The Path Integral, Perturbation Theory and Complex Actions

by

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and work in progress

- The Feynman path integral in Minkowski space is not a well defined mathematical expression.
- The integral is not absolutely convergent.
- Consider the two dimensional example:

$$\int dx dy e^{i(x^2+y^2)}$$

Changing variables to polar coordinates we have

$$2\pi \int_0^\infty dr r e^{ir^2} = (\pi/i) e^{ir^2}\Big|_0^\infty = \infty$$

- The actual definition of the path integral is via the Euclidean path integral, with imaginary time.

$$iS_{Mink.} \to -S_E$$

$$t \to -i\tau$$

$$\partial_t \to i\partial_\tau$$

$$\partial_t\phi\partial_t\phi \to -\partial_\tau\phi\partial_\tau\phi$$

$$
\begin{aligned}
iS_{Mink.} &= i\int dt d^d x (1/2)\partial_\mu\phi\partial^\mu\phi - V(\phi) \\
&= i\int dt d^d x (1/2)\partial_t\phi\partial_t\phi - (1/2)\partial_i\phi\partial_i\phi - V(\phi) \\
&\to i(-i)\int d\tau d^d x - (1/2)\partial_\tau\phi\partial_\tau\phi - (1/2)\partial_i\phi\partial_i\phi - V(\phi) \\
&\equiv -S_E
\end{aligned}
$$

- Then the Euclidean functional integral defined by:

$$
Z_E[J] = \frac{1}{\mathcal{N}}\int \mathcal{D}\phi\; e^{-S_E[\phi]+\int J\phi}
$$

Complex Actions

- The Euclidean space action is sometimes not real.
- It can have parts which are imaginary.
- If the Minkowski action has a term which is t-odd, its analytic continuation to Euclidean space generally yields an imaginary term
- Fermions contribute to the path integral with a factor that is real, but can be negative. This corresponds to an action which has and imaginary part $i\pi$

- Complex actions come in many forms, but they usually contain topological terms.
- Chern-Simons terms
- Wess-Zumino terms
- epsilon tensor related expressions, for example the theta term in four dimensions:

$$WZ = \frac{N}{24\pi^2}\int_{\frac{1}{2}S^5+S^4} d^5x\epsilon^{\mu\nu\lambda\sigma\tau} tr\left[U^\dagger\partial_\mu U U^\dagger \partial_\nu U \cdots U^\dagger \partial_\tau U\right]$$

$$CS = \lambda \int_{R^3+\infty} d^3x\epsilon^{\mu\nu\lambda} tr\left[A_\mu\partial_\nu A_\lambda + \frac{2}{3}A_\mu A_\nu A_\lambda\right]$$

$$\sim \epsilon_{\mu\nu\lambda\rho}F^{\mu\nu}F^{\lambda\rho}$$

- such terms are linear in the time derivative
- hence the i in front of the Minkowski space action is not cancelled, indeed:

$$\int dt\partial_t \to \int d\tau\partial_\tau$$

thus the Euclidean action is in general complex and the functional integral is of the form:

$$Z_E = \frac{1}{\mathcal{N}}\int \mathcal{D}\phi\ e^{-S_E[\phi]+iS_{top.}[\phi]}$$

- This is not an insurmountable problem to the proper mathematical definition of the functional integral.
- However, the usual perturbative paradigm of quantum mechanics, to find the classical critical points of the action and quantize the small oscillations, fails.
- Imagine that we have written the action strictly in terms of real fields, which is always possible.
- There are, in general, no solutions to the equations of motion.

- Classical solutions are the critical points of the action.
- The corresponding equations of motion have no solution for real fields in general
- Solutions may exist, but they are off the real axis in complexified field space.

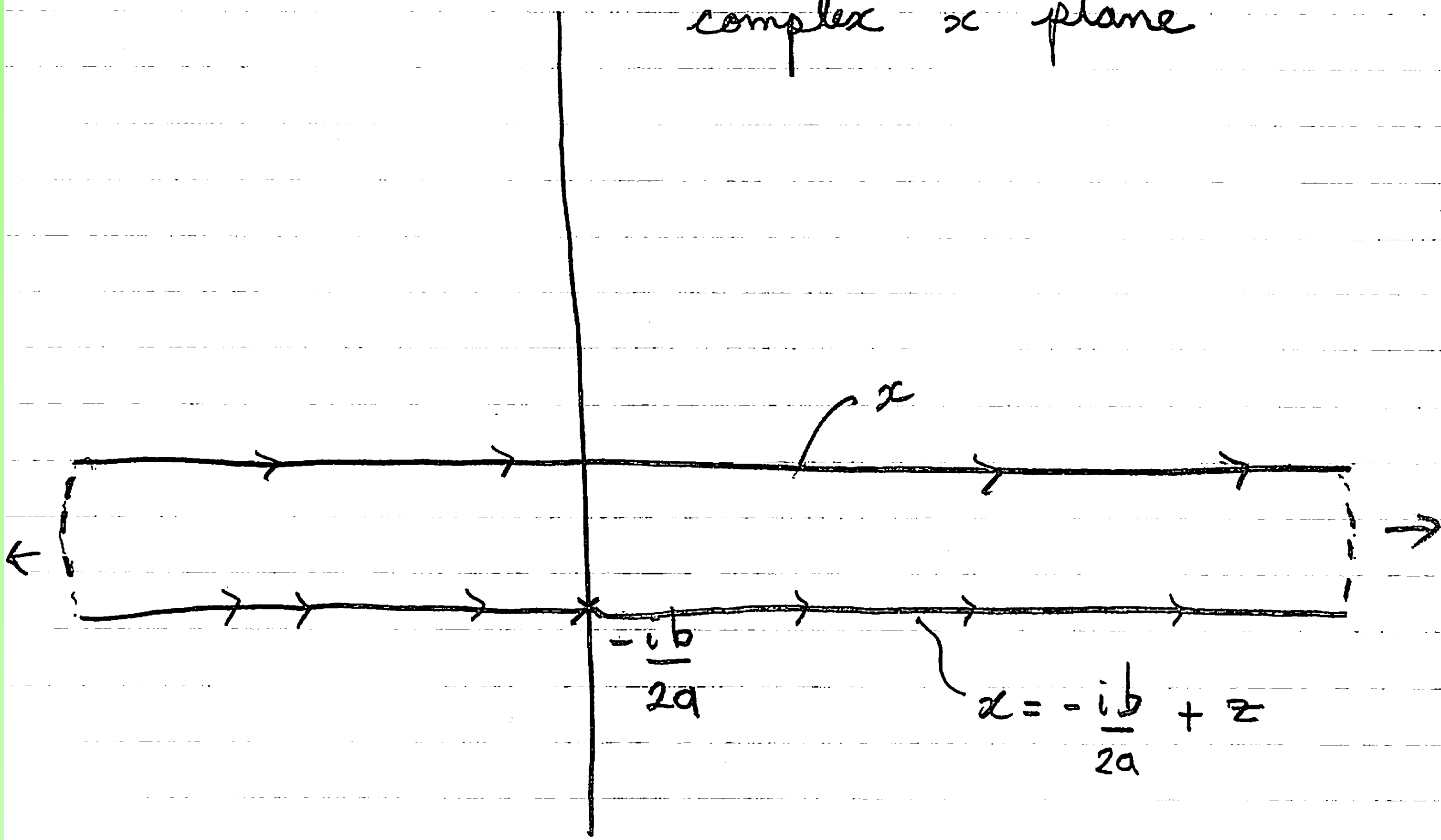
$$\frac{\delta S_E}{\delta \phi} + i \frac{\delta S_{top.}}{\delta \phi} = 0$$

