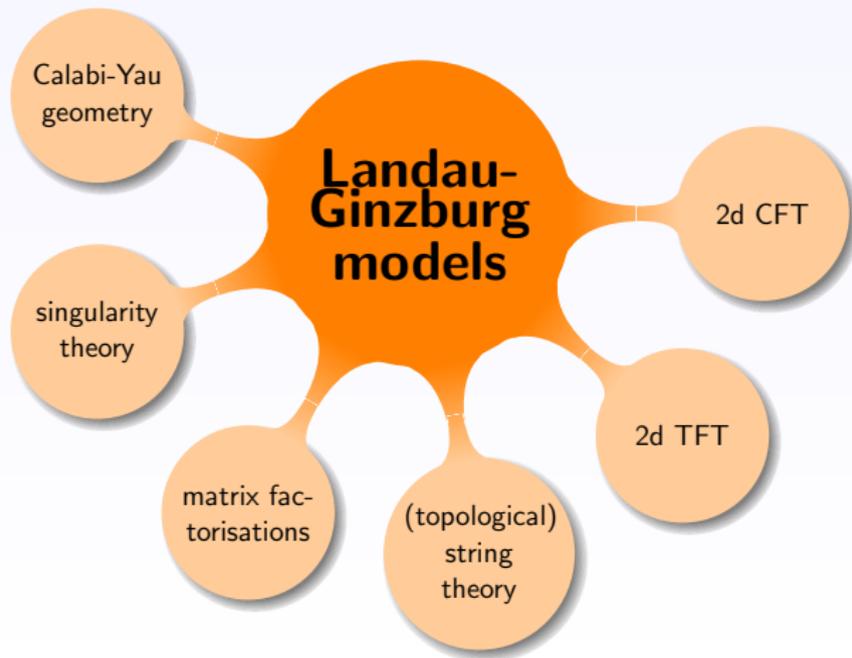
Bigger picture



Bigger picture



Bigger picture



Bigger picture



Bigger picture



Bigger picture



Bigger picture



Point

2d TFTs with defects

Point

2d TFTs with defects are naturally described in terms of **bicategories** with extra structure.

Point

2d TFTs with defects are naturally described in terms of **bicategories** with extra structure.

Theorem. The bicategory of Landau-Ginzburg models has adjoints.

Point

2d TFTs with defects are naturally described in terms of **bicategories** with extra structure.

Theorem. The bicategory of Landau-Ginzburg models has adjoints.

Plan.

- What?
- Why care?

Point

2d TFTs with defects are naturally described in terms of **bicategories** with extra structure.

Theorem. The bicategory of Landau-Ginzburg models has adjoints.
(conceptual construction, yet very “computable”)

Plan.

- What?
- Why care?

Point

2d TFTs with defects are naturally described in terms of **bicategories** with extra structure.

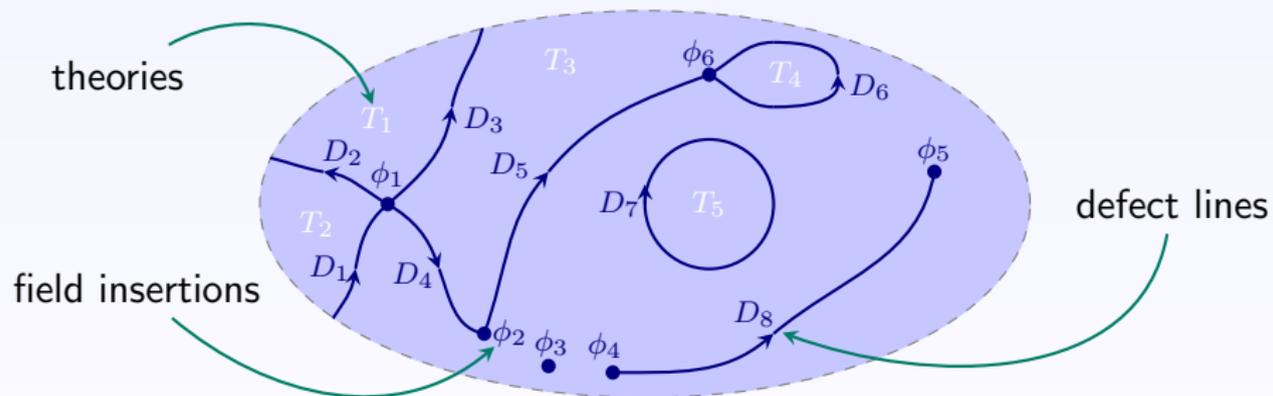
Theorem. The bicategory of Landau-Ginzburg models has adjoints.
(conceptual construction, yet very “computable”)

Plan.

- What?
- Why care?
 - ▶ see previous slide!
 - ▶ “computability” has many applications:
 - ★ **proof of Cardy condition**
 - ★ defect action on bulk fields
 - ★ **generalised orbifolds**
 - ★ ...

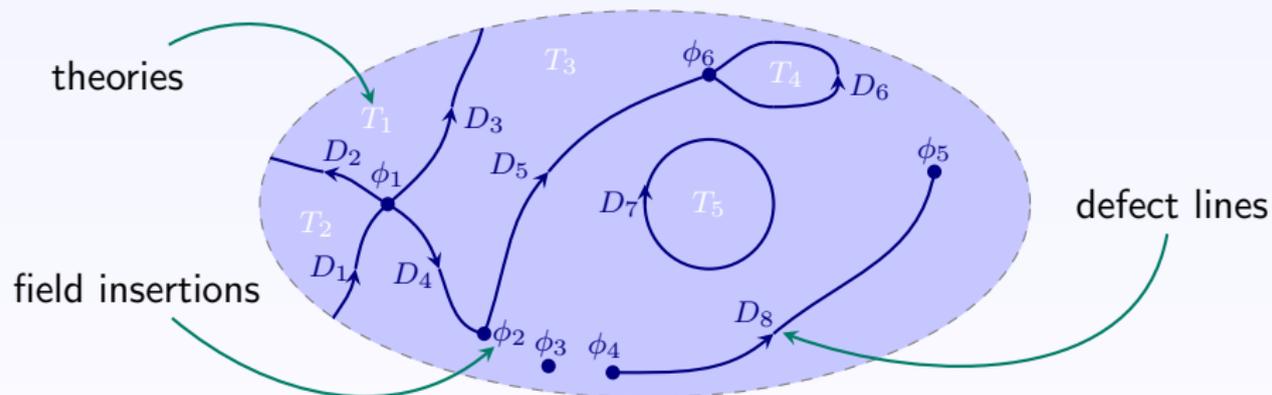
2d TFTs with defects

Worldsheet, partitioned into domains:



2d TFTs with defects

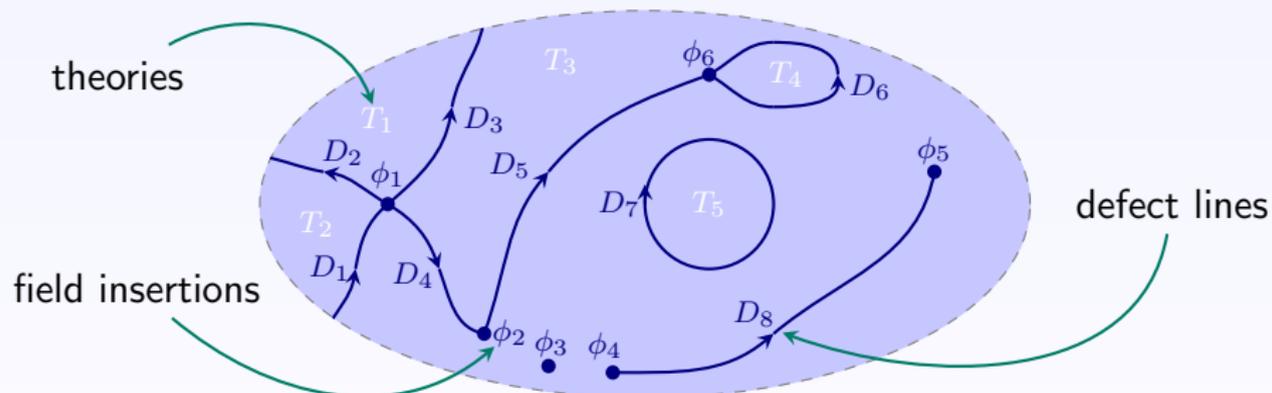
Worksheet, partitioned into domains:



A **TFT** assigns a number $\langle \dots \rangle$, the **correlator**, to any worldsheet, depending only on isotopy class of defect lines

2d TFTs with defects

Worksheet, partitioned into domains:



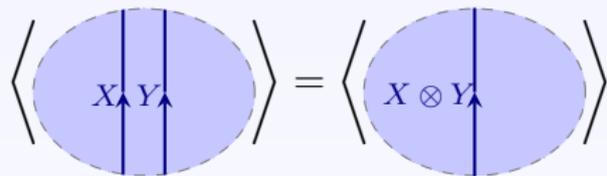
A **TFT** assigns a number $\langle \dots \rangle$, the **correlator**, to any worldsheet, depending only on isotopy class of defect lines:

$$\langle \text{Worldsheet with a vertical defect line} \rangle = \langle \text{Worldsheet with a curved defect line} \rangle$$

The equation shows two worldsheet diagrams in angle brackets, separated by an equals sign. The left diagram is a light blue oval with a vertical dark blue line with an upward-pointing arrow. The right diagram is a light blue oval with a dark blue curved line with an upward-pointing arrow.

