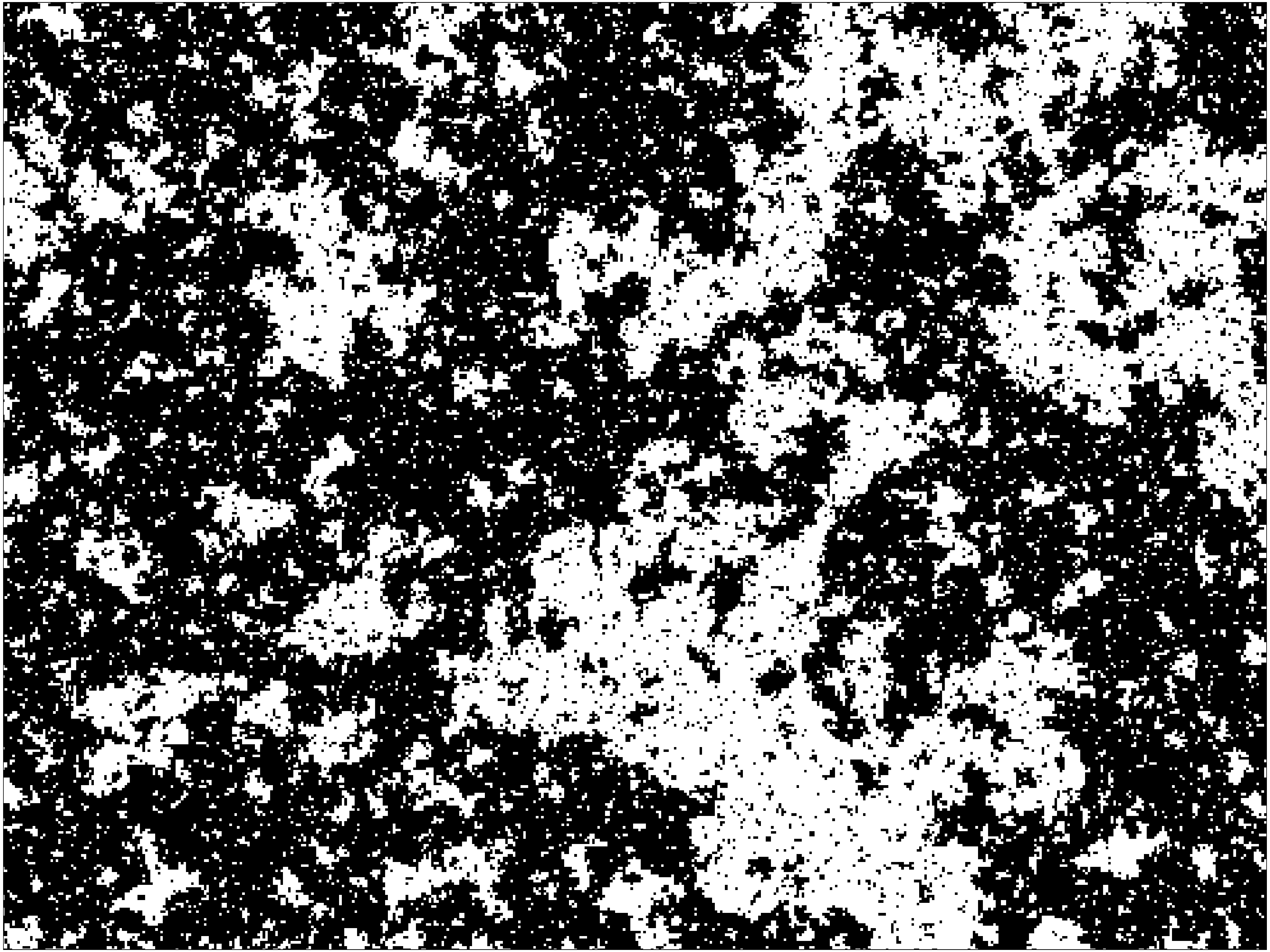
Quantum Information, the Jones Polynomial and Khovanov Homology

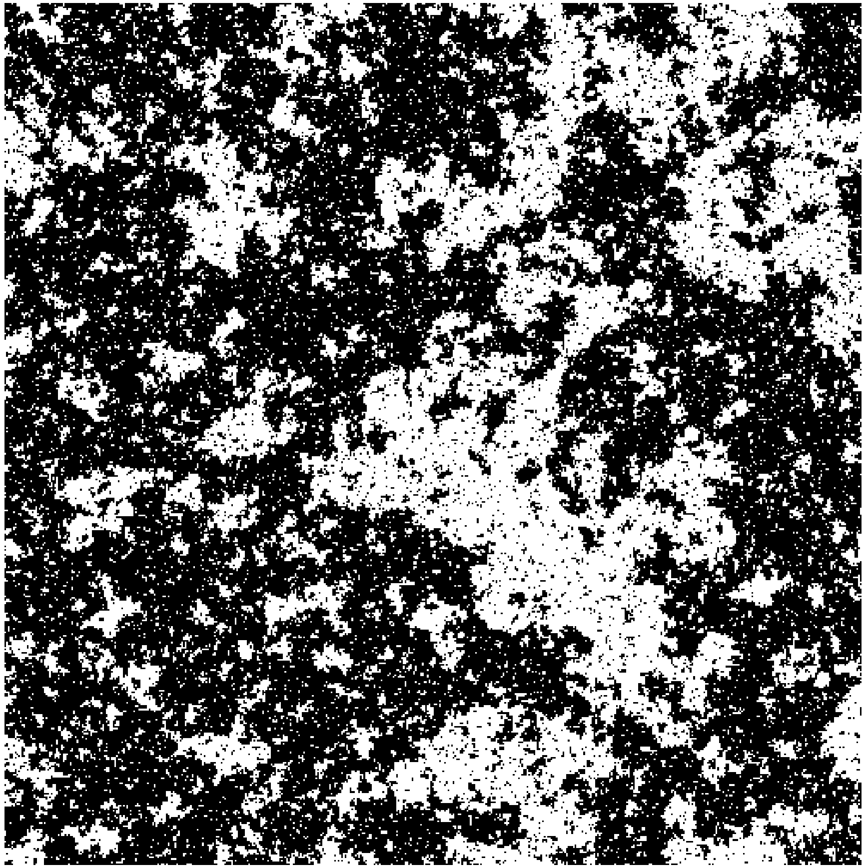
Louis H. Kauffman
Math, UIC
851 South Morgan St.
Chicago, IL 60607-7045
<kauffman@uic.edu>

www.math.uic.edu/~kauffman

http://front.math.ucdavis.edu/author/L.Kauffman



The previous slide shows a stage, near criticality, of a simulation of the Ising model for a rectangular two dimensional lattice. The space is divided into many domains of constant spin (the two colors that are indicated here).



In this lecture we will eventually discuss the Potts model (a generalization of the Ising model). The Potts model raises a question about techniques that have evolved in the knot theory. In these techniques, state loop configurations that differ by one smoothing, figure in the measurement of a homology theory -- Khovanov Homology -- that is associated with a knot diagram. Our question is -- How is Khovanov Homology related to the physics of statistical mechanics?

One clue is that these loops are the boundaries of regions of constant spin.

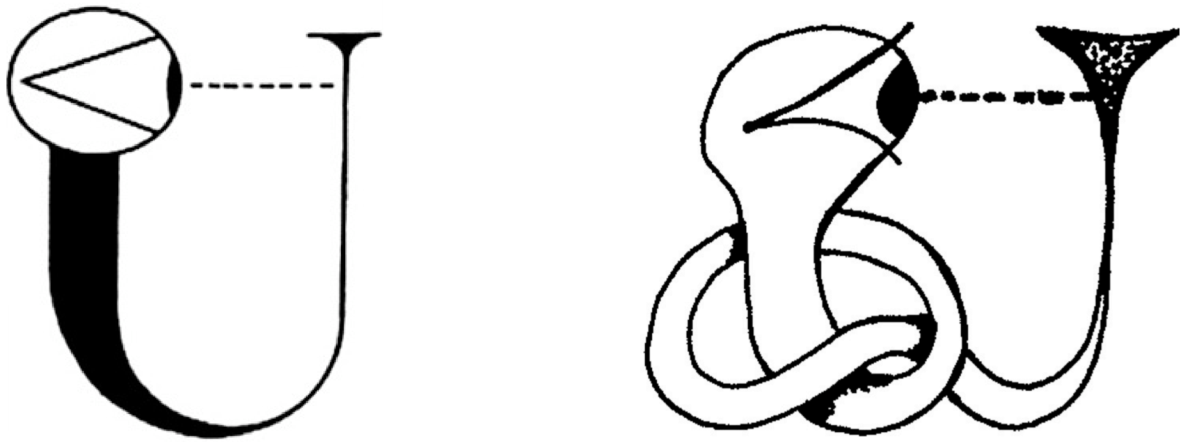
We will begin by recalling the quantum mechanical framework and how one can place the Jones polynomial into this framework. This will provide a natural transition to Khovanov homology, and let us get to the questions about statistical mechanics models.

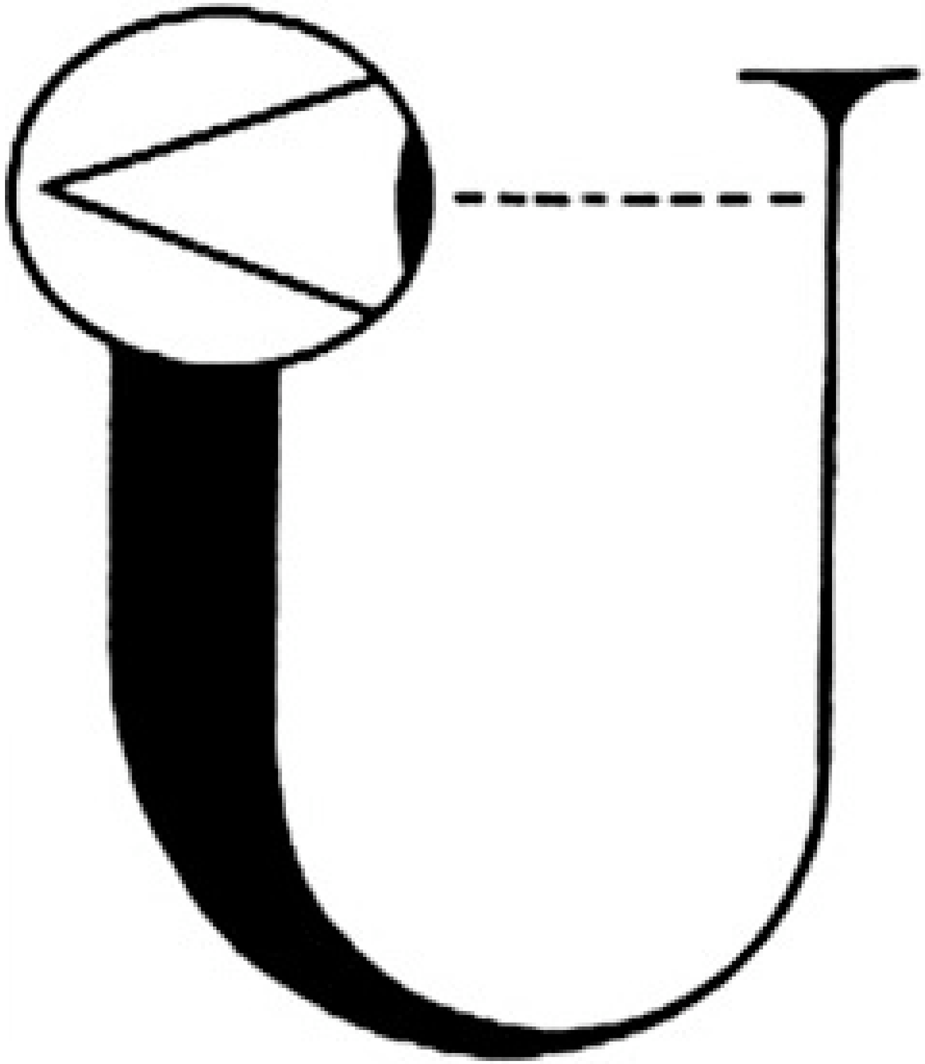
So the agenda is

Quantum Information

and

Knots in Physics.