- a trivial example is given by a simple integral

$$Z(a,b) = \int_{-\infty}^{\infty} dx\, e^{-(ax^2+ibx)} \qquad a > 0$$

$$\frac{\delta}{\delta x}(ax^2 + ibx) = 0 \qquad \Rightarrow \qquad 2ax + ib = 0$$

$$x_{crit.} = \frac{-ib}{2a}$$


complex x plane
x
$-\frac{ib}{2a}$
$x = -\frac{ib}{2a} + z$

- We consider three models:
- Georgi-Glashow model with Chern-Simons term in 2+1 dimensions.
- Abelian-Higgs model with Chern-Simons term in 0+1 dimensions
- Quantum spin models

Georgi-Glashow Model with Chern-Simons term

$$S_E = \int_{R^3+\infty} \frac{-1}{2g^2} tr(F_{\mu\nu}F_{\mu\nu}) + \frac{1}{2} D_\mu h^a D_\mu h^a + \frac{\lambda}{4}(h^a h^a - v^2)^2$$
$$+ \frac{-i\kappa}{g^2}\epsilon_{\mu\nu\lambda} tr\left[A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda\right]$$

with the definitions

$$A_\mu = (i/2)A^a_\mu \tau^a \qquad h = (i/2)h^a \tau^a$$

$$D_\mu h = \partial_\mu h + [A_\mu, h]$$

- In the Higgs phase (no CS), the symmetry is spontaneously broken to U(1)
- the usual Higgs mechanism gives mass to the $W_\mu^\pm$. The $Z_\mu^0 \to A_\mu$ remains massless.
- the vacuum solution is: $h^a = v(0,0,1) \quad A_\mu^a = 0$
- the quantized perturbative oscillations about this critical point gives rise to a U(1) gauge theory with two charged massive vector bosons and one neutral scalar.

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This description is completely mistaken!

- Contributions from the quantum fluctuations about non-trivial critical points completely reorganizes the theory, the U(1) is confined.
- Non-trivial critical points of the Euclidean action, instantons, are actually 'tHooft-Polyakov monopoles.

$$
\begin{aligned}
h^a &= \hat{x}^a h(r) \\
A_\mu^a &= \frac{1}{r}\epsilon^{a\mu\nu}\hat{x}^\nu(1-\phi(r))+\cdots
\end{aligned}
$$

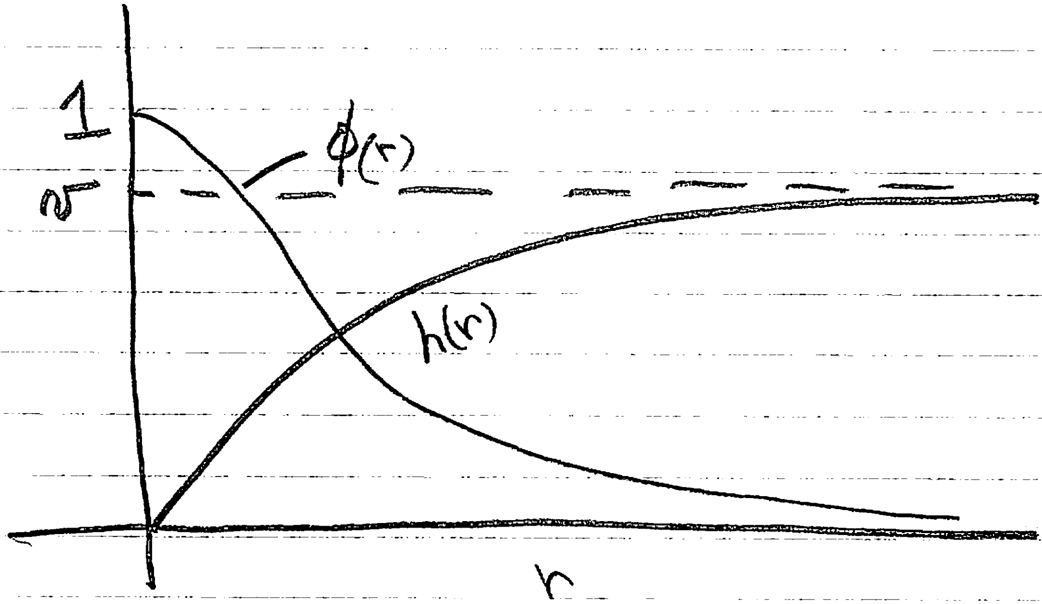
- For the monopole:

$$F^a_{\mu\nu}\Big|_{r\to\infty} \to \frac{1}{r^2}\epsilon_{\mu\nu b}\hat{x}^a\hat{x}^b$$

$$h^a \to v\hat{x}^a$$

$$F_{\mu\nu} = \frac{h^a}{v}F^a_{\mu\nu}\Bigg|_{r\to\infty} \to \frac{1}{r^2}\epsilon_{\mu\nu b}\hat{x}^b$$

$$B_\mu = 1/2\epsilon_{\mu\nu\sigma}F_{\nu\sigma} \to \frac{\hat{x}^\mu}{r^2}$$


1
v
$\phi(r)$
$h(r)$
r

- Taking into account the “Coulomb” interaction between the monopoles, Polyakov showed that the electric field is linearly confined.
- The photon becomes massive, there are no massless excitations left in the theory.
- What happens with the addition of the Chern-Simons term to the action?
- The biggest change is that all vector gauge bosons become massive.
- Moreover the magnetic monopole solution no longer exists.

- The U(1) gauge field being massive does not allow for a long range magnetic field.
- It is not obvious what happens to the critical points of the Euclidean action.
- The Chern-Simons term is complex, hence the solutions become complex monopoles, defined off the real axis of field configurations.
- Hosotani, Saririan and Tekin found such complex monopoles: hep-th/9808045

- Affleck, Harvey, Palla and Semenoff first considered the problem of what happens to Polyakov's result when a Chern-Simons term is added.
- They did not look for complex critical points, their analysis was:

$$\frac{1}{\mathcal{N}}\int \mathcal{D}(\phi, A)\ \left(e^{-S_E[\phi]} + gauge\ \ fixing\right)$$

defines a perfectly good measure on the space of (real) field configurations, then

$e^{iS_{CS}}$ is a bounded function that can simply be integrated against the measure.

- Their point was that the CS term is not invariant under certain gauge-like transformations, and integrating over these transformations gives rise to destructive interference in the presence of a monopole, annulling its contribution. For a gauge transformation:

$$\begin{aligned}\delta S_{CS} &= \frac{i\kappa}{g^2}\int d^3x \epsilon_{\mu\nu\lambda} tr\left[\frac{-1}{3}(\partial_\mu U)U^\dagger(\partial_\nu U)U^\dagger(\partial_\lambda U)U^\dagger\right] \\ &+ \frac{i\kappa}{g^2}\int_{r\to\infty} d\sigma_\mu \epsilon_{\mu\nu\lambda} tr\left[A_\nu(\partial\lambda U)U^\dagger\right]\end{aligned}$$

- The first term is the standard variation of the CS term, which is a topological invariant, and invariance of the exponential of the action imposes the quantization of the coefficient of the CS term: $\frac{\kappa}{g^2} = \frac{n}{4\pi}$
- The second term is a boundary term, which is usually zero, hence negligible.
- In the presence of a monopole, however, this term is not zero.
- The gauge group corresponds to transformations that are identity at infinity, and these are fixed by the gauge fixing condition. Thus those that do not satisfy this are field configurations that should be integrated over. Without the CS term, the action is just invariant under these transformations, and they correspond to zero modes of the monopole configuration.
- In the presence of a monopole integration over this degree of freedom simply makes the contribution vanish.

- For a transformation that is in the unbroken U(1) direction

$$U = e^{i\Lambda(r)\hat{r}\cdot\vec{\sigma}/2} \qquad \Lambda(0) = 0 \quad \Lambda(\infty) = \Lambda$$

the total change in the CS term is:

$$\delta S_{CS} = i\frac{\kappa}{g^2}4\pi\Lambda = in\Lambda$$

for $\Lambda = 2\pi$ $\qquad$ $\delta S_{CS} = i2\pi n$

and $e^{iS_{CS}}$ is invariant. But if $\Lambda \neq 2\pi$

the transformation is a zero mode of the monopole and not a gauge transformation. Consequently it must be integrated over.

- Gauge fixing constrains the form of $\Lambda(r)$.
- The normal part of the Euclidean action is simply invariant.
- Integrating over the asymptotic value Λ yields $2\pi\delta_{n,o}$
- Thus the CS term projects the integration to the zero monopole sector.
- The result seems to be correct, and consistent with other work which indicates that the monopoles are bound in pairs with anti-monopoles, with linear confinement.

- This implies that a dilute gas of monopoles and anti-monopoles is not possible.
- The mechanism of confinement of Polyakov is lost.
- The classical behaviour of charged particles should be recovered.
- The CS term gives the photon a mass, the Coulomb interaction is short ranged and even the classical logarithmic confinement is lost.
- The deconfined charged particles obtain flux and fractional statistics, becoming anyons.
- Thus the effect of a complex term in the action can radically affect the spectrum of the theory.

- However using critical points of only a part of the action is not satisfactory, it is possible that the results are deceiving.
- The CS term is not gauge invariant, the critical points of the action are affected by gauge fixing and do not transform into each other under change of gauge.
- Hosotani, Saririan and Tekin looked at this in some detail. In the Lorentz gauge the transformation