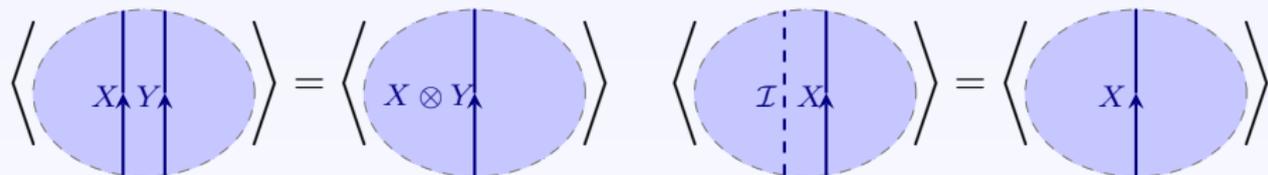
2d TFTs with defects

Defect fusion gives product



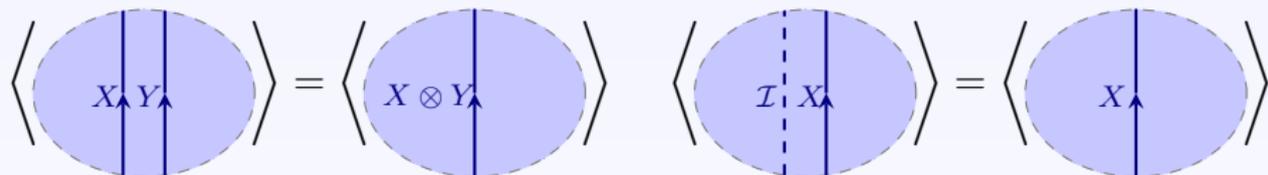
2d TFTs with defects

Defect fusion gives product, unit = “invisible” defect \mathcal{I}

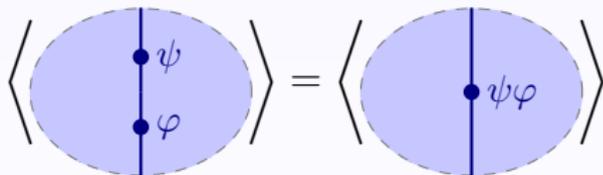


2d TFTs with defects

Defect fusion gives product, unit = “invisible” defect \mathcal{I}

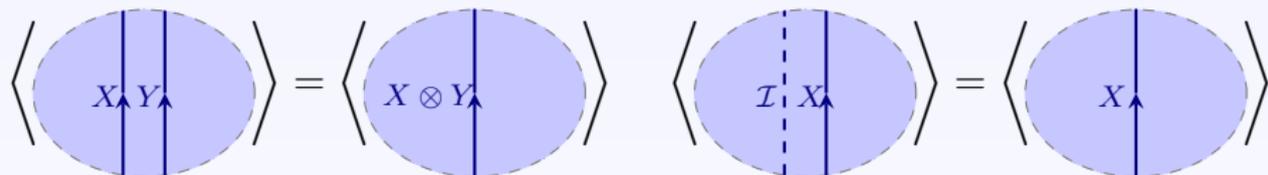


operator product of fields

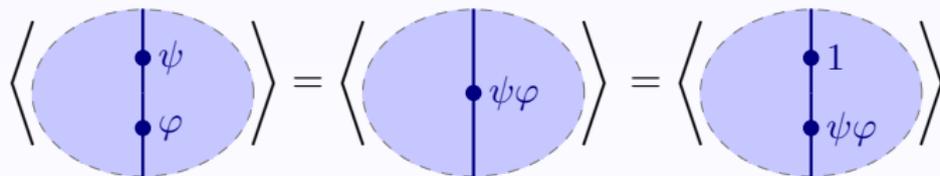


2d TFTs with defects

Defect fusion gives product, unit = “invisible” defect \mathcal{I}

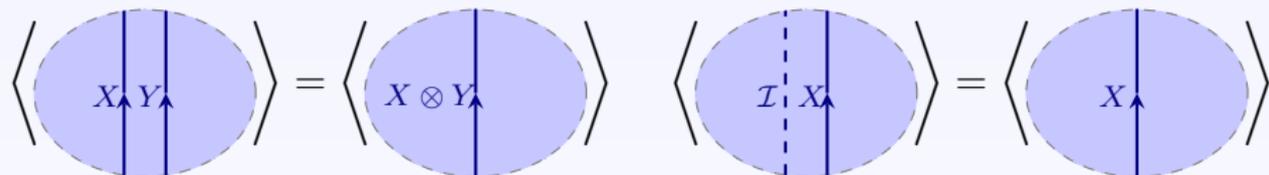


operator product of fields, unit = identity field

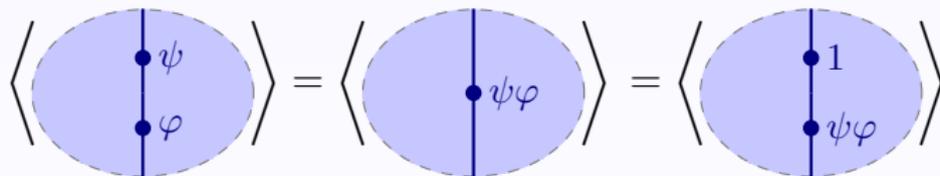


2d TFTs with defects

Defect fusion gives product, unit = “invisible” defect \mathcal{I}



operator product of fields, unit = identity field



Claim. 2d TFTs with defects give **bicategory**:

- objects (domains) = theories
- 1-morphisms (lines) = defects
- 2-morphisms (points) = fields

Diagrammatics in bicategories

$$\begin{array}{c} X \\ | \\ | \\ | \\ X \end{array} = 1_X$$

Diagrammatics in bicategories

$$\begin{array}{c} X \\ | \\ X \end{array} = 1_X$$

$$\begin{array}{c} Y \\ | \\ \bullet \\ | \\ X \end{array} = \varphi : X \longrightarrow Y$$

Diagrammatics in bicategories

$$\begin{array}{c} X \\ | \\ X \end{array} = 1_X$$

$$\begin{array}{c} Y \\ | \\ \bullet \\ | \\ X \end{array} = \varphi : X \longrightarrow Y$$

$$\begin{array}{c} Z \\ | \\ \bullet \\ | \\ \bullet \\ | \\ X \end{array} = \psi\varphi$$

Diagrammatics in bicategories

$$\begin{array}{c} X \\ | \\ X \end{array} = 1_X$$

$$\begin{array}{c} Y \\ | \\ \bullet \\ | \\ X \end{array} = \varphi : X \longrightarrow Y$$

$$\begin{array}{c} Z \\ | \\ \bullet \\ | \\ \bullet \\ | \\ X \end{array} = \psi\varphi$$

$$\begin{array}{cc} X & Y \\ | & | \\ \bullet & \bullet \\ | & | \\ X & Y \end{array} = \phi \otimes \phi'$$

Diagrammatics in bicategories

$$\begin{array}{c} X \\ | \\ X \end{array} = 1_X$$

$$\begin{array}{c} Y \\ | \\ \bullet \\ | \\ X \end{array} = \varphi : X \longrightarrow Y$$

$$\begin{array}{c} Z \\ | \\ \bullet \\ | \\ \bullet \\ | \\ X \end{array} = \psi\varphi$$

$$\begin{array}{c} X \quad Y \\ | \quad | \\ \bullet \quad \bullet \\ | \quad | \\ X \quad Y \end{array} \phi \quad \phi' = \phi \otimes \phi'$$

$$\begin{array}{c} Z \\ | \\ \bullet \\ / \quad \backslash \\ X \quad Y \end{array} \phi = \phi : X \otimes Y \longrightarrow Z$$