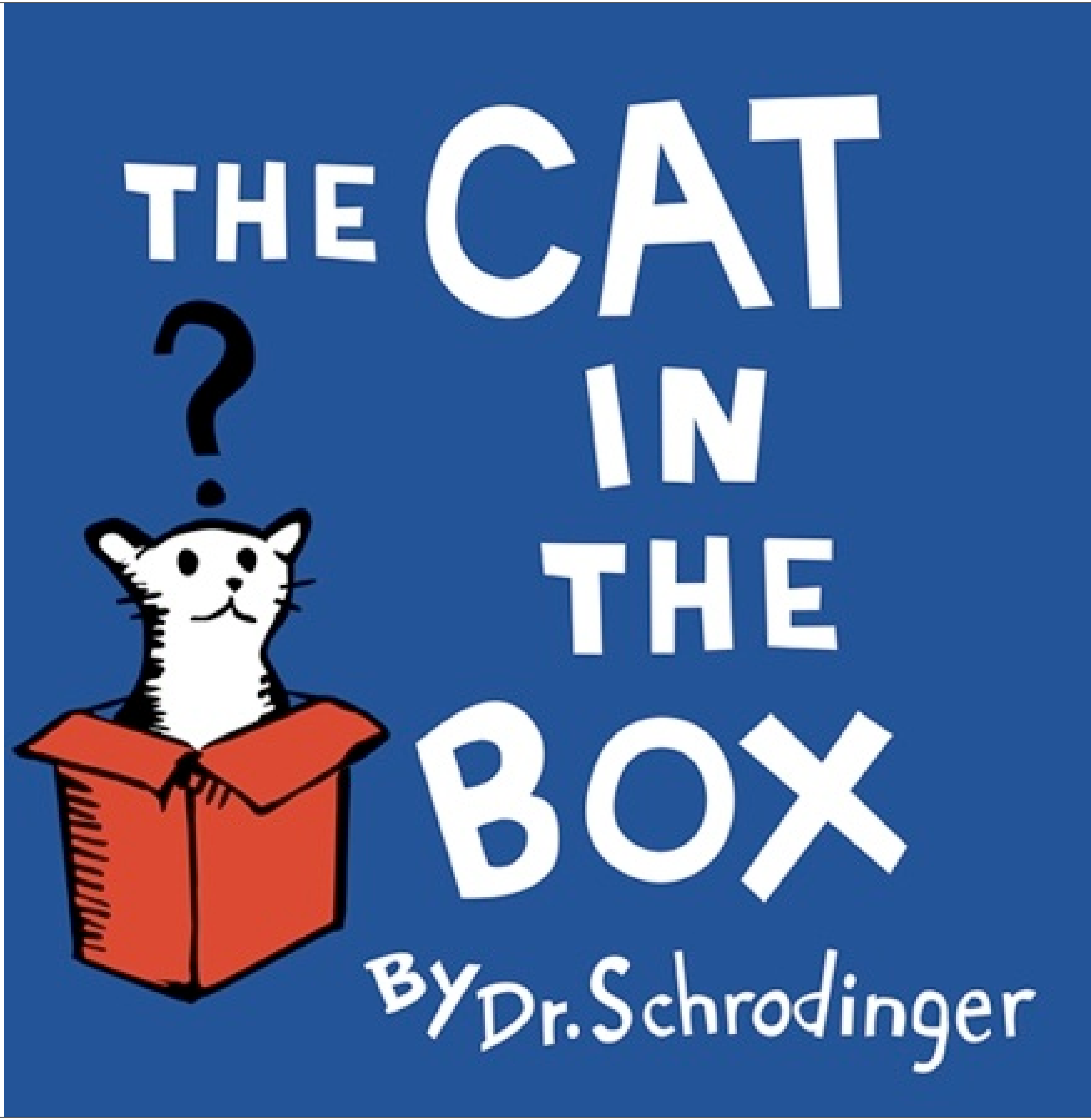
After all, it is Halloween.

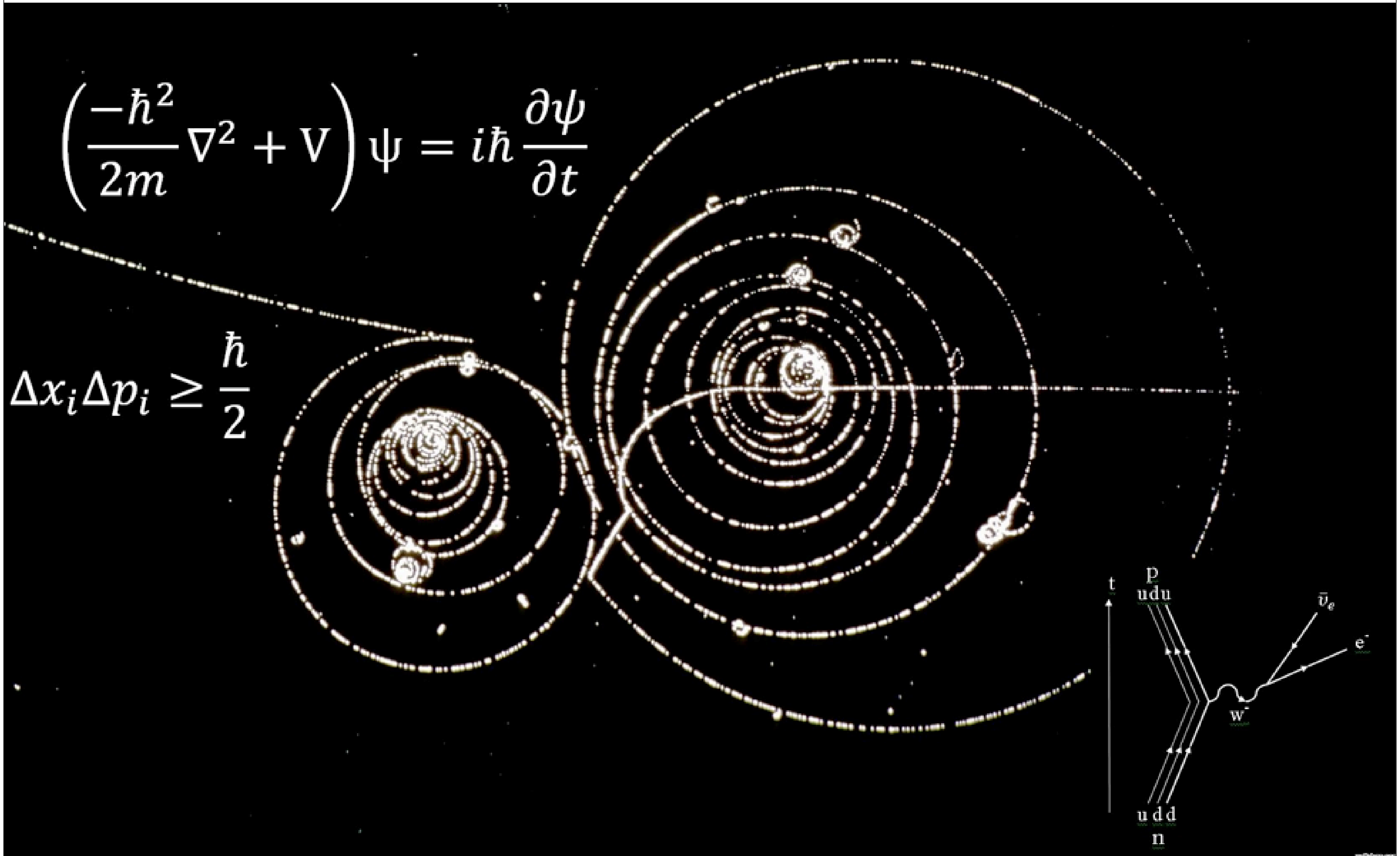

SCHRÖDINGER'S CAT IS


WANTED
DEAD & ALIVE
SCHRÖDINGER'S CAT


$\hat{H}|\psi\rangle = i\hbar\frac{d}{dt}|\psi\rangle$


QUANTUM
COMPUTING
FOR DOGS

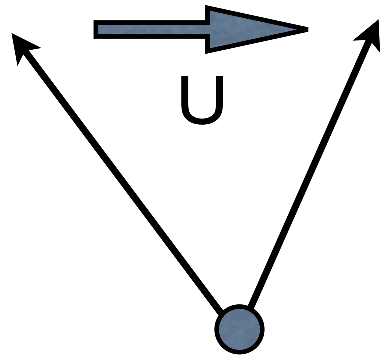

THE CAT IN THE BOX
By Dr. Schrodinger


$\left(\frac{-\hbar^2}{2m}\nabla^2 + V\right)\psi = i\hbar\frac{\partial\psi}{\partial t}$
$\Delta x_i \Delta p_i \geq \frac{\hbar}{2}$
t
p
udu
$\bar{v}_e$
e-
W-
udd
n

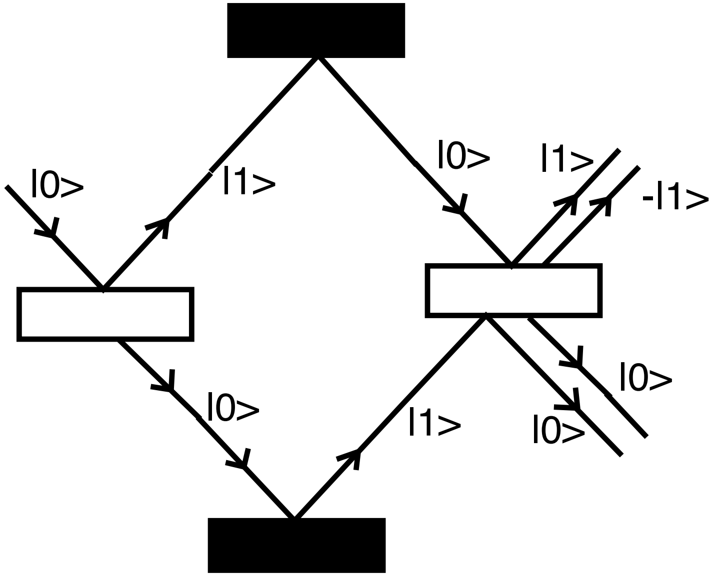
Quantum Mechanics in a Nutshell

0. A state of a physical system corresponds to a unit vector $|S\rangle$ in a complex vector space.

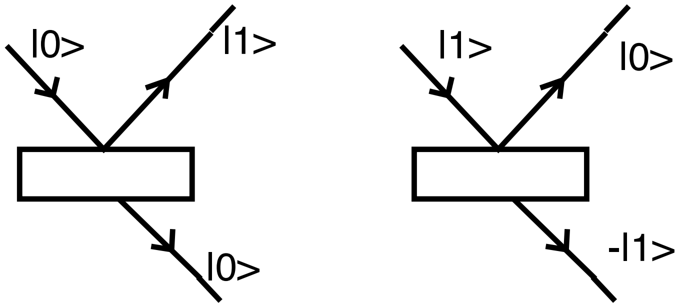
1. (measurement free) Physical processes are modeled by unitary transformations applied to the state vector: $|S\rangle \text{ -----> } U|S\rangle$



2. If $|S\rangle = z_1|1\rangle + z_2|2\rangle + \dots + z_n|n\rangle$
in a measurement basis $\{|1\rangle, |2\rangle, \dots, |n\rangle\}$, then measurement of $|S\rangle$ yields $|i\rangle$ with probability $|z_i|$^2.



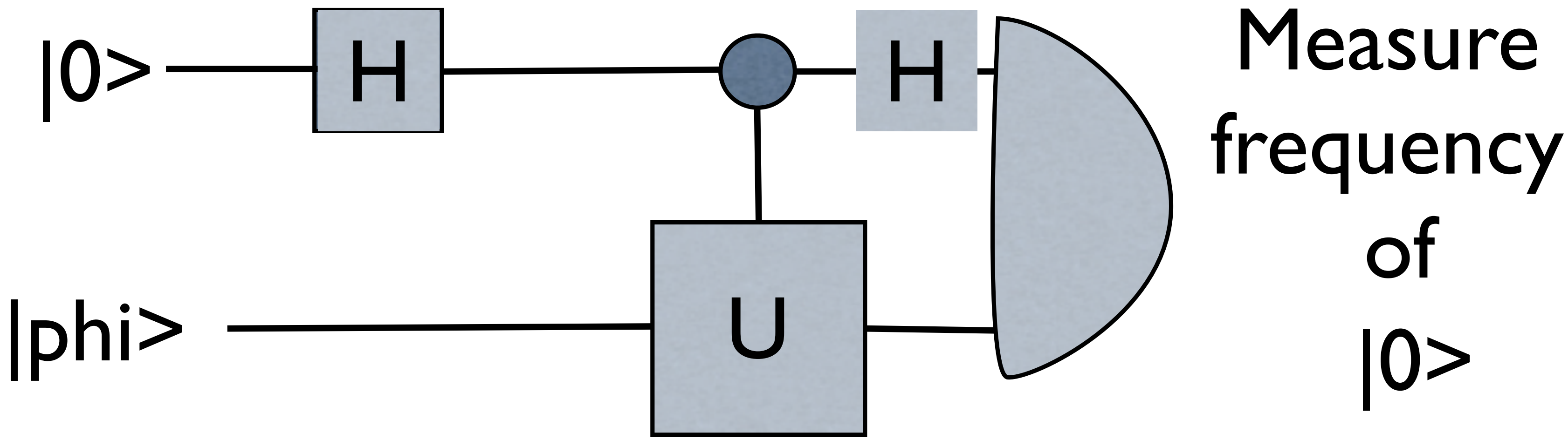
Mach-Zender Interferometer



$$H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} /\mathrm{Sqrt}(2) \qquad M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$HMH = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Hadamard Test - For Trace(U).



$$H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} / \mathrm{Sqrt}(2)$$

|0> occurs with probability
1/2 + Re[<phi|U|phi>]/2.