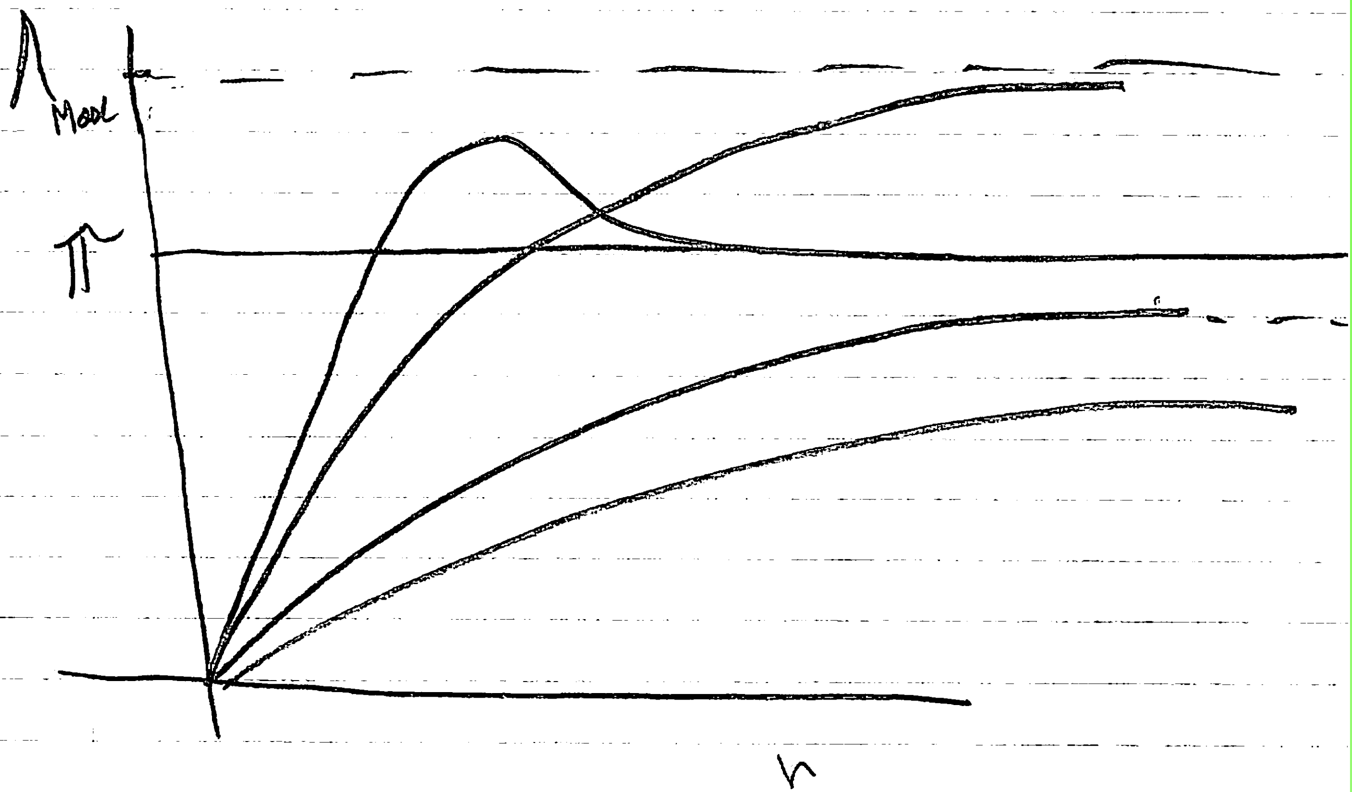
$$U = e^{i\Lambda(r)\hat{r}\cdot\vec{\sigma}/2}$$

requires the profile function satisfies the Gribov equation:

$$\Lambda''(r) + (2/r)\Lambda'(r) - (2/r^2)\phi\sin(\Lambda) = 0$$

$$\Lambda''(r) + (2/r)\Lambda'(r) - (2/r^2)\phi\sin(\Lambda) = 0$$

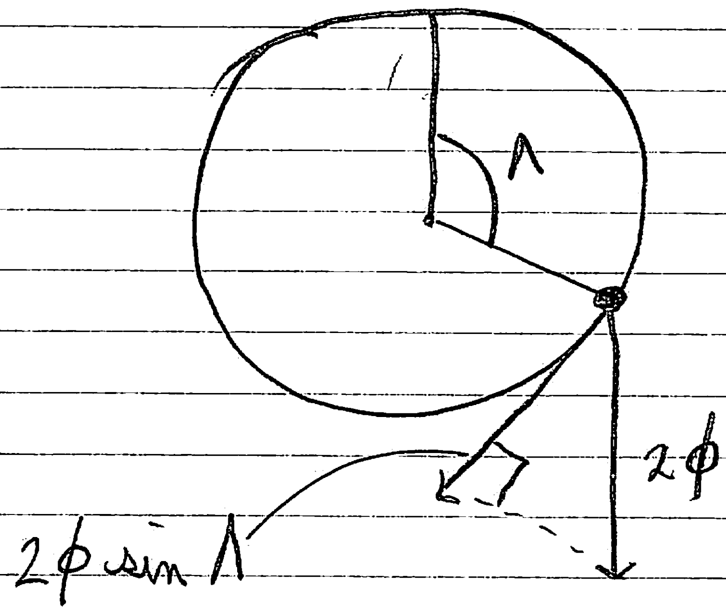
- The configuration transformed with $\Lambda(0) = 0 \quad \Lambda(\infty) = 2\pi$ is an exact Gribov copy. Intermediate transformations are not local gauge transformations.
- Hosotani et al find that solutions with arbitrary $\Lambda(\infty)$ are not allowed.
- The range of allowed $\Lambda(\infty)$ is numerically found to be between -3.98 to +3.98. (For a BPS monopole.)



Λ_{Mod}
Π
k

- This would imply the Affleck et al argument is not correct.
- We have reproduced the numerical work of Hosotani et al, the system is equivalent to a simple damped pendulum.

$$\ddot{\Lambda}(t) + \dot{\Lambda}(t) - \phi \sin(\Lambda(t)) = 0 \qquad t = \ln r$$



$\phi(-\infty) = 1$ $\qquad$ $\phi(+\infty) = 0$ $\qquad$ exponentially

- Hosotani et al also find solutions, that is, the critical points of the full action. These are complex monopole solutions.
- Only numerical solutions can be found.
- There is no Gribov ambiguity. The gauge is fixed and then solutions are found in complex field space. It is unclear how the notion of the gauge invariance should be continued analytically.
- Now the functional integration contour needs to be deformed to pass through the complex critical points, to see if it reproduces the Affleck et al result.
- It is unclear how this should be done, so we look at a simpler model.