Diagrammatics in bicategories

$$\begin{array}{c} X \\ | \\ X \end{array} = 1_X$$

$$\begin{array}{c} Y \\ | \\ \bullet \\ | \\ X \end{array} = \varphi : X \longrightarrow Y$$

$$\begin{array}{c} Z \\ | \\ \bullet \\ | \\ \bullet \\ | \\ X \end{array} = \psi\varphi$$

$$\begin{array}{cc} X & Y \\ | & | \\ \bullet & \bullet \\ | & | \\ X & Y \end{array} \phi' = \phi \otimes \phi'$$

$$\begin{array}{c} Z \\ | \\ \bullet \\ / \quad \backslash \\ X \quad Y \end{array} \phi = \phi : X \otimes Y \longrightarrow Z$$

$$\begin{array}{c} X \\ | \\ \bullet \\ | \\ X \end{array} \rho_X \quad \begin{array}{c} \mathcal{I} \\ \text{---} \\ \bullet \end{array} = \begin{array}{c} X \\ | \\ X \end{array}$$

Diagrammatics in bicategories

$$\begin{array}{c} X \\ | \\ X \end{array} = 1_X$$

$$\begin{array}{c} Y \\ | \\ \bullet \\ | \\ X \end{array} = \varphi : X \longrightarrow Y$$

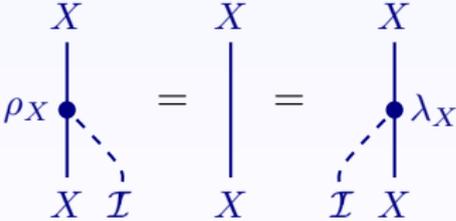
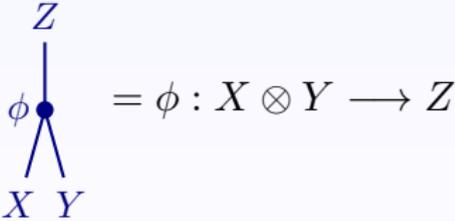
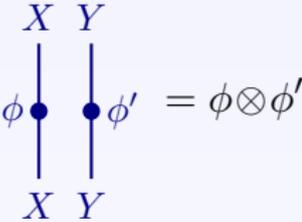
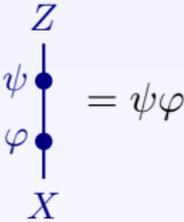
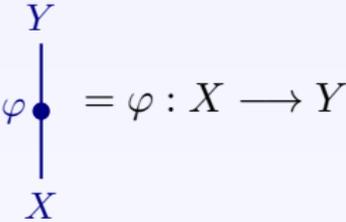
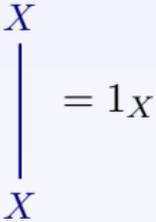
$$\begin{array}{c} Z \\ | \\ \bullet \\ | \\ \bullet \\ | \\ X \end{array} = \psi\varphi$$

$$\begin{array}{cc} X & Y \\ | & | \\ \bullet & \bullet \\ | & | \\ X & Y \end{array} \phi' = \phi \otimes \phi'$$

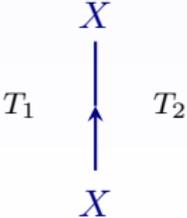
$$\begin{array}{c} Z \\ | \\ \bullet \\ / \quad \backslash \\ X \quad Y \end{array} \phi = \phi : X \otimes Y \longrightarrow Z$$

$$\begin{array}{c} X \\ | \\ \bullet \\ | \\ X \end{array} \rho_X = \begin{array}{c} X \\ | \\ X \end{array} = \begin{array}{c} X \\ | \\ \bullet \\ / \quad \backslash \\ \mathcal{I} \quad X \end{array} \lambda_X$$

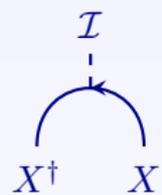
Diagrammatics in bicategories



Orientation matters:

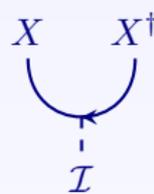


Orientation and adjoints



A diagram representing the evaluation map. It consists of a blue arc with an arrow pointing downwards from its center to a vertical dashed line. The vertical dashed line is labeled with the symbol \mathcal{I} . The left end of the arc is labeled X^\dagger and the right end is labeled X .

$$= \text{ev}_X : X^\dagger \otimes X \longrightarrow \mathcal{I}$$



A diagram representing the coevaluation map. It consists of a blue arc with an arrow pointing upwards from its center to a vertical dashed line. The vertical dashed line is labeled with the symbol \mathcal{I} . The left end of the arc is labeled X and the right end is labeled X^\dagger .

$$= \text{coev}_X : \mathcal{I} \longrightarrow X \otimes X^\dagger$$

Orientation and adjoints

$$\begin{array}{c} \mathcal{I} \\ \vdots \\ \curvearrowright \\ X^\dagger \quad X \end{array} = \text{ev}_X : X^\dagger \otimes X \longrightarrow \mathcal{I}$$

$$\begin{array}{c} X \quad X^\dagger \\ \curvearrowleft \\ \mathcal{I} \end{array} = \text{coev}_X : \mathcal{I} \longrightarrow X \otimes X^\dagger$$

Defects are **topological**:

$$\begin{array}{c} X \\ \uparrow \\ X \end{array} = \begin{array}{c} X \\ \downarrow \\ \uparrow \\ X \end{array}$$

$$\begin{array}{c} X^\dagger \\ \downarrow \\ X^\dagger \end{array} = \begin{array}{c} X^\dagger \\ \uparrow \\ \downarrow \\ X^\dagger \end{array}$$

Orientation and adjoints

$$\begin{array}{c} \mathcal{I} \\ \vdots \\ \curvearrowright \\ X^\dagger \quad X \end{array} = \text{ev}_X : X^\dagger \otimes X \longrightarrow \mathcal{I}$$

$$\begin{array}{c} X \quad X^\dagger \\ \curvearrowleft \\ \vdots \\ \mathcal{I} \end{array} = \text{coev}_X : \mathcal{I} \longrightarrow X \otimes X^\dagger$$

Defects are **topological**:

$$1_X = \begin{array}{c} X \\ \uparrow \\ X \end{array} = \begin{array}{c} X \\ \downarrow \quad \uparrow \\ X \end{array} = \rho \circ (1 \otimes \text{ev}) \circ (\text{coev} \otimes 1) \circ \lambda^{-1}$$

$$\begin{array}{c} X^\dagger \\ \downarrow \\ X^\dagger \end{array} = \begin{array}{c} X^\dagger \\ \uparrow \quad \downarrow \\ X^\dagger \end{array}$$

Orientation and adjoints

$$\begin{array}{c} \mathcal{I} \\ \vdots \\ \curvearrowright \\ X^\dagger \quad X \end{array} = \text{ev}_X : X^\dagger \otimes X \longrightarrow \mathcal{I}$$

$$\begin{array}{c} X \quad X^\dagger \\ \curvearrowleft \\ \mathcal{I} \end{array} = \text{coev}_X : \mathcal{I} \longrightarrow X \otimes X^\dagger$$

Defects are **topological**:

$$1_X = \begin{array}{c} X \\ \uparrow \\ X \end{array} = \begin{array}{c} X \\ \downarrow \quad \uparrow \\ X \end{array} = \rho \circ (1 \otimes \text{ev}) \circ (\text{coev} \otimes 1) \circ \lambda^{-1}$$

$$\begin{array}{c} X^\dagger \\ \downarrow \\ X^\dagger \end{array} = \begin{array}{c} X^\dagger \\ \uparrow \quad \downarrow \\ X^\dagger \end{array}$$

Definition. A bicategory **has adjoints** if for each 1-morphism X there is a 1-morphism X^\dagger with 2-morphisms $\text{ev}_X, \text{coev}_X$ such that the above *Zorro moves* hold.