Apply Hadamard Gate

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

to first qubit of

$$C_U \circ (H \otimes 1)|0\rangle|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |\psi\rangle + |1\rangle \otimes U|\psi\rangle$$

The resulting state is

$$\frac{1}{2}(H|0\rangle \otimes |\psi\rangle + H|1\rangle \otimes U|\psi\rangle) = \frac{1}{2}((|0\rangle + |1\rangle) \otimes |\psi\rangle + (|0\rangle - |1\rangle) \otimes U|\psi\rangle)$$

$$= \frac{1}{2}(|0\rangle \otimes (|\psi\rangle + U|\psi\rangle) + |1\rangle \otimes (|\psi\rangle - U|\psi\rangle)).$$

The expectation for |0> is

$$\frac{1}{2}(\langle\psi| + \langle\psi|U^\dagger)(|\psi\rangle + U|\psi\rangle) = \frac{1}{2} + \frac{1}{2}Re\langle\psi|U|\psi\rangle$$

The imaginary part is obtained by applying the same procedure to

$$\frac{1}{\sqrt{2}}(|0\rangle \otimes |\psi\rangle - i|1\rangle \otimes U|\psi\rangle$$

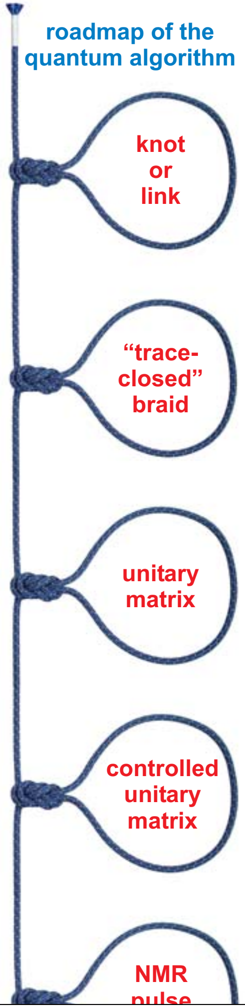
Untying Knots by NMR: first experimental implementation of a quantum algorithm for approximating the Jones polynomial

TECHNISCHE UNIVERSITÄT MÜNCHEN

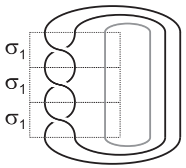
Raimund Marx[1], Andreas Spörl[1], Amr F. Fahmy[2], John M. Myers[3], Louis H. Kauffman[4], Samuel J. Lomonaco, Jr.[5], Thomas-Schulte-Herbrüggen[1], and Steffen J. Glaser[1]

[1]Department of Chemistry, Technical University Munich, Lichtenbergstr. 4, 85747 Garching, Germany
[2]Harvard Medical School, 25 Shattuck Street, Boston, MA 02115, U.S.A.
[3]Gordon McKay Laboratory, Harvard University, 29 Oxford Street, Cambridge, MA 02138, U.S.A.
[4]University of Illinois at Chicago, 851 S. Morgan Street, Chicago, IL 60607-7045, U.S.A.
[5]University of Maryland Baltimore County, 1000 Hilltop Circle, Baltimore, MD 21250, U.S.A.

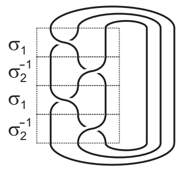
roadmap of the quantum algorithm



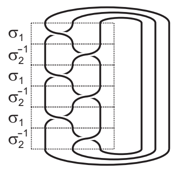
example #1 Trefoil



example #2 Figure-Eight



example #3 Borromean rings



$$U_{Trefoil} = (U_1)^3 \qquad U_{Figure-Eight} = (U_2^{-1}\cdot U_1)^2 \qquad U_{Borrom.R.} = (U_2^{-1}\cdot U_1)^3$$

$$U_1 = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & -e^{i\theta}\dfrac{\sin(4\theta)}{\sin(2\theta)} + e^{-i\theta} \end{pmatrix}$$

$$U_2 = \begin{pmatrix} -e^{i\theta}\dfrac{\sin(6\theta)}{\sin(4\theta)} + e^{-i\theta} & -e^{i\theta}\dfrac{\sqrt{\sin(6\theta)\sin(2\theta)}}{\sin(4\theta)} \\ -e^{i\theta}\dfrac{\sqrt{\sin(6\theta)\sin(2\theta)}}{\sin(4\theta)} & -e^{i\theta}\dfrac{\sin(2\theta)}{\sin(4\theta)} + e^{-i\theta} \end{pmatrix}$$

Step #1:
from the 2x2 matrix U
to the 4x4 matrix cU:

$$cU = \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}$$

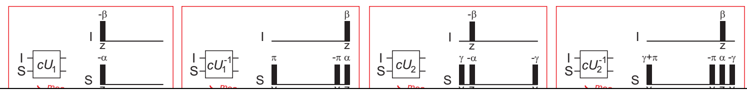
Step #2:
application of cU on the
NMR product operator I_{1x} :

$$cU\, I_{1x}\, cU^{\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}^{\dagger} = \frac{1}{2}\begin{pmatrix} 0 & U^{\dagger} \\ U & 0 \end{pmatrix}$$

Step #3:
measurement
of I_{1x} and I_{1y} :

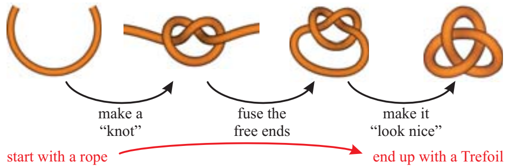
$$tr\left\{ I_{1x}\, \frac{1}{2}\begin{pmatrix} 0 & U^{\dagger} \\ U & 0 \end{pmatrix} \right\} = \frac{1}{2}\Re(tr\{U\})$$

$$tr\left\{ I_{1y}\, \frac{1}{2}\begin{pmatrix} 0 & U^{\dagger} \\ U & 0 \end{pmatrix} \right\} = \frac{1}{2}\Im(tr\{U\})$$



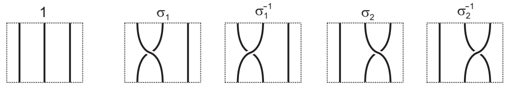
A knot is defined as a closed, non-self-intersecting curve that is embedded in three dimensions.

example: "construction" of the Trefoil knot:



J. W. Alexander proved, that any knot can be represented as a closed braid (polynomial time algorithm)

generators of the 3 strand braid group:



It is well known in knot theory, how to obtain the unitary matrix representation of all generators of a given braid goup (see "Temperley-Lieb algebra" and "path model representation"). The unitary matrices U_1 and U_2, corresponding to the generators σ_1 and σ_2 of the 3 strand braid group are shown on the left, where the variable "θ" is related to the variable "A" of the Jones polynomial by: $A = e^{-i\theta_{rad}}$. The unitary matrix representations of σ_1^{-1} and σ_2^{-1} are given by U_1^{-1} and U_2^{-1}.

The knot or link that was expressed as a product of braid group generators can therefore also be expressed as a product of the corresponding unitary matrices.

Instead of applying the unitary matrix U, we apply it's controlled variant cU. This matrix is especially suited for NMR quantum computers [4] and other thermal state expectation value quantum computers: you only have to apply cU to the NMR product operator I_{1x} and measure I_{1x} and I_{1y} in order to obtain the trace of the original matrix U.

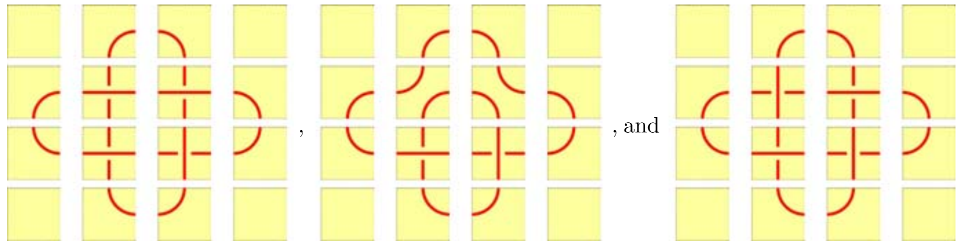
Independent of the dimension of matrix U you only need ONE extra qubit for the implementation of cU as compared to the implementation of U itself.

The measurement of I_{1x} and I_{1y} can be accomplished in one single-scan experiment.

All knots and links can be expressed as a product of braid group generators (see above). Hence the corresponding NMR pulse sequence can also be expressed as a sequence of NMR pulse sequence blocks, where each block corresponds to the controlled unitary matrix cU of one braid group generator.

Quantum knots and mosaics

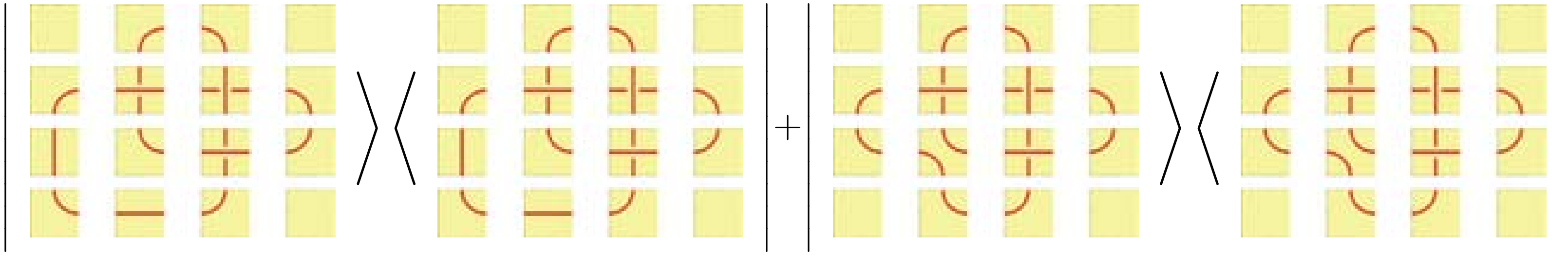
with
Sam
Lomonaco



Each of these knot mosaics is a string made up of the following 11 symbols

called *mosaic tiles*.

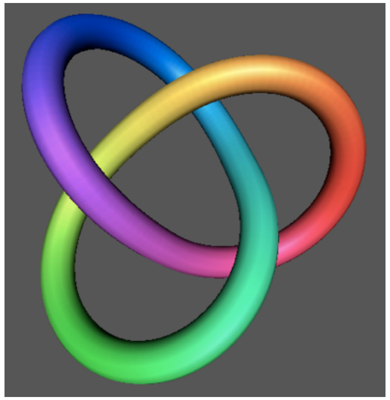
Each mosaic is a tensor product of elementary tiles.

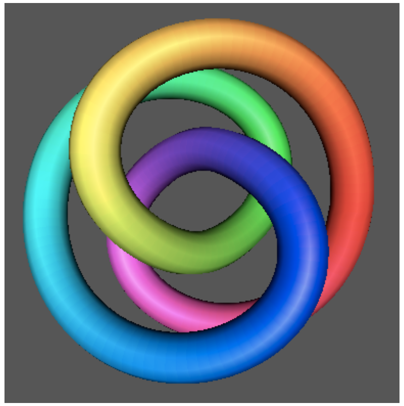
$\Omega =$ 

This observable is a quantum knot invariant for 4x4 tile space. Knots have characteristic invariants in nxn tile space.