Abelian Higgs model with Chern-Simons term in 0+1 dimensions

- The Lagrangian of this model (*N* scalars):

$$\mathcal{L} = \sum_{i=1}^{N} \left(|(\partial_\tau + iA)\phi_i|^2 + m^2 \, |\phi_i|^2 \right) + i\lambda A$$

We take compact Euclidean time

$$\tau \, : \;\; 0 \longrightarrow \beta$$

which is the same as finite temperature.
We take the gauge choice:

$$\partial_\tau A = 0 \Rightarrow A = const.$$

- With the topologically non-trivial gauge transformation

$$U = e^{i2\pi n\tau/\beta} \Rightarrow A \to A - 2\pi n\tau/\beta$$

this implies that we can restrict $A: \; 0 \to 2\pi/\beta$

The CS term is not invariant:

$$i\lambda \int_0^\beta d\tau A \to i\lambda \int_0^\beta d\tau (A - \partial_\tau \Lambda)$$

$$= i\lambda(\beta A - 2n\pi)$$

hence we must have $\lambda 2n\pi = 2M\pi \quad \forall \quad n$

$$\Rightarrow \lambda \in Z$$

- The equations of motion are:

$$-D_\tau^2\phi_i + m^2\phi_i = 0$$

$$\int_0^\beta d\tau \left(i((\partial_\tau\vec{\phi})^* \cdot \vec{\phi} - \vec{\phi}^* \cdot \partial_\tau\vec{\phi}) + 2A\vec{\phi}^* \cdot \vec{\phi} \right) - iN\beta = 0$$

$$\Rightarrow \quad A = \frac{iN\beta - \int_0^\beta d\tau i((\partial_\tau\vec{\phi})^* \cdot \vec{\phi} - \vec{\phi}^* \cdot \partial_\tau\vec{\phi})}{2\int_0^\beta d\tau\vec{\phi}^* \cdot \vec{\phi}}$$

$$\approx \frac{i\gamma + \alpha}{\delta}$$

- The scalar field equation has no solution in the space of periodic field configurations for non-trivial holonomy: $A \neq 2n\pi/\beta$
- However the functional integral can be exactly evaluated. Each scalar field gives

$$\phi(\tau) = \sum_{n=-\infty}^{\infty} \phi_n e^{i2\pi n\tau/\beta}$$

$$\int_0^\beta |D_\tau \phi|^2 = \beta \sum_{n=-\infty}^{\infty} \phi_n^* \phi_n ((2\pi n/\beta) + A)^2$$

$$Z(\beta, m, A) = \left(\int_{-\infty}^{\infty} \prod_n d\{\phi_n^* \phi_n\} e^{-\beta \sum_n |\phi_n|^2 ((\frac{2\pi n}{\beta}) + A)^2 + m^2) + i\beta A} \right)^N$$

$$Z(\beta, m, A) = \prod_{n=-\infty}^{\infty} \left(\frac{1}{\beta((\frac{2\pi n}{\beta} + A)^2 + m^2)} \right)^N e^{iN\beta A}$$

$$\frac{Z(\beta, m, A)}{Z(\beta, m, 0)} = \left(\prod_{n=-\infty}^{\infty} \frac{(\frac{2\pi n}{\beta})^2 + m^2}{(\frac{2\pi n}{\beta} + A)^2 + m^2} \right)^N e^{iN\beta A}$$

The infinite product can be exactly done, adapting methods of Jackiw and Dunne, Lee and Lu. We find:

$$\frac{Z(\beta, m, A)}{Z(\beta, m, 0)} = \left(\frac{\cosh \beta m - 1}{\cosh \beta m - \cos \beta A} \right)^N e^{iN\beta A}$$

- The "functional" integral that we are left with is:

$$\mathcal{I}(N,\beta,m) = \int_0^{2\pi/\beta} dA \left(\frac{\cosh \beta m - 1}{\cosh \beta m - \cos \beta A} \right)^N e^{iN\beta A}$$

There is a slight analogy with the integration over $\Lambda(\infty)$ in Affleck et al:

$$\int_0^{2\pi} d\Lambda(\infty) e^{-S_E + iS_{CS}} e^{in\Lambda(\infty)} \to \delta_{n,0}$$

- Our integral can actually be done exactly.
- Also we can use the saddle point approximation through the complex critical point and compare with the approximation of just using the critical point of the real part of the Euclidean action.
- To perform the integral exactly we use the complex variable

$$z = e^{i\beta A}$$

$$dA = dz/i\beta z$$

$$\cos\beta A = (e^{i\beta A} + e^{-i\beta A})/2 = (z + 1/z)/2$$

and the integration contour is the unit circle in the complex z plane.

$$\mathcal{I}(N,\beta,m) = \oint \frac{dz}{i\beta}\left(\frac{2(\cosh\beta m - 1)}{2z\cosh\beta m - z^2 - 1}\right)^N z^{2N-1}$$

the poles are at

$$z_\pm = \cosh\beta m \pm \sqrt{(\cosh\beta m)^2 - 1} = e^{\pm\beta m}$$

and Cauchy's theorem gives the result

The exact result is:

$$\mathcal{I}(N,\beta,m) = \frac{2\pi(\cosh\beta m - 1)^N 2^N}{\beta(-1)^N(N-1)!}\sum_{k=0}^{N-1}\binom{N-1}{k}\left(\frac{d^k z^{2N-1}}{dz^k}\right)\left(\frac{d^{N-1-k}}{dz^{N-1-k}}\frac{1}{(z-e^{\beta m})^N}\right)\Bigg|_{e^{-\beta m}}$$

In the limit $\beta m \to \infty$ we obtain:

$$\mathcal{I}(N,\beta,m) \approx \frac{e^{-N\beta m}2^{2N}}{\beta}\sqrt{\frac{\pi}{N}}.$$

However, looking at it as a path integral over the gauge field:

$$\mathcal{I}(N,\beta,m) = \int_0^{2\pi/\beta} dA\, e^{-f(A)} e^{iN\beta A}$$

with

$$f(A) - iN\beta A = -N\left(\ln\frac{\cosh\beta m - 1}{\cosh\beta m - \cos\beta A} + i\beta A\right)$$