Landau-Ginzburg models

- theories: **potentials** $W \in R = \mathbb{C}[x_1, \dots, x_n]$, $\dim(R/(\partial W)) < \infty$

Landau-Ginzburg models

- theories: **potentials** $W \in R = \mathbb{C}[x_1, \dots, x_n]$, $\dim(R/(\partial W)) < \infty$
- defects between $W \in R$ and $V \in S$: **matrix factorisations** of $V - W$, i. e. free \mathbb{Z}_2 -graded $(R \otimes S)$ -modules $X = X^0 \oplus X^1$ with

$$d_X = \begin{pmatrix} 0 & d_X^1 \\ d_X^0 & 0 \end{pmatrix} \in \text{End}_{R \otimes S}^1(X), \quad d_X^2 = (V - W) \cdot 1_X$$

Landau-Ginzburg models

- theories: **potentials** $W \in R = \mathbb{C}[x_1, \dots, x_n]$, $\dim(R/(\partial W)) < \infty$
- defects between $W \in R$ and $V \in S$: **matrix factorisations** of $V - W$, i. e. free \mathbb{Z}_2 -graded $(R \otimes S)$ -modules $X = X^0 \oplus X^1$ with

$$d_X = \begin{pmatrix} 0 & d_X^1 \\ d_X^0 & 0 \end{pmatrix} \in \text{End}_{R \otimes S}^1(X), \quad d_X^2 = (V - W) \cdot 1_X$$

- fields between X and Y : **(BRST) cohomology** of

$$\text{Hom}(X, Y) \ni \psi \longmapsto d_Y \psi - (-1)^{|\psi|} \psi d_X$$

Landau-Ginzburg models

- theories: **potentials** $W \in R = \mathbb{C}[x_1, \dots, x_n]$, $\dim(R/(\partial W)) < \infty$
- defects between $W \in R$ and $V \in S$: **matrix factorisations** of $V - W$, i. e. free \mathbb{Z}_2 -graded $(R \otimes S)$ -modules $X = X^0 \oplus X^1$ with

$$d_X = \begin{pmatrix} 0 & d_X^1 \\ d_X^0 & 0 \end{pmatrix} \in \text{End}_{R \otimes S}^1(X), \quad d_X^2 = (V - W) \cdot 1_X$$

- fields between X and Y : **(BRST) cohomology** of

$$\text{Hom}(X, Y) \ni \psi \longmapsto d_Y \psi - (-1)^{|\psi|} \psi d_X$$

Example. $R = \mathbb{C}[x]$, $W = x^d$, $X = R \oplus R$

Landau-Ginzburg models

- theories: **potentials** $W \in R = \mathbb{C}[x_1, \dots, x_n]$, $\dim(R/(\partial W)) < \infty$
- defects between $W \in R$ and $V \in S$: **matrix factorisations** of $V - W$, i. e. free \mathbb{Z}_2 -graded $(R \otimes S)$ -modules $X = X^0 \oplus X^1$ with

$$d_X = \begin{pmatrix} 0 & d_X^1 \\ d_X^0 & 0 \end{pmatrix} \in \text{End}_{R \otimes S}^1(X), \quad d_X^2 = (V - W) \cdot 1_X$$

- fields between X and Y : **(BRST) cohomology** of

$$\text{Hom}(X, Y) \ni \psi \longmapsto d_Y \psi - (-1)^{|\psi|} \psi d_X$$

Example. $R = \mathbb{C}[x]$, $W = x^d$, $X = R \oplus R$,

$$d_X = \begin{pmatrix} 0 & x^n \\ x^{d-n} & 0 \end{pmatrix}, \quad \psi = \begin{pmatrix} x^i & 0 \\ 0 & x^i \end{pmatrix}$$

Landau-Ginzburg models

- defect fusion: $X \otimes Y, d_{X \otimes Y} = d_X \otimes 1 + 1 \otimes d_Y$

Landau-Ginzburg models

- defect fusion: $X \otimes Y$, $d_{X \otimes Y} = d_X \otimes 1 + 1 \otimes d_Y$
- invisible defect:

$$\mathcal{I}_W = (R \otimes R)^{\oplus 2}, \quad d_{\mathcal{I}_W} = \begin{pmatrix} 0 & x - y \\ \frac{W(x) - W(y)}{x - y} & 0 \end{pmatrix}$$

Landau-Ginzburg models

- defect fusion: $X \otimes Y$, $d_{X \otimes Y} = d_X \otimes 1 + 1 \otimes d_Y$
- invisible defect:

$$\mathcal{I}_W = (R \otimes R)^{\oplus 2}, \quad d_{\mathcal{I}_W} = \begin{pmatrix} 0 & x - y \\ \frac{W(x) - W(y)}{x - y} & 0 \end{pmatrix}$$

for $n = 1$, in general:

$$\mathcal{I}_W = \bigwedge \left(\bigoplus_{i=1}^n (R \otimes R) \cdot \theta_i \right), \quad d_{\mathcal{I}_W} = \sum_{i=1}^n \left((x_i - y_i) \cdot \theta_i^* + \partial_{[i]} W \cdot \theta_i \right)$$

Landau-Ginzburg models

- defect fusion: $X \otimes Y$, $d_{X \otimes Y} = d_X \otimes 1 + 1 \otimes d_Y$
- invisible defect:

$$\mathcal{I}_W = (R \otimes R)^{\oplus 2}, \quad d_{\mathcal{I}_W} = \begin{pmatrix} 0 & x - y \\ \frac{W(x) - W(y)}{x - y} & 0 \end{pmatrix}$$

for $n = 1$, in general:

$$\mathcal{I}_W = \bigwedge \left(\bigoplus_{i=1}^n (R \otimes R) \cdot \theta_i \right), \quad d_{\mathcal{I}_W} = \sum_{i=1}^n \left((x_i - y_i) \cdot \theta_i^* + \partial_{[i]} W \cdot \theta_i \right)$$

Fact. $\text{End}(\mathcal{I}_W) \cong R/(\partial W)$

Landau-Ginzburg models

- defect fusion: $X \otimes Y$, $d_{X \otimes Y} = d_X \otimes 1 + 1 \otimes d_Y$
- invisible defect:

$$\mathcal{I}_W = (R \otimes R)^{\oplus 2}, \quad d_{\mathcal{I}_W} = \begin{pmatrix} 0 & x - y \\ \frac{W(x) - W(y)}{x - y} & 0 \end{pmatrix}$$

for $n = 1$, in general:

$$\mathcal{I}_W = \bigwedge \left(\bigoplus_{i=1}^n (R \otimes R) \cdot \theta_i \right), \quad d_{\mathcal{I}_W} = \sum_{i=1}^n \left((x_i - y_i) \cdot \theta_i^* + \partial_{[i]} W \cdot \theta_i \right)$$

Fact. $\text{End}(\mathcal{I}_W) \cong R/(\partial W) = \text{bulk space}$

Landau-Ginzburg models

- defect fusion: $X \otimes Y$, $d_{X \otimes Y} = d_X \otimes 1 + 1 \otimes d_Y$
- invisible defect:

$$\mathcal{I}_W = (R \otimes R)^{\oplus 2}, \quad d_{\mathcal{I}_W} = \begin{pmatrix} 0 & x - y \\ \frac{W(x) - W(y)}{x - y} & 0 \end{pmatrix}$$

for $n = 1$, in general:

$$\mathcal{I}_W = \bigwedge \left(\bigoplus_{i=1}^n (R \otimes R) \cdot \theta_i \right), \quad d_{\mathcal{I}_W} = \sum_{i=1}^n \left((x_i - y_i) \cdot \theta_i^* + \partial_{[i]} W \cdot \theta_i \right)$$

Fact. $\text{End}(\mathcal{I}_W) \cong R/(\partial W) = \text{bulk space}$

$$\lambda_X : \mathcal{I} \otimes X \longrightarrow (R \otimes R) \otimes X \xrightarrow{\text{mult.}} X$$

Landau-Ginzburg models

- defect fusion: $X \otimes Y$, $d_{X \otimes Y} = d_X \otimes 1 + 1 \otimes d_Y$
- invisible defect:

$$\mathcal{I}_W = (R \otimes R)^{\oplus 2}, \quad d_{\mathcal{I}_W} = \begin{pmatrix} 0 & x - y \\ \frac{W(x) - W(y)}{x - y} & 0 \end{pmatrix}$$

for $n = 1$, in general:

$$\mathcal{I}_W = \bigwedge \left(\bigoplus_{i=1}^n (R \otimes R) \cdot \theta_i \right), \quad d_{\mathcal{I}_W} = \sum_{i=1}^n \left((x_i - y_i) \cdot \theta_i^* + \partial_{[i]} W \cdot \theta_i \right)$$

Fact. $\text{End}(\mathcal{I}_W) \cong R/(\partial W) = \text{bulk space}$

$$\lambda_X : \mathcal{I} \otimes X \longrightarrow (R \otimes R) \otimes X \xrightarrow{\text{mult.}} X, \quad \rho_X : X \otimes \mathcal{I} \longrightarrow X$$

Landau-Ginzburg models

- defect fusion: $X \otimes Y$, $d_{X \otimes Y} = d_X \otimes 1 + 1 \otimes d_Y$
- invisible defect:

$$\mathcal{I}_W = (R \otimes R)^{\oplus 2}, \quad d_{\mathcal{I}_W} = \begin{pmatrix} 0 & x - y \\ \frac{W(x) - W(y)}{x - y} & 0 \end{pmatrix}$$

for $n = 1$, in general:

$$\mathcal{I}_W = \bigwedge \left(\bigoplus_{i=1}^n (R \otimes R) \cdot \theta_i \right), \quad d_{\mathcal{I}_W} = \sum_{i=1}^n \left((x_i - y_i) \cdot \theta_i^* + \partial_{[i]} W \cdot \theta_i \right)$$

Fact. $\text{End}(\mathcal{I}_W) \cong R/(\partial W) = \text{bulk space}$

$$\lambda_X : \mathcal{I} \otimes X \dashrightarrow (R \otimes R) \otimes X \xrightarrow{\text{mult.}} X, \quad \rho_X : X \otimes \mathcal{I} \dashrightarrow X$$

- operator product: matrix multiplication

Main result

Theorem. Landau-Ginzburg models give a bicategory, called \mathcal{LG} .