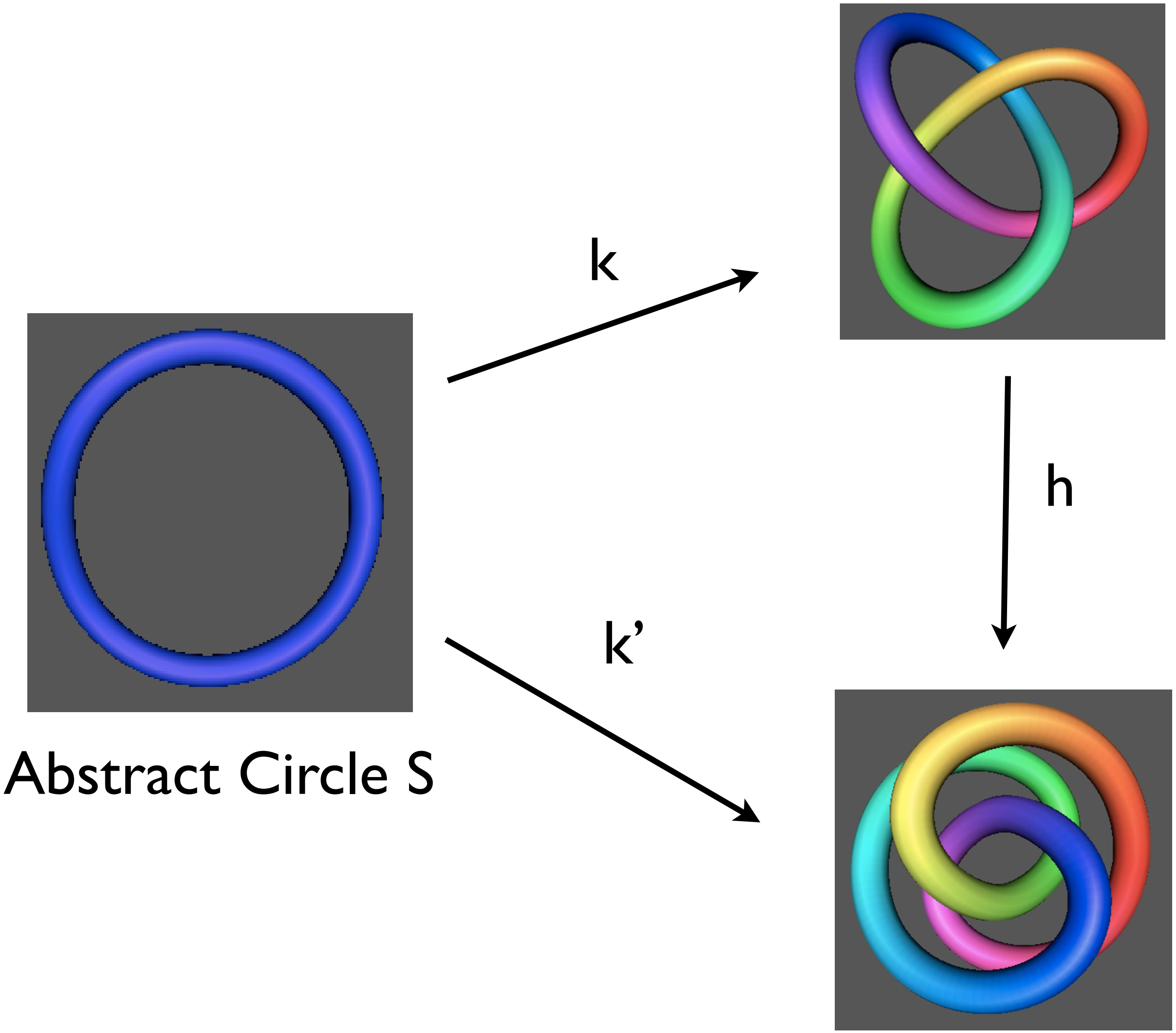
Superpositions of combinatorial knot configurations are seen as quantum states.

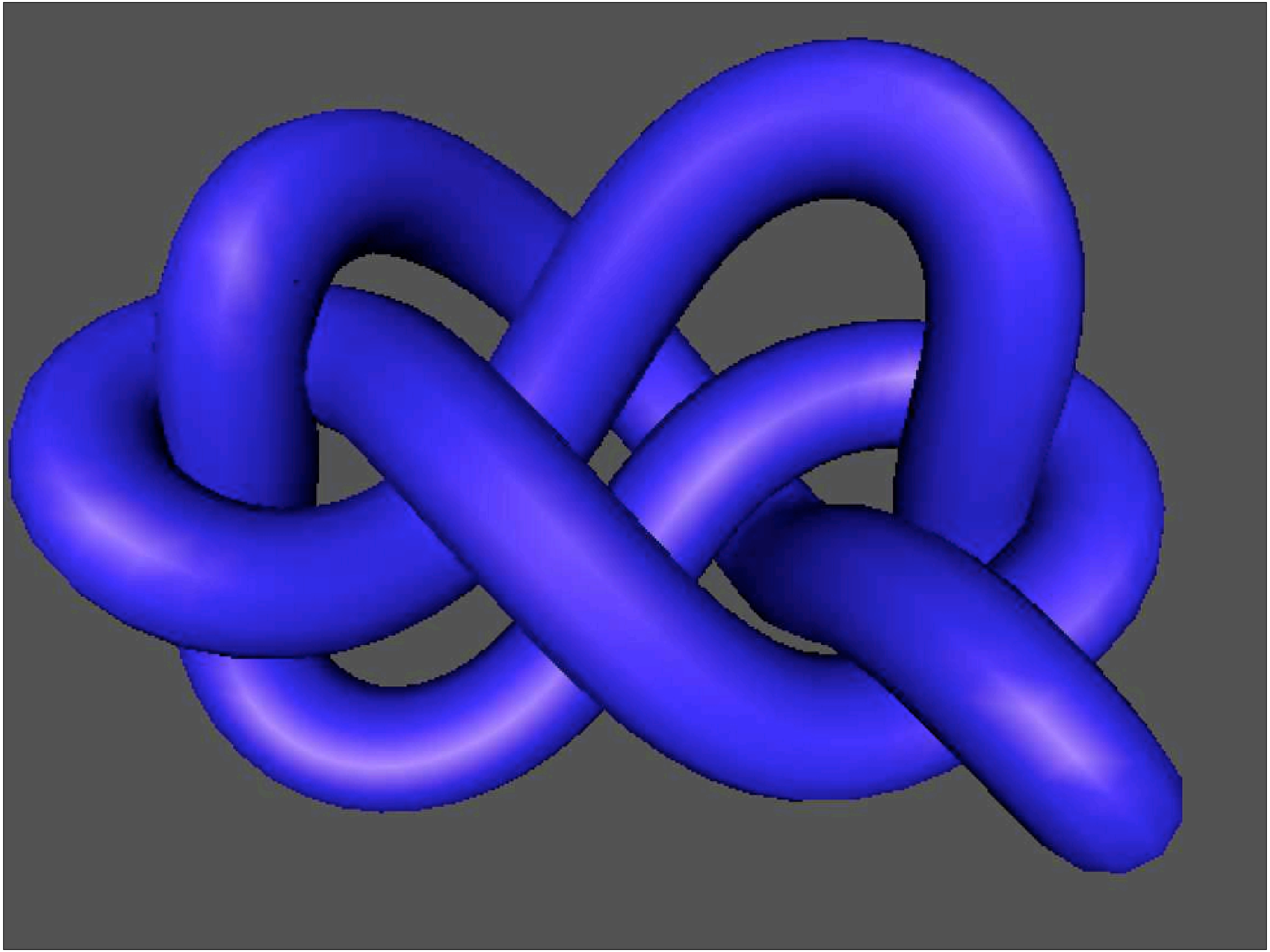
A knot is an embedding of a simple closed curve in three dimensional space.

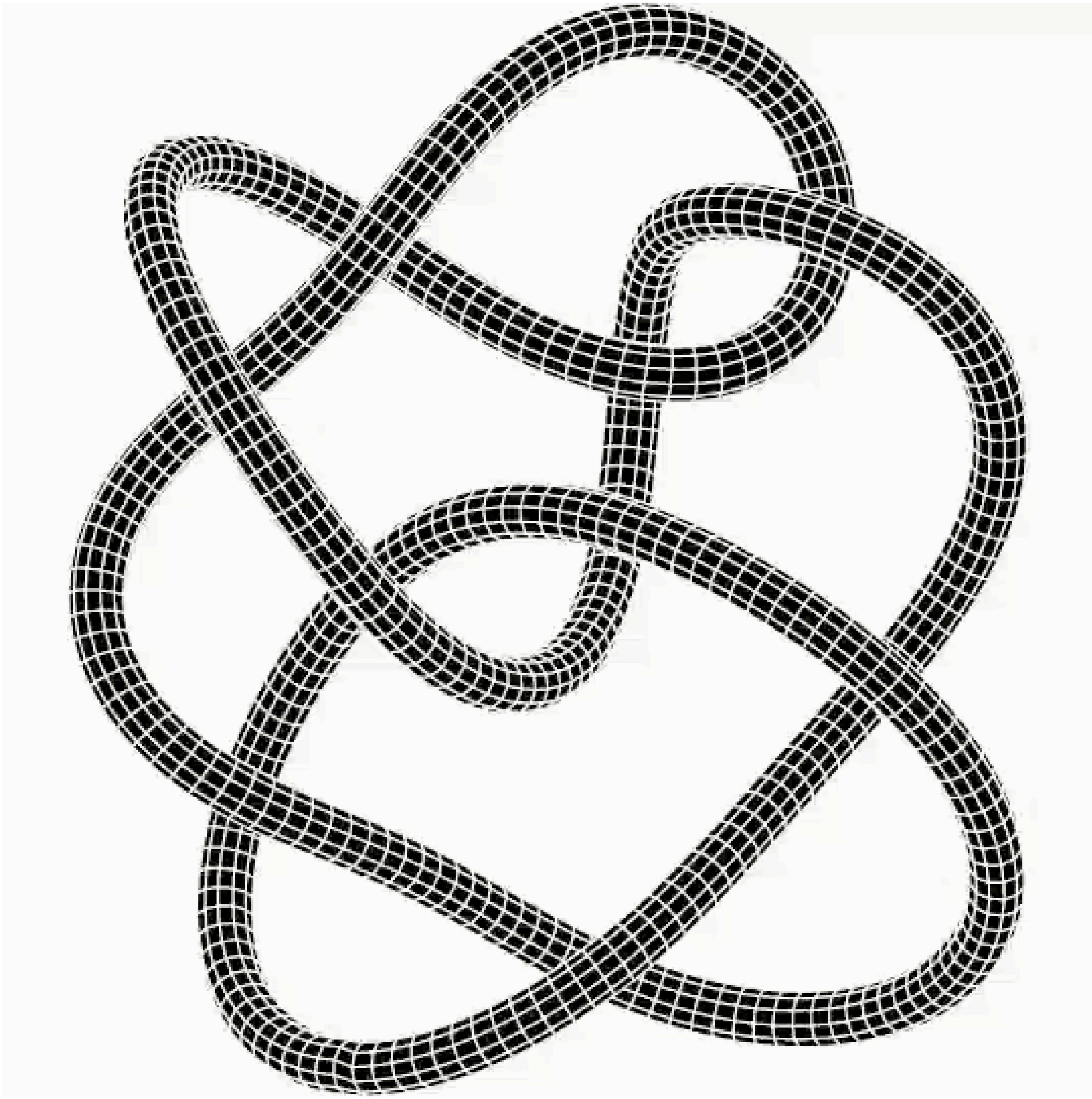


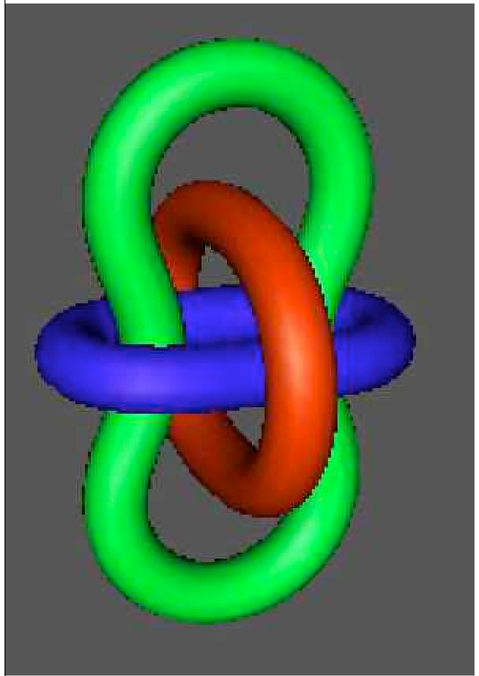


Two knots K, L are equivalent if there is a homeomorphism h:R3 ----> R3 so that h(K) = L.