- The critical points are given by

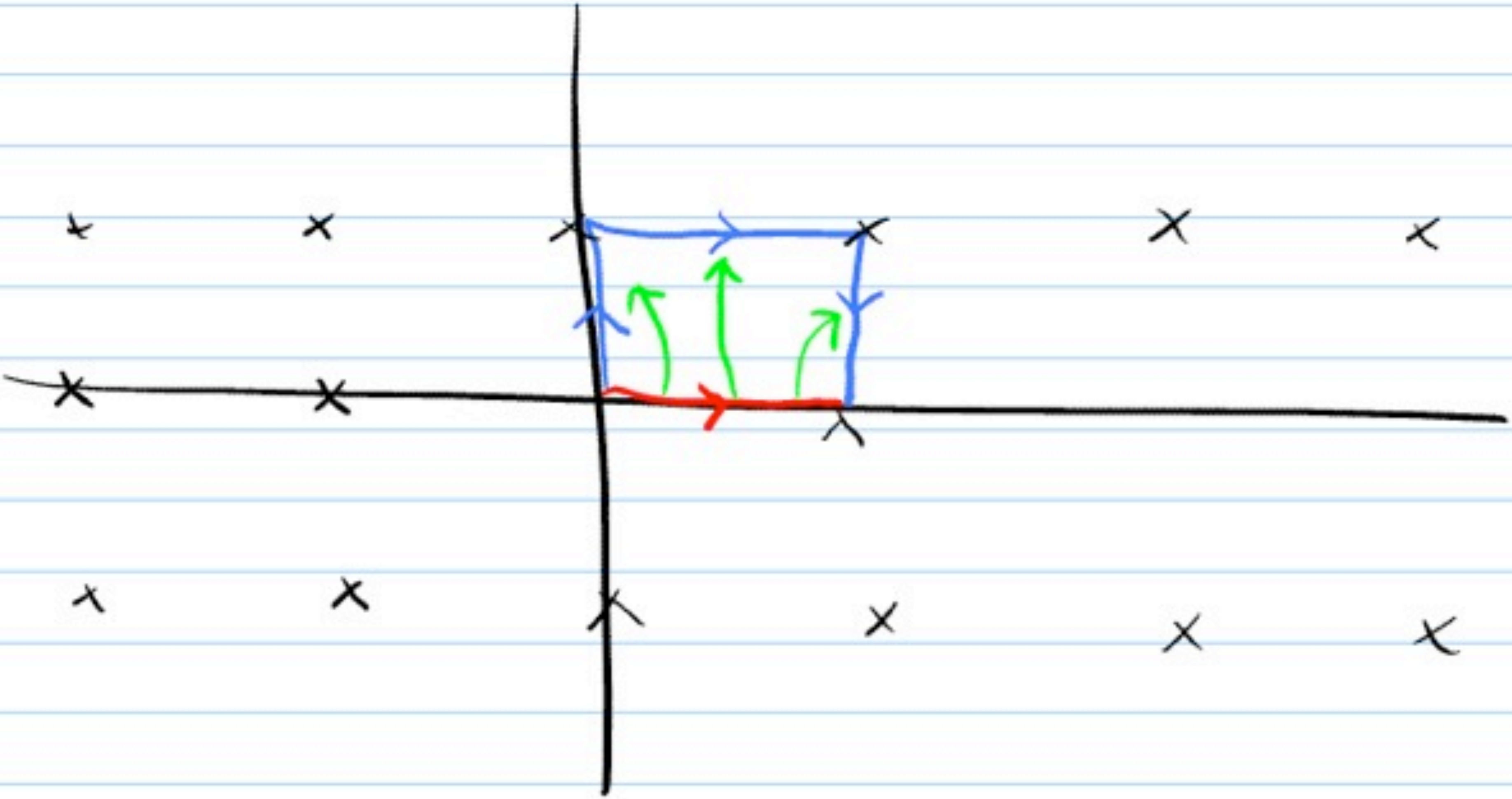
$$\frac{d}{dA}(f(A) - iN\beta A) = 0 \quad \Rightarrow \frac{\beta \sin \beta A}{\cosh \beta m - \cos \beta A} - i\beta = 0$$

ie.

$$\cos \beta A - i \sin \beta A = \cosh \beta m$$

with solutions

$$\beta A^* = i \ln(\cosh \beta m) + 2\pi k$$



- Integrating along the blue contour, the two vertical sections exactly cancel.
- The integral (in saddle point approximation) from one side of the first critical point plus the contribution from the other side of the next periodic critical point just corresponds to integrating through just one critical point.
- The result is of the form:

$$\mathcal{I}(N,\beta,m) = e^{-f(A^*)+iN\beta A^*}\sqrt{\frac{2\pi}{f''(A^*)}}$$

- with

$$e^{-(fA^*)-i\beta NA^*)} = \frac{2^N(\cosh\beta m - 1)^N}{\sinh^{2N}\beta m}$$

$$f''(A^*) = 2N\beta^2\cosh^2\beta m/\sinh^2\beta m$$

which gives in the limit $\beta m \to \infty$

$$e^{-(fA^*)-i\beta NA^*)} \approx 2^{2N}e^{-N\beta m}$$

$$f''(A^*) \approx 2N\beta^2$$

yielding exactly as before

$$\mathcal{I}(N,\beta,m) \approx \frac{e^{-N\beta m}2^{2N}}{\beta}\sqrt{\frac{\pi}{N}}.$$

- On the other hand the critical points of the real part of the action is just

$$A^* = 2\pi k$$

$$e^{-f(A^*)} = 1$$

$$f''(A^*) = \beta^2/(\cosh\beta m - 1)$$

which gives

$$\mathcal{I}(N,\beta,m) = \int_0^{2\pi/\beta} dA e^{-(N/2)(\beta^2/(\cosh\beta m - 1))A^2} e^{iN\beta A}$$

the Gaussian integral gives the measure against which the oscillatory phase is integrated:

$$\mathcal{I}(N,\beta,m) \approx \int_{-\infty}^{\infty} dx e^{-(N\alpha/2)\left(x^2+2i\beta x/\alpha-\beta^2/\alpha^2\right)} e^{-N\beta^2/2\alpha}$$

$$\alpha = \beta^2/(\cosh\beta m - 1)$$

$$\mathcal{I}(N,\beta,m) = \sqrt{\frac{2\pi}{N\alpha}} e^{-N\beta^2/2\alpha}$$

$$\mathcal{I}(N,\beta,m) = \sqrt{\frac{2\pi(\cosh\beta m - 1)}{N\beta^2}} e^{-N(\cosh\beta m - 1)/2}$$

Quantum Spin tunnelling

- We consider a path integral description of a quantum spin, a 0+1 dimensional problem.
- This corresponds to a dynamical system of a particle on a two sphere with a Wess-Zumino term.
- In Euclidean time, the Wess-Zumino term remains imaginary.