Main result

Theorem. Landau-Ginzburg models give a bicategory, called \mathcal{LG} .

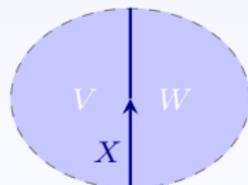
Theorem. \mathcal{LG} has adjoints

Main result

Theorem. Landau-Ginzburg models give a bicategory, called \mathcal{LG} .

Theorem. \mathcal{LG} has adjoints:

Let $W \in \mathbb{C}[x_1, \dots, x_n]$, $V \in \mathbb{C}[z_1, \dots, z_m]$, X matrix fact. of $V - W$:

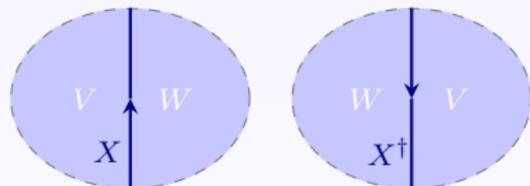


Main result

Theorem. Landau-Ginzburg models give a bicategory, called \mathcal{LG} .

Theorem. \mathcal{LG} has adjoints:

Let $W \in \mathbb{C}[x_1, \dots, x_n]$, $V \in \mathbb{C}[z_1, \dots, z_m]$, X matrix fact. of $V - W$:

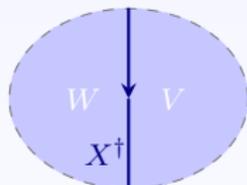
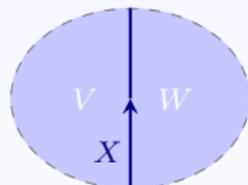


Main result

Theorem. Landau-Ginzburg models give a bicategory, called \mathcal{LG} .

Theorem. \mathcal{LG} has adjoints:

Let $W \in \mathbb{C}[x_1, \dots, x_n]$, $V \in \mathbb{C}[z_1, \dots, z_m]$, X matrix fact. of $V - W$:



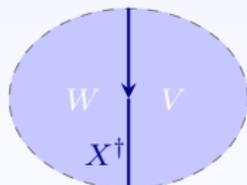
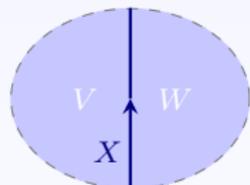
$$X^\dagger = X^\vee[n] \quad d_{X^\dagger} = \begin{pmatrix} 0 & (d_X^0)^\top \\ -(d_X^1)^\top & 0 \end{pmatrix}[n]$$

Main result

Theorem. Landau-Ginzburg models give a bicategory, called \mathcal{LG} .

Theorem. \mathcal{LG} has adjoints:

Let $W \in \mathbb{C}[x_1, \dots, x_n]$, $V \in \mathbb{C}[z_1, \dots, z_m]$, X matrix fact. of $V - W$:



$$X^\dagger = X^\vee[n] \quad d_{X^\dagger} = \begin{pmatrix} 0 & (d_X^0)^\top \\ -(d_X^1)^\top & 0 \end{pmatrix}[n]$$

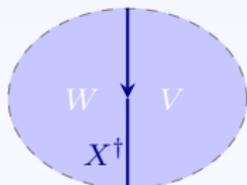
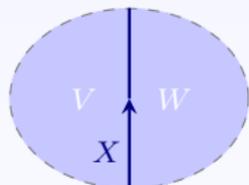
$$= \text{coev}_X : 1 \mapsto \partial_{[1]} d_X \dots \partial_{[n]} d_X \in X \otimes X^\vee$$

Main result

Theorem. Landau-Ginzburg models give a bicategory, called \mathcal{LG} .

Theorem. \mathcal{LG} has adjoints:

Let $W \in \mathbb{C}[x_1, \dots, x_n]$, $V \in \mathbb{C}[z_1, \dots, z_m]$, X matrix fact. of $V - W$:



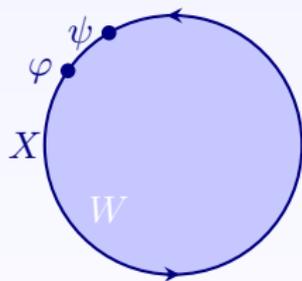
$$X^\dagger = X^\vee[n] \quad d_{X^\dagger} = \begin{pmatrix} 0 & (d_X^0)^\top \\ -(d_X^1)^\top & 0 \end{pmatrix}[n]$$

$$= \text{coev}_X : 1 \mapsto \partial_{[1]} d_X \dots \partial_{[n]} d_X \in X \otimes X^\vee$$

$$= \text{ev}_X = \text{Res} \left[\frac{\text{str} \left((-) \circ \partial_{z_1} d_X \dots \partial_{z_m} d_X \right) dz}{\partial_{z_1} V \dots \partial_{z_m} V} \right] + \mathcal{O}(\theta)$$

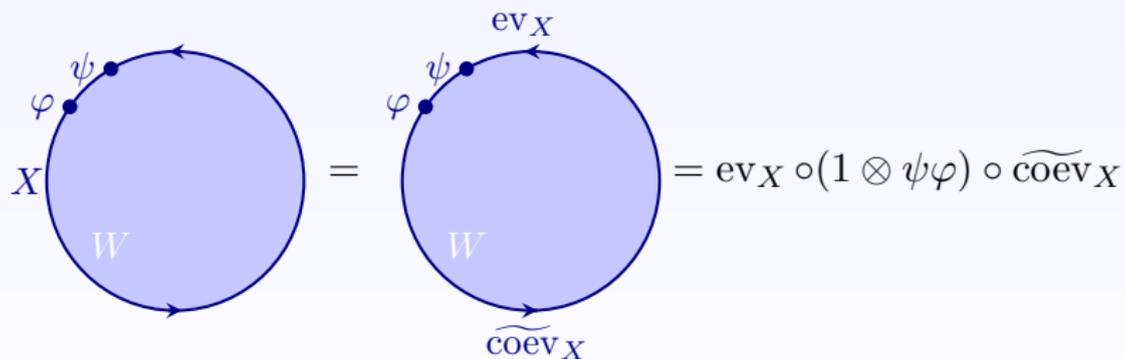
Applications

(1) **Boundary topological metric** / 2-point disk correlator



Applications

(1) **Boundary topological metric** / 2-point disk correlator



The diagram shows an equality between two circular diagrams representing a 2-point disk correlator. Both circles are filled with light blue and have a counter-clockwise arrow on the boundary. The left circle has two points on its boundary labeled φ and ψ , with the label X to its left and W inside. The right circle has the same two points φ and ψ , but with an additional label $\widetilde{\text{coev}}_X$ at the bottom and ev_X at the top. The right circle is followed by the expression $= \text{ev}_X \circ (1 \otimes \psi\varphi) \circ \widetilde{\text{coev}}_X$.

$$\text{Diagram 1} = \text{Diagram 2} = \text{ev}_X \circ (1 \otimes \psi\varphi) \circ \widetilde{\text{coev}}_X$$

Applications

(1) **Boundary topological metric** / 2-point disk correlator

$$\begin{aligned} & \text{Left circle} = \text{Right circle} = \text{ev}_X \circ (1 \otimes \psi\varphi) \circ \widetilde{\text{coev}}_X \\ & \text{Left circle: } X \text{ on the left, } W \text{ inside, } \varphi \text{ and } \psi \text{ on the boundary.} \\ & \text{Right circle: } W \text{ inside, } \varphi \text{ and } \psi \text{ on the boundary, } \widetilde{\text{coev}}_X \text{ at the bottom, } \text{ev}_X \text{ at the top.} \end{aligned}$$

$$= \text{Res} \left[\frac{\text{str} (\psi\varphi \partial_{x_1} d_X \dots \partial_{x_n} d_X) dx}{\partial_{x_1} W \dots \partial_{x_n} W} \right]$$

Applications

(1) **Boundary topological metric** / 2-point disk correlator

$$X \circlearrowleft \begin{matrix} \psi \\ \varphi \\ W \end{matrix} = \begin{matrix} \text{ev}_X \\ \psi \\ \varphi \\ W \\ \text{coev}_X \end{matrix} \circlearrowright = \text{ev}_X \circ (1 \otimes \psi\varphi) \circ \widetilde{\text{coev}}_X$$

$$= \text{Res} \left[\frac{\text{str} (\psi\varphi \partial_{x_1} d_X \dots \partial_{x_n} d_X) dx}{\partial_{x_1} W \dots \partial_{x_n} W} \right]$$

Recover **Kapustin-Li pairing** as a 2-morphism!