k
h
k'
Abstract Circle S







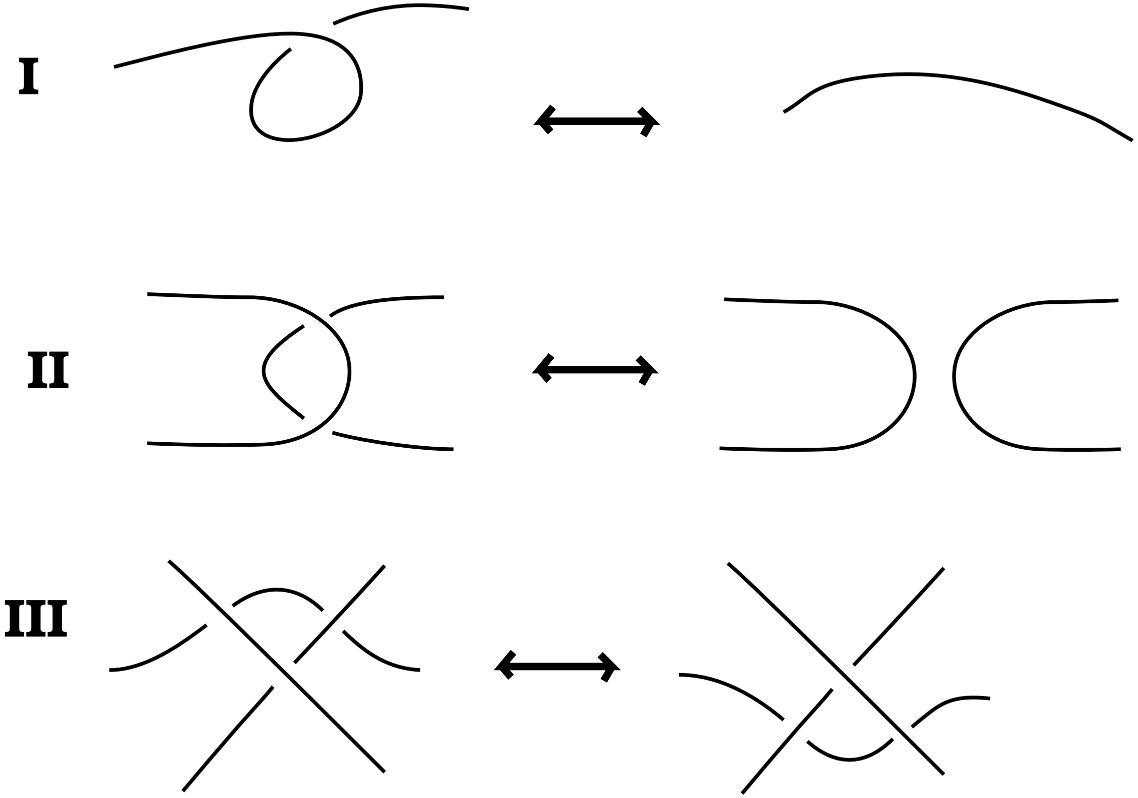
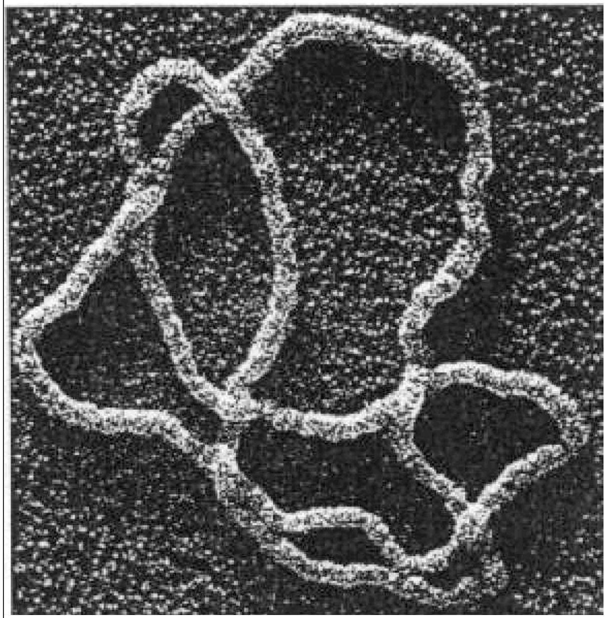
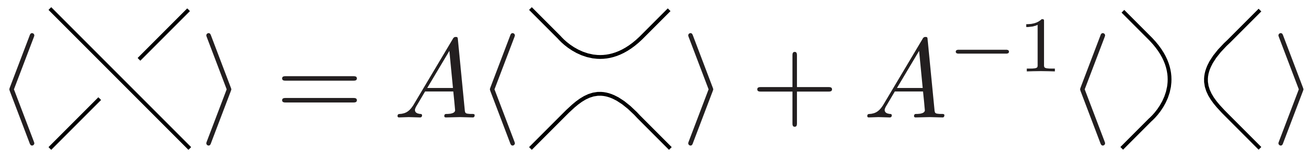


Figure 2 - The Reidemeister Moves.

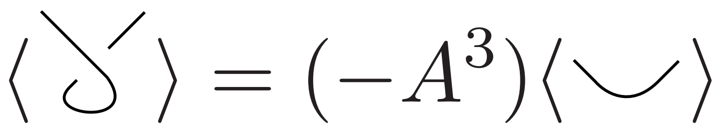


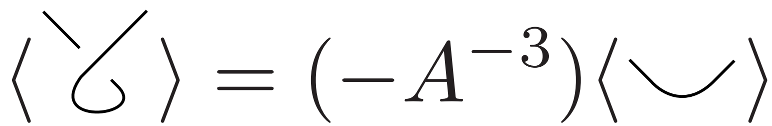
Reidemeister Moves reformulate knot theory in terms of graph combinatorics.

Bracket Polynomial Model for the Jones Polynomial



$$\langle K \bigcirc \rangle = (-A^2 - A^{-2})\langle K \rangle$$





Bracket Polynomial of the Trefoil Knot

$\Big\langle$  $\Big\rangle = A \Big\langle$  $\Big\rangle + A^{-1} \Big\langle$  $\Big\rangle$

$$= A(-A^{4} - A^{-4}) + A^{-1}(-A^{-3})^{2}$$

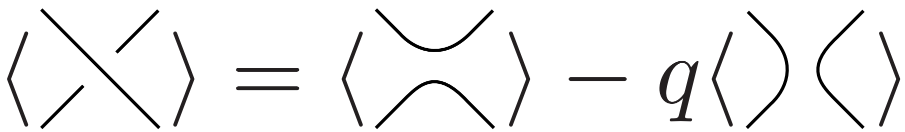
$$\langle T \rangle = -A^{5} - A^{-3} + A^{-7}$$

Reformulating the Bracket

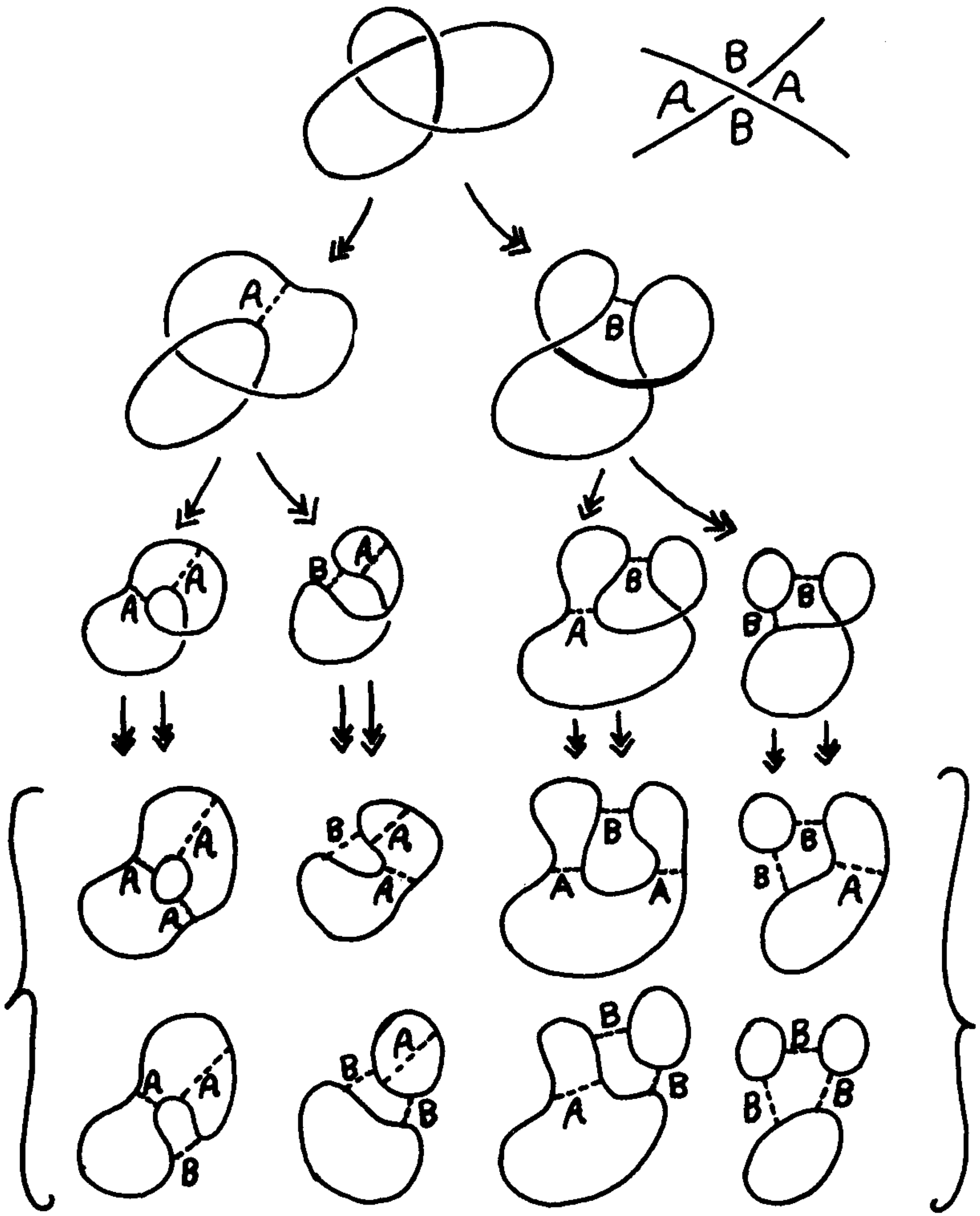
Let c(K) = number of crossings on link K.

Form $A^{-c(K)}$<K> and replace A^{-2} by -q .

Then the skein relation for <K> will be replaced by:

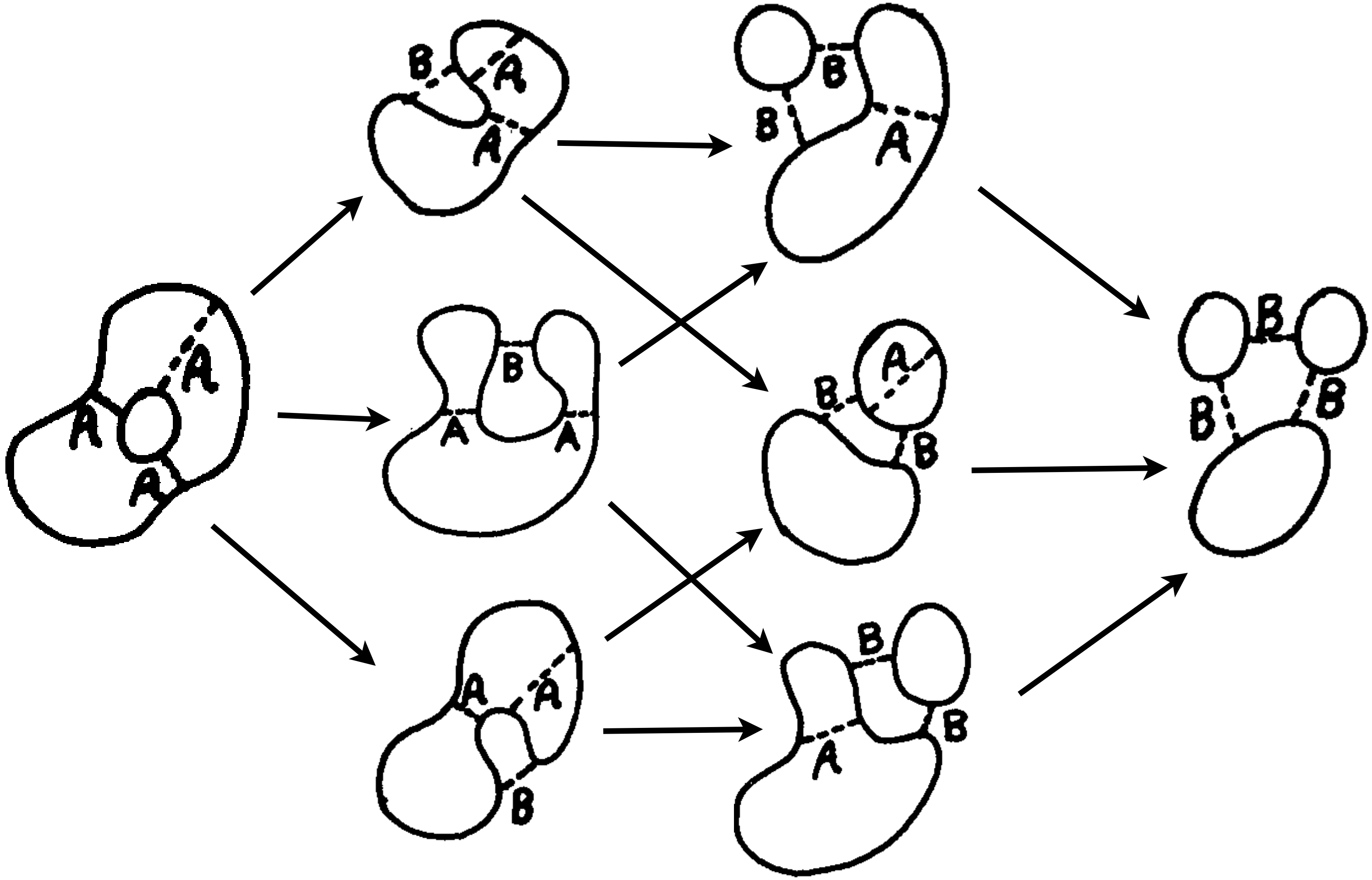


$$\langle \bigcirc \rangle = (q + q^{-1})$$

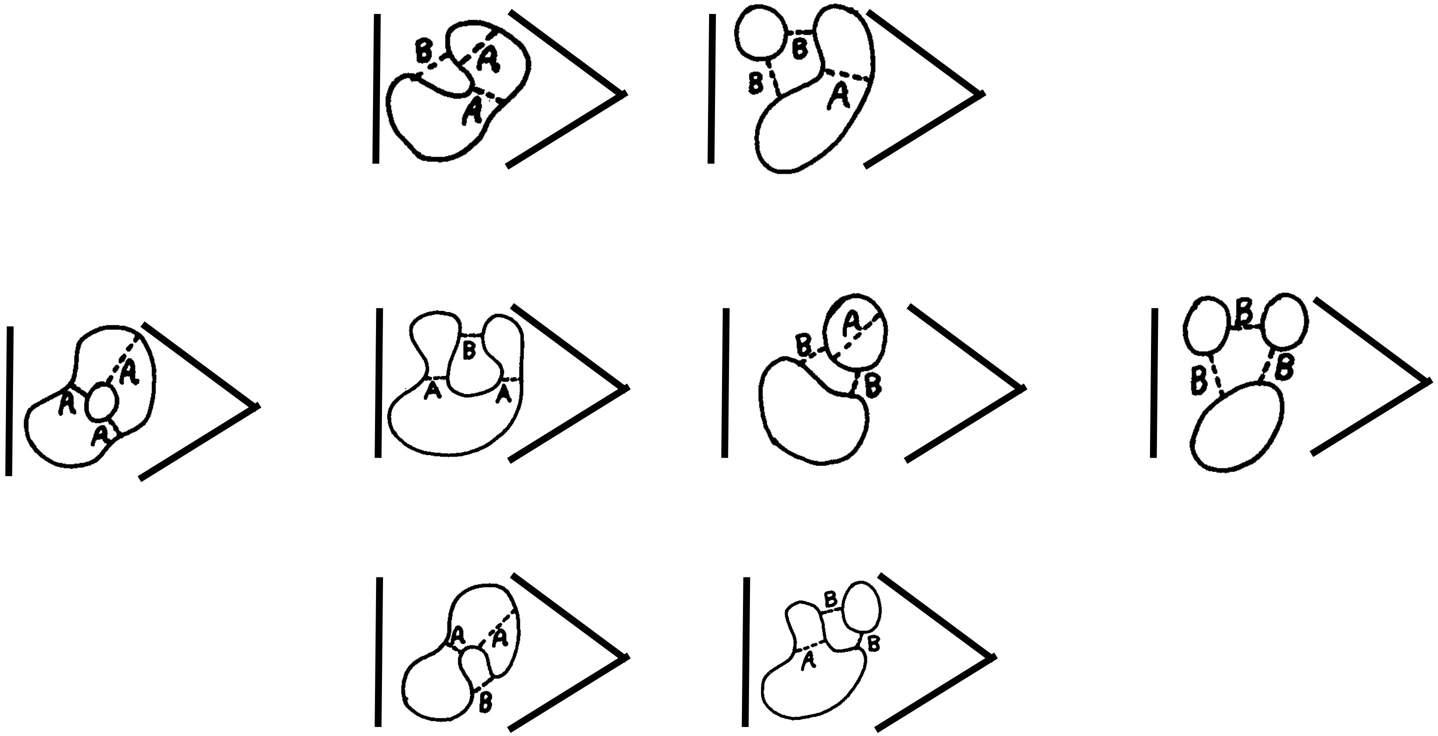


The form of the
expansion is the
same as
a loop expansion
of the Potts
model,
where the loops
are
boundaries of
regions of
constant spin.

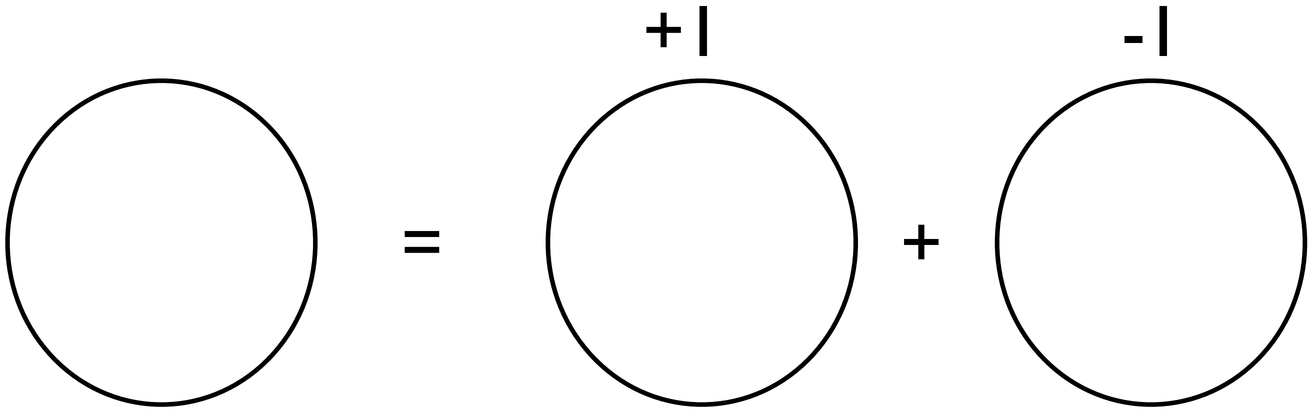
The Khovanov Cubical Organization of Bracket States



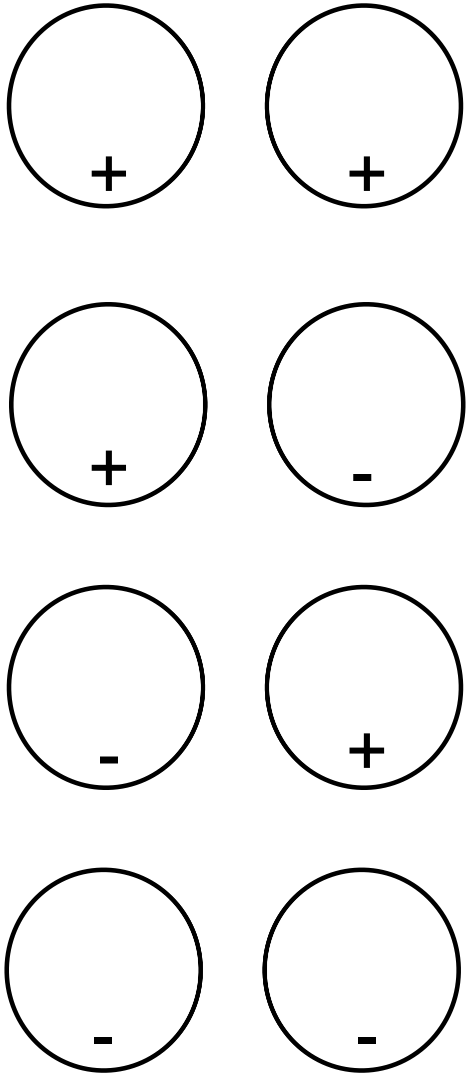
We make a Hilbert space whose basis is a set of (enhanced) states of the bracket polynomial for a given knot diagram K.



Enhanced states label each loop with
+1 or -1.



$$\longleftrightarrow \qquad q + q^{-1}$$

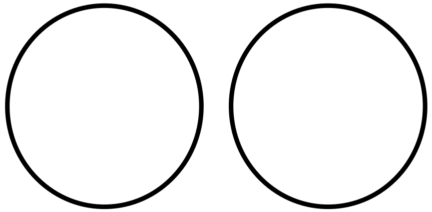


q^2

1

1

q^{-2}

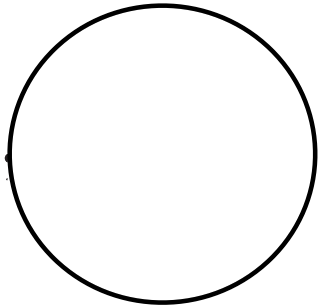


$(q + q^{-1})^2$

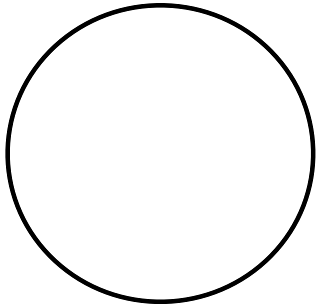
Enhanced States circumvent the binomial theorem.

Enhanced States

$$q^{-1} \Longleftrightarrow -1 \Longleftrightarrow X$$

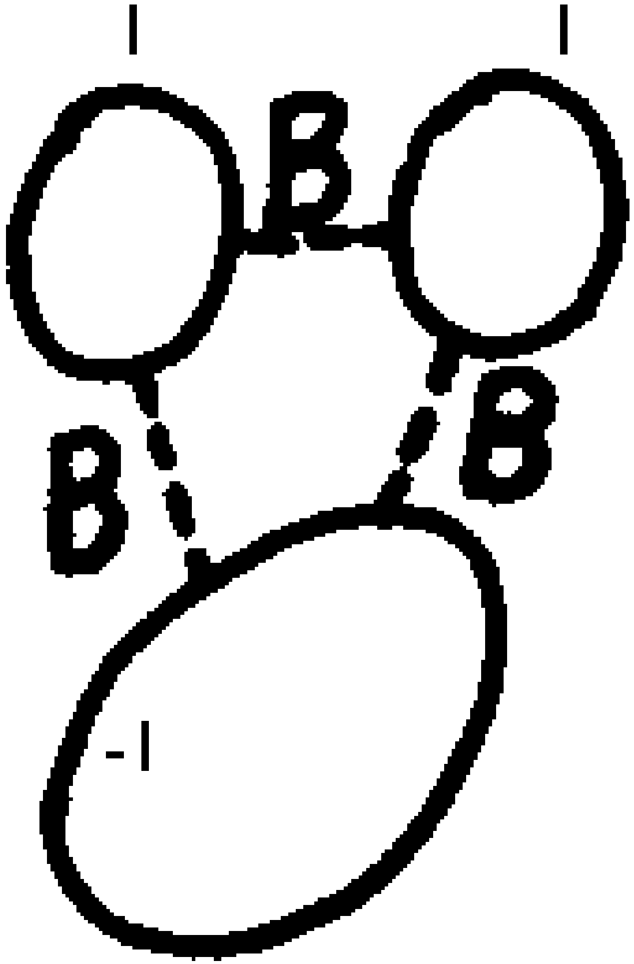


$$q^{+1} \Longleftrightarrow +1 \Longleftrightarrow 1$$



For reasons that will soon become apparent, we let -1 be denoted by X and +1 be denoted by 1.

(The module V will be generated by 1 and X.)



An enhanced state
that contributes

[(q)(q)(1/q)] [(-q) (-q) (-q)]

1 1 -1 **B B B**

to the revised
bracket state sum.

Enhanced State Sum Formula for the Bracket

$$\langle K \rangle = \sum_{s} q^{j(s)} (-1)^{i(s)}$$

A Quantum Statistical Model for the Bracket Polynonmial.

Let C(K) denote a Hilbert space
with basis |s> where s runs over the
enhanced states of a knot or link diagram K.

We define a unitary transformation.

$$U : \mathcal{C}(K) \longrightarrow \mathcal{C}(K)$$

$$U|s\rangle = (-1)^{i(s)} q^{j(s)} |s\rangle$$

q is chosen on the unit circle in the
complex plane.

<K> = Trace(U).

$$|\psi\rangle = \sum_{s} |s\rangle$$

Lemma. The evaluation of the bracket polynomial is given by the following formula

$$\langle K \rangle = \langle \psi | U | \psi \rangle.$$

This gives a new quantum algorithm for the Jones polynomial (via Hadamard Test).

Khovanov Homology - Jones Polynomial as an Euler Characteristic

Two key motivating ideas are involved in finding the Khovanov invariant. First of all, one would like to *categorify* a link polynomial such as $\langle K \rangle$. There are many meanings to the term categorify, but here the quest is to find a way to express the link polynomial as a *graded Euler characteristic* $\langle K \rangle = \chi_q \langle H(K) \rangle$ for some homology theory associated with $\langle K \rangle$.