- A quantum spin is described by the Lagrangian of a particle on a sphere, with the addition of the Wess-Zumino term.

$$S_{Mink.} = \int dt \left(\frac{I}{2} \partial_t \hat{s} \cdot \partial_t \hat{s} - V(\hat{s}) \right) + \sigma \int d^2 x \epsilon^{ij} (\hat{s} \cdot \partial_i \hat{s} \times \partial_j \hat{s})$$

Which yields a Euclidean Lagrangian

$$S_E = \int d\tau \left(\frac{I}{2} \partial_\tau \hat{s} \cdot \partial_t \tau \hat{s} + V(\hat{s}) \right) - i\sigma \int d^2 x \epsilon^{ij} (\hat{s} \cdot \partial_i \hat{s} \times \partial_j \hat{s})$$

Tunnelling is mediated by instantons, the solutions of the Euclidean equations of motion.

The second term is integrated over a two dimensional manifold whose boundary corresponds to the time variable.

For convenience we take this to be periodic, ie. a circle. Then the 2-d manifold can be taken as half a 2 sphere, the equator of which is the time variable.

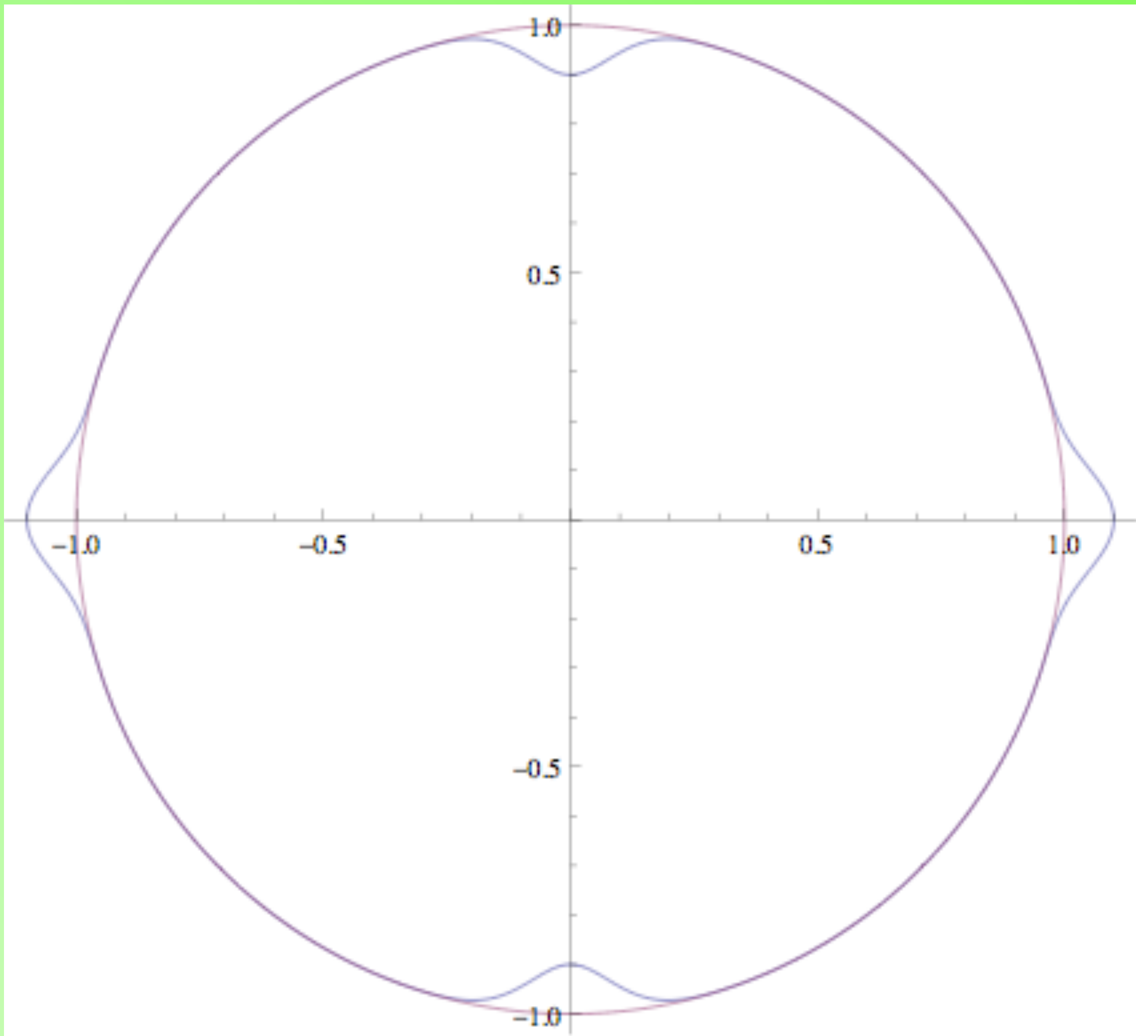
The integral is ambiguous by an integer, hence the coefficient is quantized.

- The potential is assumed to be easy-axis, (ie. azimuthally symmetric), reflection symmetric, with two classically degenerate minima at the two poles:

$$V(\theta,\phi)\equiv V(\theta)=V(\pi-\theta)$$

$$V(\hat{s})\equiv V(\theta,\phi)=\frac{1}{2}\gamma\sin^2\theta$$

Example of a suitable potential:



- The Wess-Zumino term can be actually written as a local 1-d density
- We must fix a filling in for the half sphere
- We take:

$$\hat{s} \equiv (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$\theta(t,x) = (1-x)\times\theta(t), \quad \varphi(t,x) = \varphi(t)$$

- And $\partial_x \hat{s} = -\hat{\theta}\theta(t), \quad \partial_t \hat{s} = (1-x)\dot{\theta}\hat{\theta} + \sin((1-x)\theta)\dot{\varphi}\hat{\varphi}$

$$\hat{s}\cdot(\partial_t \hat{s} \times \partial_x \hat{s}) = \theta\sin((1-x)\theta)\dot{\varphi}$$

- Then the integral over *x* can be done explicitly, giving:

$$S_{WZNW} = 2i\sigma \int dt \cos\theta \dot{\varphi}$$

A local expression dependent only on the time coordinate.