Applications

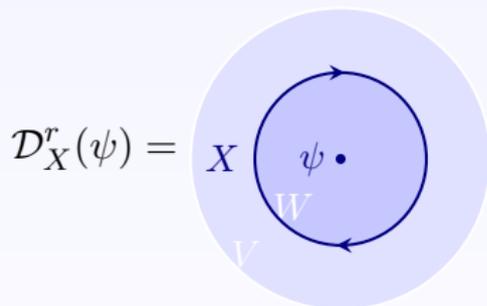
(2) **Defect action on bulk fields**

Applications

(2) **Defect action on bulk fields:** for defect X between $W(x)$ and $V(z)$ write $\Lambda_X^{(x)} = \prod_i \partial_{x_i} d_X$ and $\Lambda_X^{(z)} = \prod_j \partial_{z_j} d_X$.

Applications

(2) **Defect action on bulk fields:** for defect X between $W(x)$ and $V(z)$ write $\Lambda_X^{(x)} = \prod_i \partial_{x_i} d_X$ and $\Lambda_X^{(z)} = \prod_j \partial_{z_j} d_X$.



Applications

(2) **Defect action on bulk fields:** for defect X between $W(x)$ and $V(z)$ write $\Lambda_X^{(x)} = \prod_i \partial_{x_i} d_X$ and $\Lambda_X^{(z)} = \prod_j \partial_{z_j} d_X$.

$$\mathcal{D}_X^r(\psi) = \begin{array}{c} \text{Diagram 1: A light blue circle with a central dot labeled } \psi. \text{ A solid blue circle is inside it, with a counter-clockwise arrow. The region between the circles is labeled } X. \text{ The region inside the inner circle is labeled } W. \text{ The region outside the outer circle is labeled } V. \end{array} = \begin{array}{c} \text{Diagram 2: Similar to Diagram 1, but with a vertical dashed line passing through the center. The inner circle is dashed. The top-right and bottom-right parts of the inner circle are labeled } \lambda_{X^\dagger} \text{ and } \lambda_{X^\dagger}^{-1} \text{ respectively.} \end{array} = \text{Res} \left[\frac{\psi \text{str} (\Lambda_X^{(x)} \Lambda_X^{(z)}) \underline{dx}}{\partial_{x_1} W \dots \partial_{x_n} W} \right]$$

Applications

(2) **Defect action on bulk fields:** for defect X between $W(x)$ and $V(z)$ write $\Lambda_X^{(x)} = \prod_i \partial_{x_i} d_X$ and $\Lambda_X^{(z)} = \prod_j \partial_{z_j} d_X$.

$$\mathcal{D}_X^r(\psi) = \begin{array}{c} \text{Diagram 1: Circle with center } \psi \cdot \text{ and boundary } X. \text{ Region } W \text{ is inside, } V \text{ is outside.} \\ \text{Diagram 2: Circle with center } \psi \cdot \text{ and boundary } X. \text{ A vertical dashed line } X \text{ passes through the center.} \\ \text{Region } W \text{ is inside, } V \text{ is outside.} \\ \text{Dashed paths connect } \psi \cdot \text{ to the boundary } X \text{ at points } \lambda_{X^\dagger} \text{ and } \lambda_{X^\dagger}^{-1}. \end{array} = \text{Res} \left[\frac{\psi \text{ str} (\Lambda_X^{(x)} \Lambda_X^{(z)}) \underline{dx}}{\partial_{x_1} W \dots \partial_{x_n} W} \right]$$

$$\mathcal{D}_X^l(\phi) = \begin{array}{c} \text{Diagram 1: Circle with center } \phi \cdot \text{ and boundary } X. \text{ Region } V \text{ is inside, } W \text{ is outside.} \\ \text{Diagram 2: Circle with center } \phi \cdot \text{ and boundary } X. \text{ A vertical dashed line } X \text{ passes through the center.} \\ \text{Region } V \text{ is inside, } W \text{ is outside.} \\ \text{Dashed paths connect } \phi \cdot \text{ to the boundary } X \text{ at points } \lambda_X \text{ and } \lambda_X^{-1}. \end{array} = \text{Res} \left[\frac{\phi \text{ str} (\Lambda_X^{(x)} \Lambda_X^{(z)}) \underline{dz}}{\partial_{z_1} V \dots \partial_{z_m} V} \right]$$

Applications

(2) **Defect action on bulk fields:** for defect X between $W(x)$ and $V(z)$ write $\Lambda_X^{(x)} = \prod_i \partial_{x_i} d_X$ and $\Lambda_X^{(z)} = \prod_j \partial_{z_j} d_X$.

$$\mathcal{D}_X^r(\psi) = \begin{array}{c} \text{Diagram 1: Circle with center } \psi \cdot \text{ and boundary } X. \text{ Region } W \text{ is inside, } V \text{ is outside.} \\ \text{Diagram 2: Circle with center } \psi \cdot \text{ and boundary } X. \text{ A vertical dashed line } X \text{ passes through the center.} \\ \text{Region } W \text{ is inside, } V \text{ is outside.} \\ \text{A dashed path } \lambda_{X^\dagger} \text{ goes from } \psi \cdot \text{ to the boundary } X \text{ on the right.} \\ \text{A dashed path } \lambda_{X^\dagger}^{-1} \text{ goes from } \psi \cdot \text{ to the boundary } X \text{ on the left.} \end{array} = \text{Res} \left[\frac{\psi \text{ str} (\Lambda_X^{(x)} \Lambda_X^{(z)}) \underline{dx}}{\partial_{x_1} W \dots \partial_{x_n} W} \right]$$

$$\mathcal{D}_X^l(\phi) = \begin{array}{c} \text{Diagram 1: Circle with center } \phi \cdot \text{ and boundary } X. \text{ Region } V \text{ is inside, } W \text{ is outside.} \\ \text{Diagram 2: Circle with center } \phi \cdot \text{ and boundary } X. \text{ A vertical dashed line } X \text{ passes through the center.} \\ \text{Region } V \text{ is inside, } W \text{ is outside.} \\ \text{A dashed path } \lambda_X \text{ goes from } \phi \cdot \text{ to the boundary } X \text{ on the right.} \\ \text{A dashed path } \lambda_X^{-1} \text{ goes from } \phi \cdot \text{ to the boundary } X \text{ on the left.} \end{array} = \text{Res} \left[\frac{\phi \text{ str} (\Lambda_X^{(x)} \Lambda_X^{(z)}) \underline{dz}}{\partial_{z_1} V \dots \partial_{z_m} V} \right]$$

Special cases $\mathcal{D}_X^l(1)$ and $\mathcal{D}_X^r(1)$ are left and right **quantum dimensions**.

Applications

$$\mathcal{D}_X^r(\psi) = \text{Res} \left[\frac{\psi \text{str} (\Lambda_X^{(x)} \Lambda_X^{(z)}) dx}{\partial_{x_1} W \dots \partial_{x_n} W} \right], \quad \mathcal{D}_X^l(\phi) = \text{Res} \left[\frac{\phi \text{str} (\Lambda_X^{(x)} \Lambda_X^{(z)}) dz}{\partial_{z_1} V \dots \partial_{z_m} V} \right]$$

Applications

$$\mathcal{D}_X^r(\psi) = \text{Res} \left[\frac{\psi \text{str} (\Lambda_X^{(x)} \Lambda_X^{(z)}) dx}{\partial_{x_1} W \dots \partial_{x_n} W} \right], \quad \mathcal{D}_X^l(\phi) = \text{Res} \left[\frac{\phi \text{str} (\Lambda_X^{(x)} \Lambda_X^{(z)}) dz}{\partial_{z_1} V \dots \partial_{z_m} V} \right]$$

Left and right defect actions are adjoint with respect to the bulk topological metric:

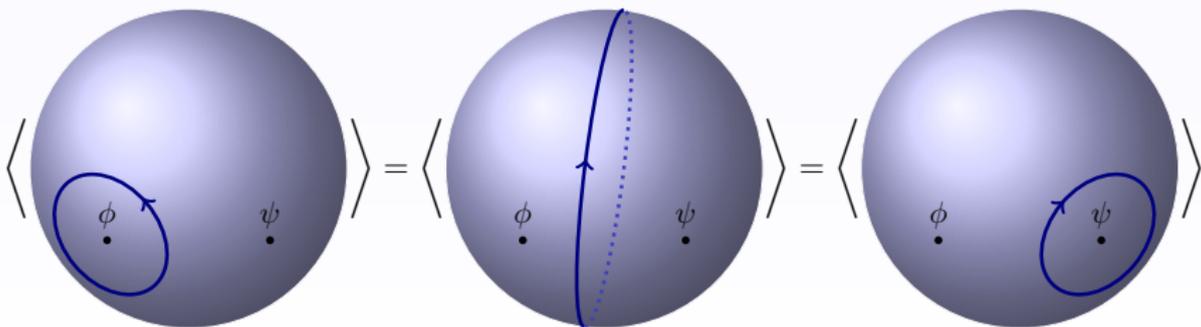
$$\langle \mathcal{D}_X^l(\phi), \psi \rangle_W = \langle \phi, \mathcal{D}_X^r(\psi) \rangle_V, \quad \langle \alpha, \beta \rangle_V = \text{Res} \left[\frac{\alpha \beta dz}{\partial_{z_1} V \dots \partial_{z_m} V} \right]$$

Applications

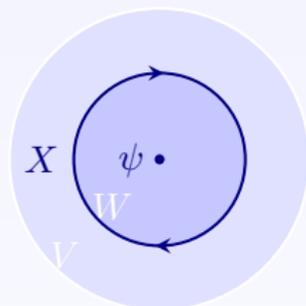
$$\mathcal{D}_X^r(\psi) = \text{Res} \left[\frac{\psi \text{ str} (\Lambda_X^{(x)} \Lambda_X^{(z)}) dx}{\partial_{x_1} W \dots \partial_{x_n} W} \right], \quad \mathcal{D}_X^l(\phi) = \text{Res} \left[\frac{\phi \text{ str} (\Lambda_X^{(x)} \Lambda_X^{(z)}) dz}{\partial_{z_1} V \dots \partial_{z_m} V} \right]$$

Left and right defect actions are adjoint with respect to the bulk topological metric:

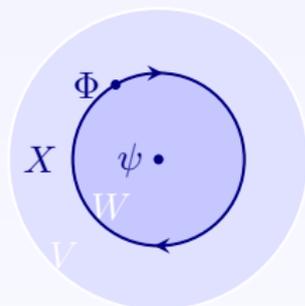
$$\langle \mathcal{D}_X^l(\phi), \psi \rangle_W = \langle \phi, \mathcal{D}_X^r(\psi) \rangle_V, \quad \langle \alpha, \beta \rangle_V = \text{Res} \left[\frac{\alpha \beta dz}{\partial_{z_1} V \dots \partial_{z_m} V} \right]$$



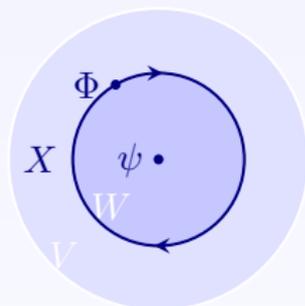
Applications


$$= \text{Res} \left[\frac{\psi \text{str} \left(\Lambda_X^{(x)} \Lambda_X^{(z)} \right) \underline{dx}}{\partial_{x_1} W \dots \partial_{x_n} W} \right]$$

Applications


$$= \text{Res} \left[\frac{\psi \text{str} (\Phi \Lambda_X^{(x)} \Lambda_X^{(z)}) \underline{dx}}{\partial_{x_1} W \dots \partial_{x_n} W} \right]$$

Applications



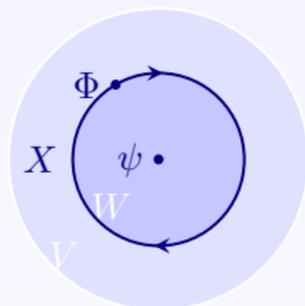
The diagram shows a light blue circular region representing a disk. Inside the disk is a smaller dark blue circle with a counter-clockwise arrow. A point labeled ψ is marked with a dot at the center of this inner circle. A point labeled Φ is marked with a dot on the boundary of the inner circle. The label X is placed to the left of the inner circle, and V is placed to the left of the outer disk boundary. The label W is placed below the inner circle.

$$= \text{Res} \left[\frac{\psi \text{str} (\Phi \Lambda_X^{(x)} \Lambda_X^{(z)}) dx}{\partial_{x_1} W \dots \partial_{x_n} W} \right]$$

Special cases:

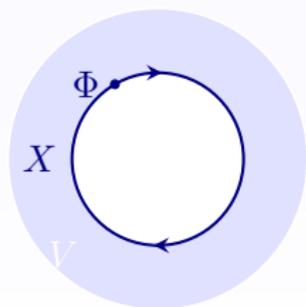
- $V = 0$ gives Kapustin-Li **disk correlator**

Applications

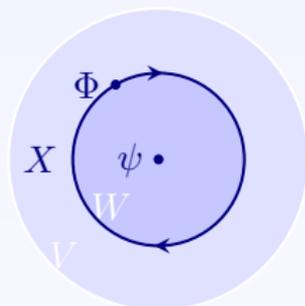

$$= \text{Res} \left[\frac{\psi \text{str} (\Phi \Lambda_X^{(x)} \Lambda_X^{(z)}) dx}{\partial_{x_1} W \dots \partial_{x_n} W} \right]$$

Special cases:

- $V = 0$ gives Kapustin-Li **disk correlator**
- $W = 0$ gives **boundary-bulk map** β^X :

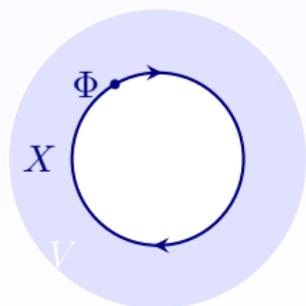

$$= \text{str} (\Phi \partial_{z_1} d_X \dots \partial_{z_m} d_X) =: \beta^X(\Phi)$$

Applications


$$= \text{Res} \left[\frac{\psi \text{str} (\Phi \Lambda_X^{(x)} \Lambda_X^{(z)}) dx}{\partial_{x_1} W \dots \partial_{x_n} W} \right]$$

Special cases:

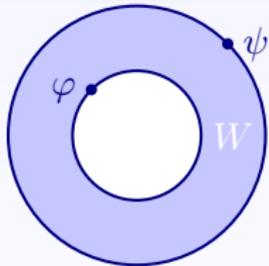
- $V = 0$ gives Kapustin-Li **disk correlator**
- $W = 0$ gives **boundary-bulk map** β^X :


$$= \text{str} (\Phi \partial_{z_1} d_X \dots \partial_{z_m} d_X) =: \beta^X(\Phi)$$

$\text{ch}(X) := \beta^X(1) = \text{str} (\partial_{z_1} d_X \dots \partial_{z_m} d_X)$ is the **Chern character**

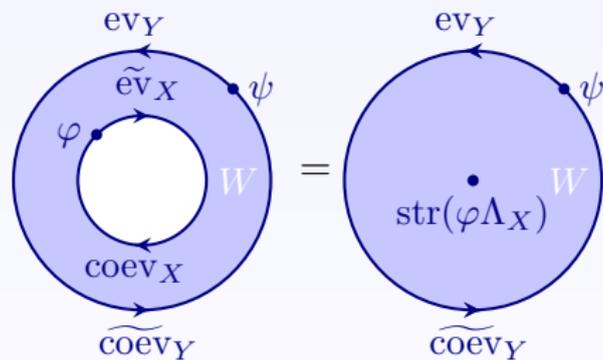
Applications

(3) Cardy condition



Applications

(3) Cardy condition



Applications

(3) Cardy condition

$$= \text{Res} \left[\frac{\beta^X(\phi) \beta^Y(\psi) \underline{dx}}{\partial_{x_1} W \dots \partial_{x_n} W} \right]$$

Applications

(3) Cardy condition

$$\begin{array}{c} \text{ev}_Y \\ \leftarrow \\ \text{ev}_X \\ \rightarrow \\ \text{coev}_X \\ \rightarrow \\ \text{coev}_Y \end{array} \text{ (annulus with } \phi, \psi \text{)} = \begin{array}{c} \text{ev}_Y \\ \leftarrow \\ \text{str}(\phi \Lambda_X) \\ \bullet \\ \text{coev}_Y \end{array} \text{ (disk with } \psi \text{)} = \text{Res} \left[\frac{\beta^X(\phi) \beta^Y(\psi) \underline{dx}}{\partial_{x_1} W \dots \partial_{x_n} W} \right]$$

$$= \begin{array}{c} \text{ev}_{X \dagger \otimes Y} \\ \leftarrow \\ 1 \otimes \phi \quad 1 \otimes \psi \\ \rightarrow \\ \text{coev}_{X \dagger \otimes Y} \end{array}$$

(\mathcal{LG} is also pivotal)