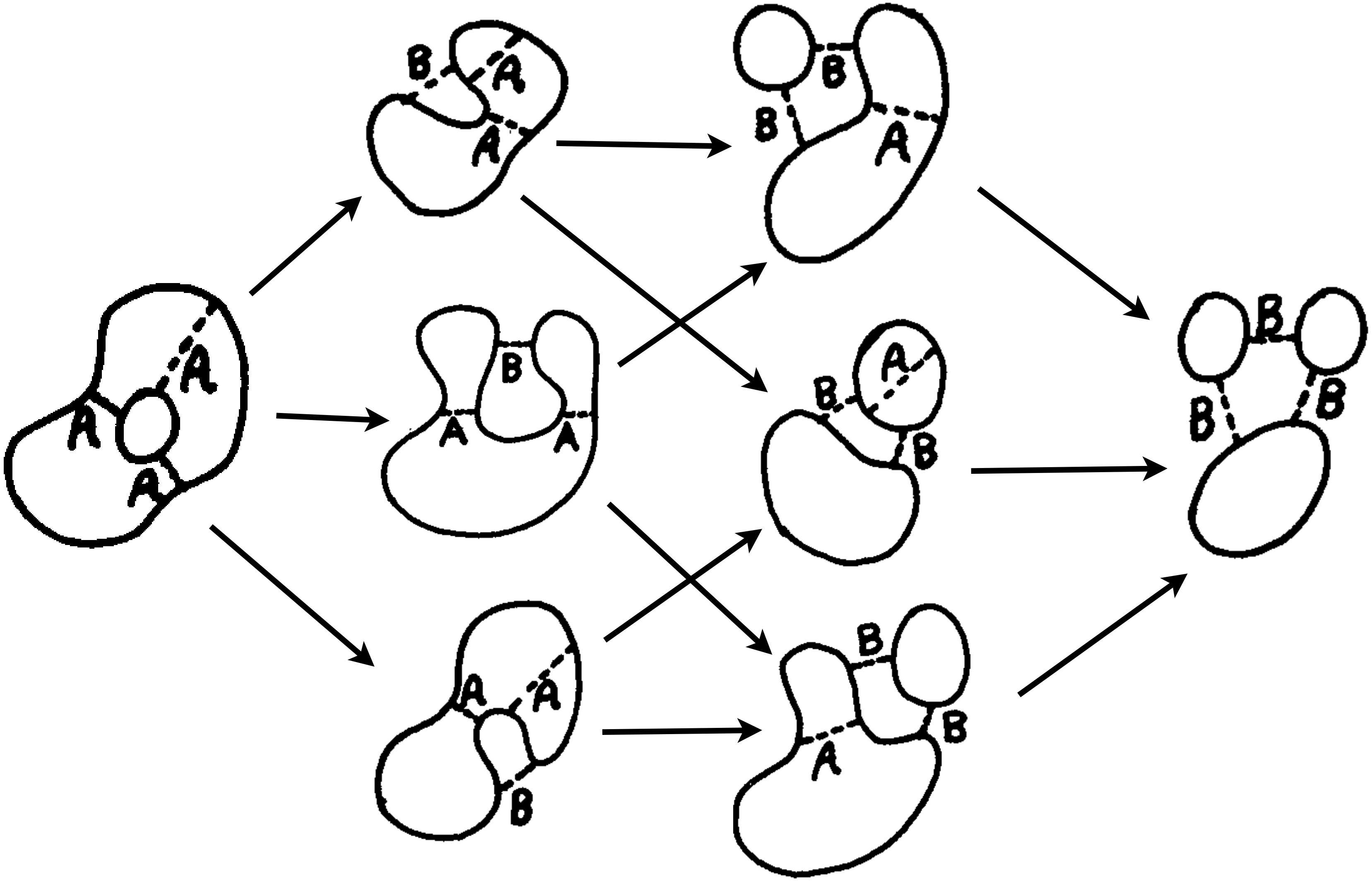
We will formulate Khovanov
Homology
in the context of our quantum
statistical model for the bracket
polynomial.

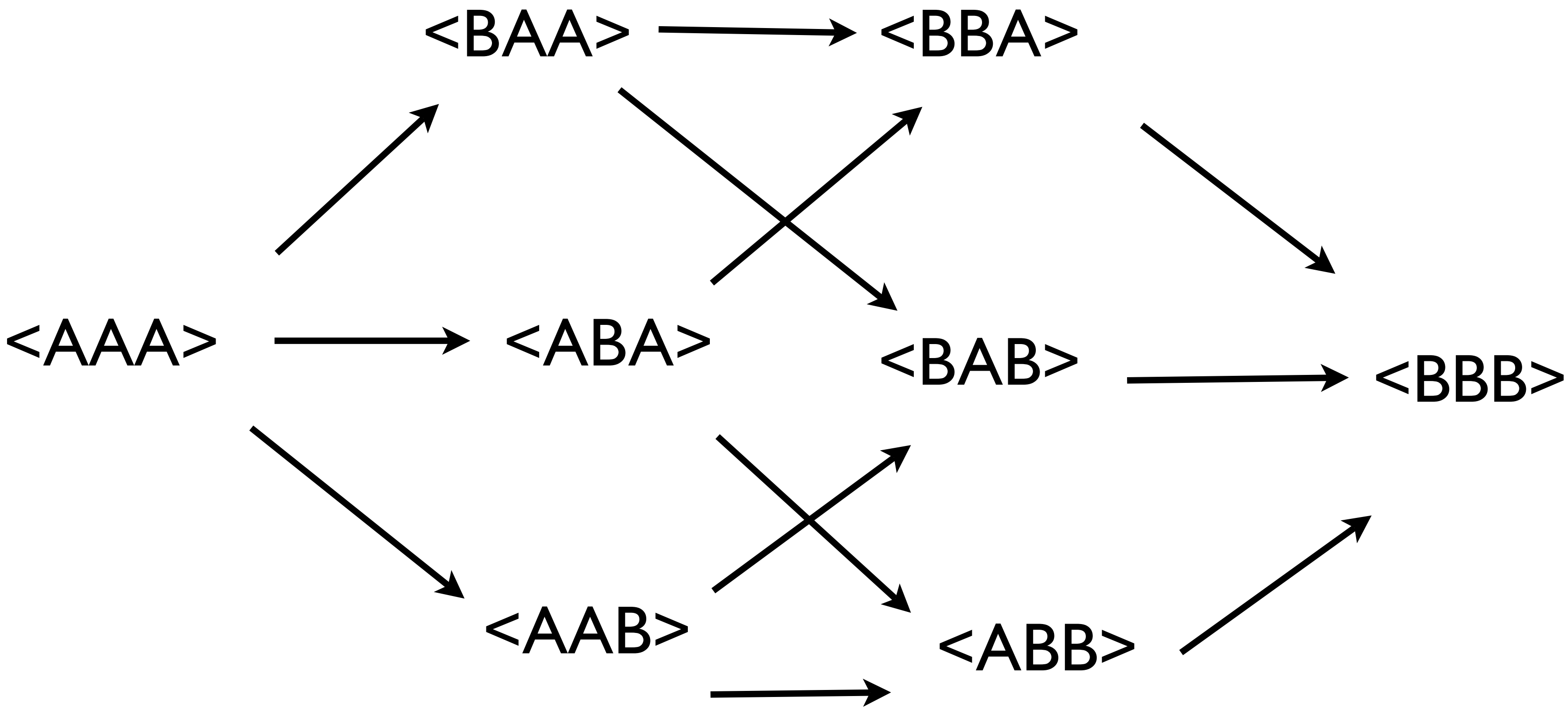
CATEGORIFICATION

View the next slide as a category.

The cubical shape of this category suggests making a homology theory.

The Khovanov Category




<BAA>
<BBA>
<AAA>
<ABA>
<BAB>
<BBB>
<AAB>
<ABB>

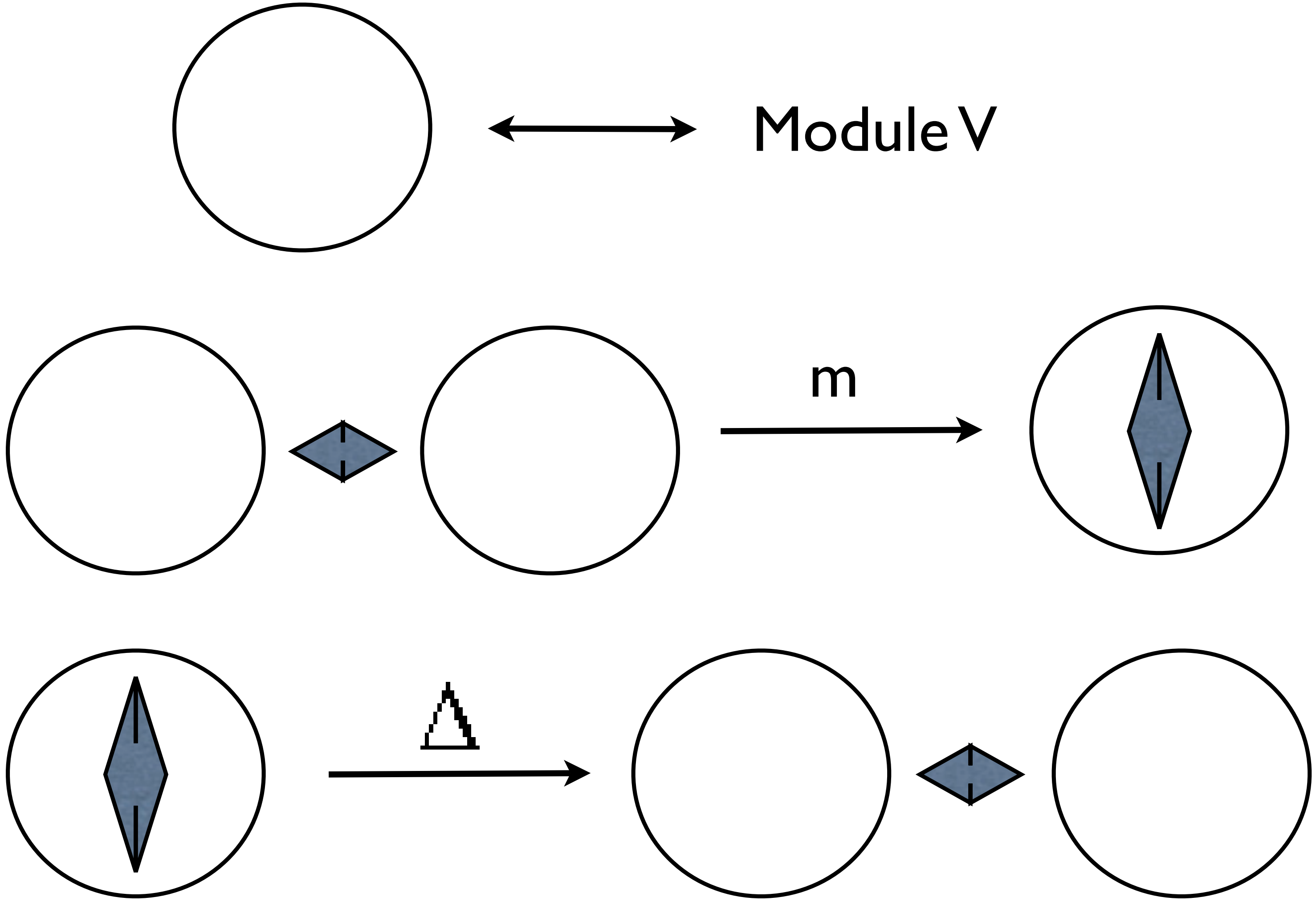
Functor : Cubical Category -------> Module Category.

In order to make a non-trivial homology theory
we need a functor from this cubical category of states
to a module category.

Each state loop will map to a module V.

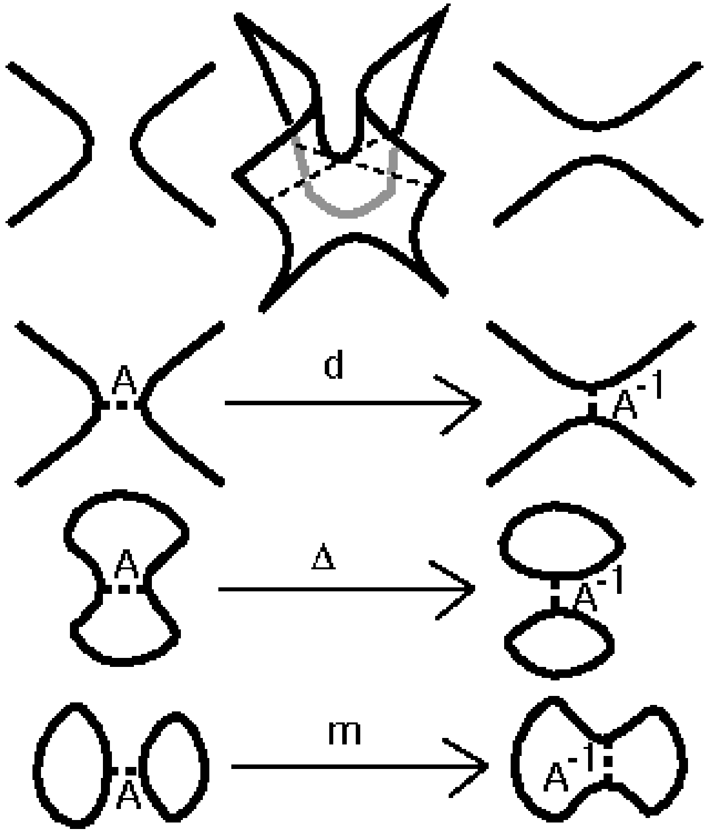
Unions of loops will map to
tenor products of this module.