The equations of motion arising from this Lagrangian are completely integrable.

Thus we can find explicitly if there are any instanton like solutions to the Euclidean equations of motion.

First with the Wess-Zumino term absent, where we have already integrated the azimuthal equation:

- the equations of motion are

$$I\ddot{\theta} - I\sin\theta\cos\theta\dot{\varphi}^2 - \frac{\partial V(\theta)}{\partial\theta} = 0$$

$$I\sin^2\theta\dot{\varphi} = l \qquad\qquad \dot{\varphi} = 0$$

which gives the conserved "energy":

$$\frac{I}{2}\dot{\theta}^2 - V(\theta) = const. = 0$$

with solution

$$\int_{\pi/2}^{\theta(\tau)} d\theta \sqrt{\frac{I}{2V(\theta)}} = \tau - \tau_0$$

and finite Euclidean action

$$S_0 = \int d\tau \left(\frac{I}{2}\dot{\theta}^2 + V(\theta) \right) = \int_0^{\pi} d\theta \sqrt{2IV(\theta)}$$

which implies tunnelling!

Looking at the equations with the WZ term:

$$I\ddot{\theta} - I\sin\theta\cos\theta\dot{\varphi}^2 - \frac{\partial V(\theta)}{\partial\theta} + i\sigma\sin\theta\dot{\varphi} = 0$$

$$I\frac{d}{d\tau}\left(\sin^2\theta\dot{\varphi}\right) - i\sigma\sin\theta\dot{\theta} = 0$$

the last equation integrates as

$$I\sin^2\theta\dot{\varphi} + i\sigma\cos\theta = il$$

which yields:

$$\frac{I}{2}\dot{\theta}^2 - \frac{1}{2}\frac{(l-\sigma\cos\theta)^2}{I\sin^2\theta} - V(\theta) = \mathcal{S}$$

and an effective potential (the motion is in minus):

$$V_{eff.}(\theta) = \frac{1}{2}\frac{(l-\sigma\cos\theta)^2}{I\sin^2\theta} + V(\theta)$$

The action for the instanton is:

$$S_0 = \int d\tau \left(\frac{I}{2}\dot{\theta}^2 + V_{eff.}(\theta) \right) = \int_0^\pi d\theta \sqrt{2IV_{eff.}(\theta)}$$

$$= \int_0^\pi d\theta \sqrt{\sigma^2 \tan^2(\theta/2) + 2IV(\theta)} = \infty$$

$$e^{-S_0/\hbar} = 0$$

Thus the tunneling is completely suppressed for all values of sigma.

Another way to see the lack of tunneling is through the equivalent Schrödinger quantum mechanical problem:

$$\left(-\frac{\hbar^2}{2I}\left(\hat{s}\times(\vec{\nabla}-i\sigma\vec{A}^{\pm})\right)^2+V(\theta)\right)\Psi^{\pm}=E\Psi^{\pm}$$

with

$$A_{\varphi}^{+}=\sigma(1-\cos\theta)/\sin\theta \quad \theta\in[0,\pi)$$

$$A_{\varphi}^{-}=-\sigma(1+\cos\theta)/\sin\theta \quad \theta\in(0,\pi]$$

- writing the wave sections as:

$$\Psi^{\pm}_{n,m} = \begin{cases} \sum_{l\geq|m|} \psi_{n,l,m}\Theta_{n,l,m}(\theta)e^{i(m+\sigma)\varphi} & \theta \in [0,\pi) \\ \sum_{l\geq|m|} \psi_{n,l,m}\Theta_{n,l,m}(\theta)e^{i(m-\sigma)\varphi} & \theta \in (0,\pi] \end{cases}$$

$$\equiv \begin{cases} \psi_{n,m}(\theta)e^{i(m+\sigma)\varphi} & \theta \in [0,\pi) \\ \psi_{n,m}(\theta)e^{i(m-\sigma)\varphi} & \theta \in (0,\pi] \end{cases}$$

which satisfies the Schrödinger equation

$$\left(-\frac{\hbar^2}{2I}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}-\frac{(m+\sigma\cos\theta)^2}{\sin^2\theta}\right)\right)\psi_{n,m}(\theta)$$
$$+V(\theta)\psi_{n,m}(\theta)=E_{n,m}\psi_{n,m}(\theta)$$

This equation is symmetric under:

$$m\to -m, \theta\to\pi-\theta$$

- Thus the ground state is doubly degenerate, evidently for fermions, but also for bosons
- This is because the ground state occurs for

$$m=\pm\sigma$$

For each choice, the potential is no longer divergent at one of the poles, and the wave function localizes to the pole without the divergence. The symmetry assures the existence of a doubly degenerate ground state.

The unpaired state at $m = 0$ for bosonic spins is not the ground state as the divergent potential at both poles forces the spin to be away from the poles, to the region where $V(\theta)$ is non-zero, thus raising the energy.

Conclusions

- Thus it seems the tunnelling is suppressed for the azimuthally and reflection symmetric potentials, as it should be.
- Deforming the contour of path integration through the critical points of the full action and the corresponding saddle point approximation seems to be the right procedure.
- Tunnelling of spins waves seems to follow generally with previous analyses of Josephson type tunnelling
- Similar results follow for tunnelling in systems with two spins with ferromagnetic interactions.

Conclusions and problems

- It seems that the saddle point approximation with just the real part of the action does not give the right answer.
- Deforming the contour through the critical points of the full action and the corresponding saddle point approximation is the right procedure.
- The question of what is the appropriate gauge invariance for complexified field configurations needs to be thought out.
- Applications to theories with fermions and CP violations in 4 dimensions needs to be worked out.