Applications

(3) Cardy condition

$$\begin{aligned}
 & \text{Diagram 1} = \text{Diagram 2} = \text{Res} \left[\frac{\beta^X(\phi) \beta^Y(\psi) \underline{dx}}{\partial_{x_1} W \dots \partial_{x_n} W} \right]
 \end{aligned}$$

$$\begin{aligned}
 & = \text{Diagram 3} = \text{str}(\psi \circ (-) \circ \phi)
 \end{aligned}$$

Applications

Theorem. The **Cardy condition** holds in \mathcal{LG}

Applications

Theorem. The **Cardy condition** holds in \mathcal{LG} : for matrix factorisations X, Y of W and maps $\varphi : X \rightarrow X$, $\psi : Y \rightarrow Y$ we have

$$\text{str}(\psi \circ (-) \circ \varphi) = \text{Res} \left[\frac{\beta^X(\varphi) \beta^Y(\psi) \underline{dx}}{\partial_1 W \dots \partial_n W} \right]$$

Applications

Theorem. The **Cardy condition** holds in \mathcal{LG} : for matrix factorisations X, Y of W and maps $\varphi : X \rightarrow X$, $\psi : Y \rightarrow Y$ we have

$$\text{str}(\psi \circ (-) \circ \varphi) = \text{Res} \left[\frac{\beta^X(\varphi) \beta^Y(\psi) \underline{dx}}{\partial_1 W \dots \partial_n W} \right]$$

Special case $\varphi = 1_X$, $\psi = 1_Y$ gives the Landau-Ginzburg version of the **Hirzebruch-Riemann-Roch theorem**:

$$\chi(\text{Hom}(\mathcal{E}, \mathcal{F})) = \int \text{ch}(\mathcal{E}^*) \text{ch}(\mathcal{F}) \text{Td}(X)$$

Applications

(4) **Generalised orbifolds** (work with **Ingo Runkel**)

Applications

(4) **Generalised orbifolds** (work with **Ingo Runkel**)

Theorem. Let $X \in \mathcal{LG}(W, V)$ have invertible quantum dimensions.

- $A = X^\dagger \otimes X$ is a special symmetric Frobenius algebra.

Applications

(4) **Generalised orbifolds** (work with **Ingo Runkel**)

Theorem. Let $X \in \mathcal{LG}(W, V)$ have invertible quantum dimensions.

- $A = X^\dagger \otimes X$ is a special symmetric Frobenius algebra.
- **Everything about theory V can be recovered from A**

Applications

(4) **Generalised orbifolds** (work with **Ingo Runkel**)

Theorem. Let $X \in \mathcal{LG}(W, V)$ have invertible quantum dimensions.

- $A = X^\dagger \otimes X$ is a special symmetric Frobenius algebra.
- **Everything about theory V can be recovered from A :**
 - ▶ $\mathcal{LG}(0, V) \cong \text{mod}(A)$ (boundary sector)
 - ▶ $\mathcal{LG}(V, V) \cong \text{bimod}(A)$ (defect sector)

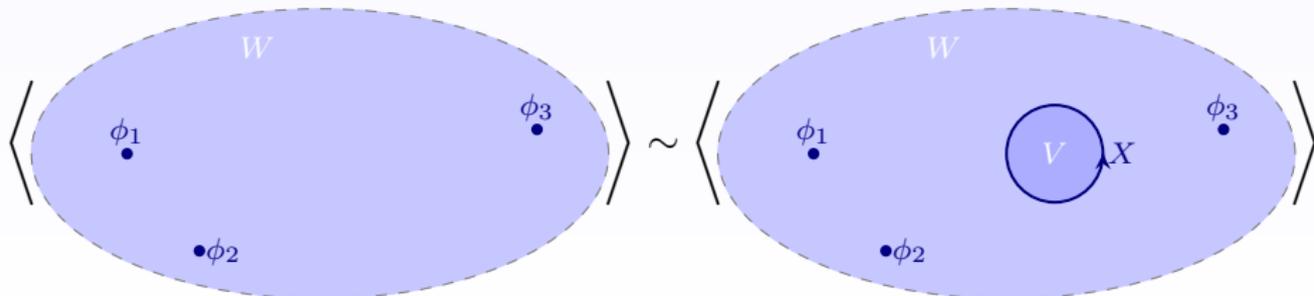
Applications

(4) Generalised orbifolds (work with Ingo Runkel)

Theorem. Let $X \in \mathcal{LG}(W, V)$ have invertible quantum dimensions.

- $A = X^\dagger \otimes X$ is a special symmetric Frobenius algebra.
- **Everything about theory V can be recovered from A :**
 - ▶ $\mathcal{LG}(0, V) \cong \text{mod}(A)$ (boundary sector)
 - ▶ $\mathcal{LG}(V, V) \cong \text{bimod}(A)$ (defect sector)

Idea. Introducing X -bubbles in W -correlator is scaling by $\text{qdim}(X)$.
Blowing up all X -bubbles produces V -correlator with defect network.



Applications

Examples for generalised orbifolds:

- “ordinary” orbifolds: for discrete symmetry group G of W we have $\mathcal{LG}(0, W)^G \cong \text{mod}(\bigoplus_{g \in G} \mathcal{I}_g)$

Applications

Examples for generalised orbifolds:

- “ordinary” orbifolds: for discrete symmetry group G of W we have $\mathcal{LG}(0, W)^G \cong \text{mod}(\bigoplus_{g \in G} \mathcal{I}_g)$
- \mathbb{Z}_2 -orbifold between A- and D-type minimal models

Applications

Examples for generalised orbifolds:

- “ordinary” orbifolds: for discrete symmetry group G of W we have $\mathcal{LG}(0, W)^G \cong \text{mod}(\bigoplus_{g \in G} \mathcal{I}_g)$
- \mathbb{Z}_2 -orbifold between A- and D-type minimal models:

$$X = \begin{pmatrix} 0 & \frac{x^d - u^{2d}}{x - u^2} - y^2 \\ x - u^2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & z + uy \\ z - uy & 0 \end{pmatrix}$$

is defect between $W = u^{2d}$ and $V = x^d - xy^2 + z^2$, has invertible quantum dimensions

Applications

Examples for generalised orbifolds:

- “ordinary” orbifolds: for discrete symmetry group G of W we have $\mathcal{LG}(0, W)^G \cong \text{mod}(\bigoplus_{g \in G} \mathcal{I}_g)$
- \mathbb{Z}_2 -orbifold between A- and D-type minimal models:

$$X = \begin{pmatrix} 0 & \frac{x^d - u^{2d}}{x - u^2} - y^2 \\ x - u^2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & z + uy \\ z - uy & 0 \end{pmatrix}$$

is defect between $W = u^{2d}$ and $V = x^d - xy^2 + z^2$, has invertible quantum dimensions

- similar equivalences expected e. g. between A- and E-type

Applications

Examples for generalised orbifolds:

- “ordinary” orbifolds: for discrete symmetry group G of W we have $\mathcal{LG}(0, W)^G \cong \text{mod}(\bigoplus_{g \in G} \mathcal{I}_g)$
- \mathbb{Z}_2 -orbifold between A- and D-type minimal models:

$$X = \begin{pmatrix} 0 & \frac{x^d - u^{2d}}{x - u^2} - y^2 \\ x - u^2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & z + uy \\ z - uy & 0 \end{pmatrix}$$

is defect between $W = u^{2d}$ and $V = x^d - xy^2 + z^2$, has invertible quantum dimensions

- similar equivalences expected e. g. between A- and E-type

Task. Classify all defects with invertible quantum dimensions (and find new equivalences this way)!

Conclusions

“2d TFT with defects = bicategory + x ”

Conclusions

“2d TFT with defects = bicategory + x ”: natural, easy, useful

Conclusions

“2d TFT with defects = bicategory + x ”: natural, easy, useful

Theorem. The bicategory of **Landau-Ginzburg models** has adjoints.
(conceptual construction, yet very “computable”)

Conclusions

“2d TFT with defects = bicategory + x ”: natural, easy, useful

Theorem. The bicategory of **Landau-Ginzburg models** has adjoints.
(conceptual construction, yet very “computable”)

Description naturally incorporates known structure:

- disk correlators
- boundary-bulk maps
- defect action on bulk fields, quantum dimensions
- Cardy condition
- ...

Conclusions

“2d TFT with defects = bicategory + x ”: natural, easy, useful

Theorem. The bicategory of **Landau-Ginzburg models** has adjoints.
(conceptual construction, yet very “computable”)

Description naturally incorporates known structure:

- disk correlators
- boundary-bulk maps
- defect action on bulk fields, quantum dimensions
- Cardy condition
- ...

Also allows to find new structure: **generalised orbifolds**