The Functor from the cubical category to the module category demands multiplication and comultiplication in the module.



$$\partial(s) = \sum_{\tau} \partial_{\tau}(s)$$

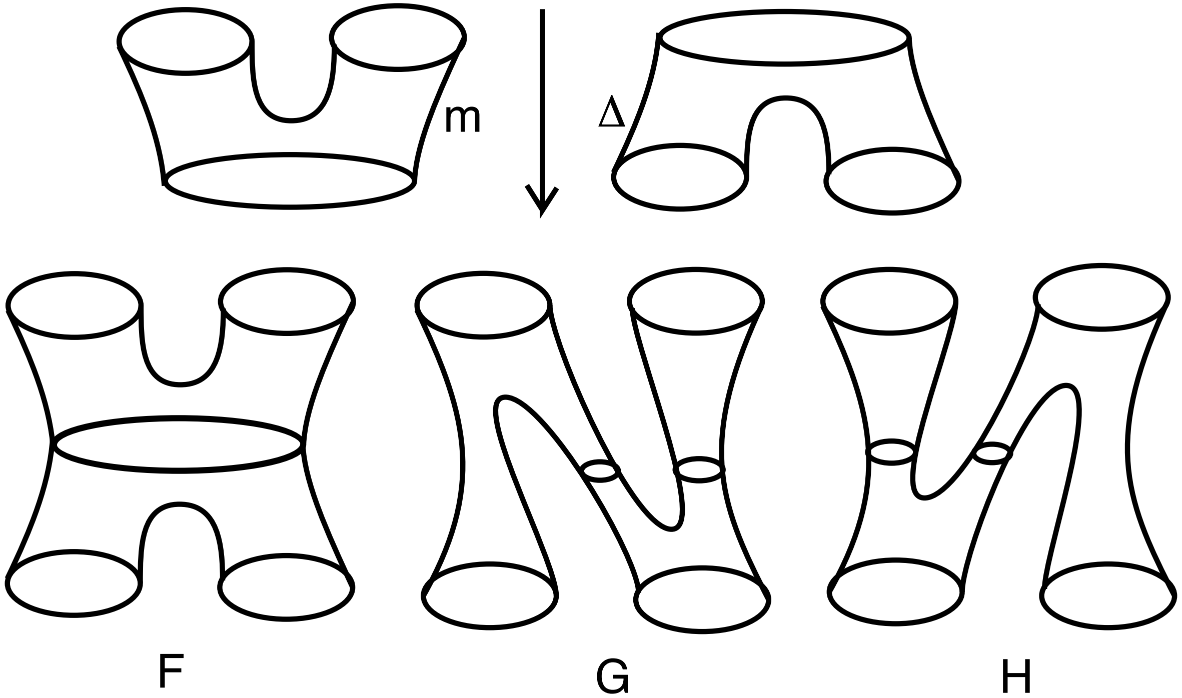
The boundary is a sum of partial differentials corresponding to resmoothings on the states.



Each state loop is a module.

A collection of state loops corresponds to a tensor product of these modules.

The commutation of the partial boundaries demands a structure of Frobenius algebra for the algebra associated to a state circle.



It turns out that one can take the algebra generated by 1 and X with
$X^2 = 0$ and

$$\Delta(X) = X \otimes X \text{ and } \Delta(1) = 1 \otimes X + X \otimes 1.$$

The chain complex is then generated by enhanced states with loop labels 1 and X.

Enhanced State Sum Formula for the Bracket

$$\langle K \rangle = \sum_{s} q^{j(s)}(-1)^{i(s)}$$

$$j(s) = n_B(s) + \lambda(s)$$

$i(s) = n_B(s)$ = number of B-smoothings in the state s.

$\lambda(s)$ = number of +1 loops minus number of -1 loops.

$$\langle K \rangle = \sum_{i\,,j} (-1)^i q^j dim(\mathcal{C}^{ij})$$

C^{ij} = module generated by enhanced states with $i = n_B$ and j as above.

$$\langle K \rangle = \sum_{i\,,j} (-1)^i q^j dim(\mathcal{C}^{ij})$$

The Khovanov differential acts in the form

$$\partial : \mathcal{C}^{ij} \longrightarrow \mathcal{C}^{i+1\,j}$$

(For j to be constant as i increases by 1,

$\lambda(s)$ decreases by 1.)

The differential increases the homological grading i by 1 and leaves fixed the quantum grading j.

Then

$$\langle K \rangle = \sum_j q^j \sum_i (-1)^i dim(\mathcal{C}^{ij}) = \sum_j q^j \chi(\mathcal{C}^{\bullet\, j})$$

$$\chi(H(\mathcal{C}^{\bullet\, j})) = \chi(\mathcal{C}^{\bullet\, j})$$

$$\langle K \rangle = \sum_j q^j \chi(H(\mathcal{C}^{\bullet\, j}))$$

$$\langle K \rangle = \sum_j q^j \chi(H(\mathcal{C}^{\bullet j}))$$

With $U|s\rangle = (-1)^{i(s)} q^{j(s)} |s\rangle,$

$$\partial : \mathcal{C}^{ij} \longrightarrow \mathcal{C}^{i+1\,j}$$

$$U\partial + \partial U = 0.$$

This means that the unitary transformation
U acts on the homology so that

U: H(C(K)) -----> H(C(K))

Eigenspace Picture

$$\mathcal{C}^0 = \oplus_\lambda \mathcal{C}^0_\lambda$$

$$\mathcal{C}^\bullet_\lambda : \mathcal{C}^0_\lambda \longrightarrow \mathcal{C}^1_{-\lambda} \longrightarrow \mathcal{C}^2_{+\lambda} \longrightarrow \cdots \mathcal{C}^n_{(-1)^n\lambda}$$

$$\mathcal{C} \ = \ \oplus_\lambda \mathcal{C}^\bullet_\lambda$$

$$\langle\psi|U|\psi\rangle = \sum_\lambda \lambda\chi(H(C^\bullet_\lambda))$$

SUMMARY

We have interpreted the bracket polynomial as a quantum amplitude by making a Hilbert space C(K) whose basis is the collection of enhanced states of the bracket.

This space C(K) is naturally intepreted as the chain space for the Khovanov homology associated with the bracket polynomial.

$$\langle K \rangle = \langle \psi | U | \psi \rangle .$$

The homology and the unitary transformation U speak to one another via the formula

$$U\partial + \partial U = 0.$$

Questions

We have shown how Khovanov homology fits into the context of quantum information related to the Jones polynomial and how the polynomial is replaced in this context by a unitary transformation U on the Hilbert space of the model. This transformation U acts on the homology, and its eigenspaces give a natural decomposition of the homology that is related to the quantum amplitude corresponding to the Jones polynomial.

The states of the model are intensely combinatorial, related to the representation of the knot or link.

How can this formulation be used in quantum information theory and in statistical mechanics?!

Potts Model and Statistical Mechanics

Partition Function

Recall that the partition function of a physical system has the form of the sum over all states s of the system the quantity

$$\sum \exp[(J/kT)E(s)]$$

where

J = +1 or -1 (ferromagnetic or antiferromagnetic models)

k = Boltzmann's constant

T = Temperature

E(s) = energy of the state s

Potts Model

In the Potts model, one has a graph G and assigns labels (spins, charges) to each node of the graph from a label set $\{1,2,...,Q\}$.
A state s is such a labeling.
The energy E(s) is equal to the number of edges in the graph where the endpoints of the edge receive the same label.

For Q = 2, the Potts model is equivalent to the Ising model. The Ising model was shown by Osager to have a phase transition in the limit of square planar lattices (in the the 1940's).

The partition function $P_G(Q,T)$ for the Q-state Potts model on a graph G is given by the dichromatic polynomial

$$Z[G](v,Q)$$

where

$$v = e^{J\frac{1}{kT}} - 1$$

J = +1 or -1 (ferromagnetic or antiferromagnetic models)

k = Boltzmann's constant

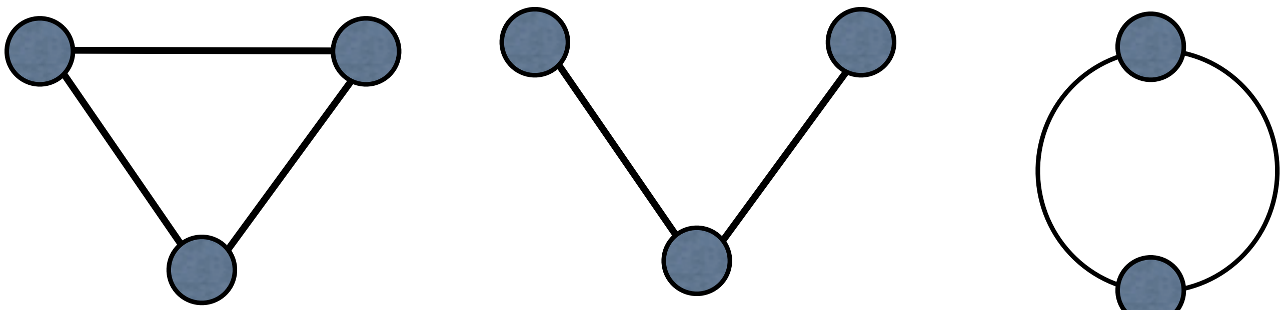
T = Temperature

The Dichromatic Polynomial and the Potts Model

Dichromatic Polynomial

$$Z[G](v,Q) = Z[G'](v,Q) + vZ[G''](v,Q)$$

$$Z[\bullet \sqcup G] = QZ[G].$$



G

G'
Delete

G''
Contract

v = e^K - 1

Z  = Z  + v Z 

= q^2 + vq

= q(q + v)

= q(q - 1) + q e^K

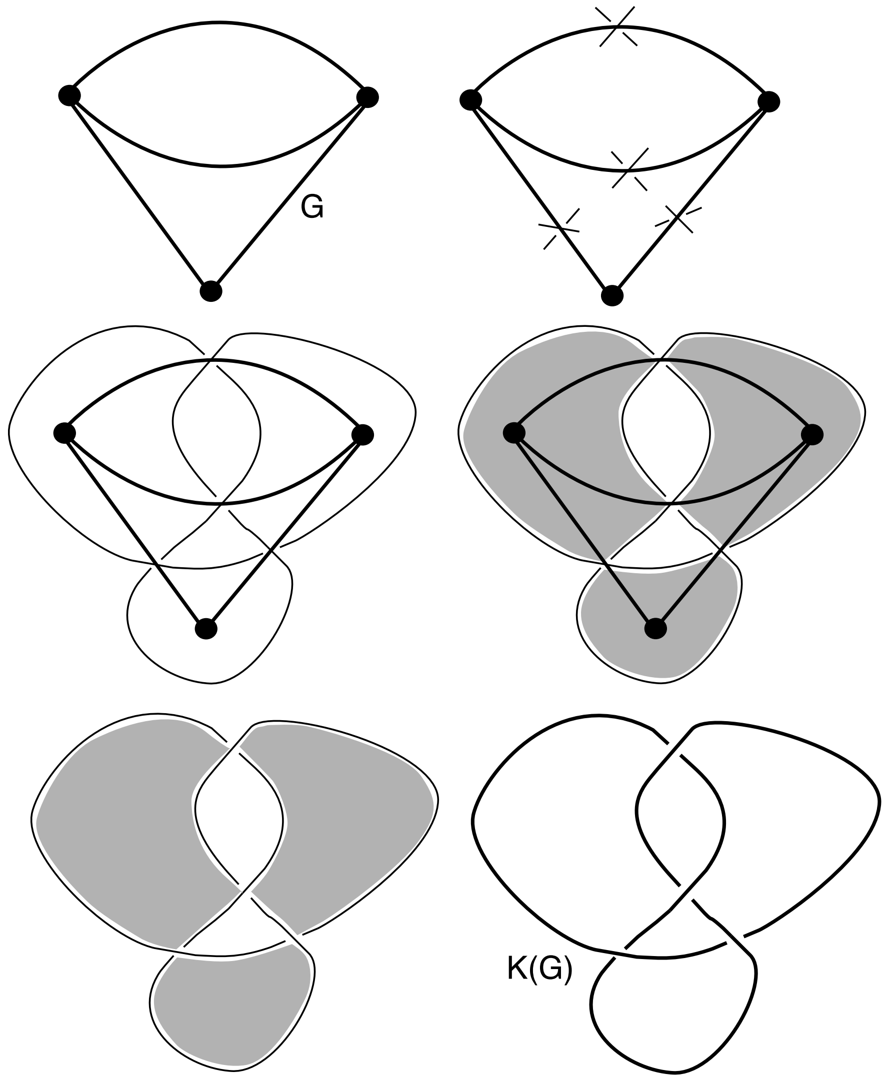
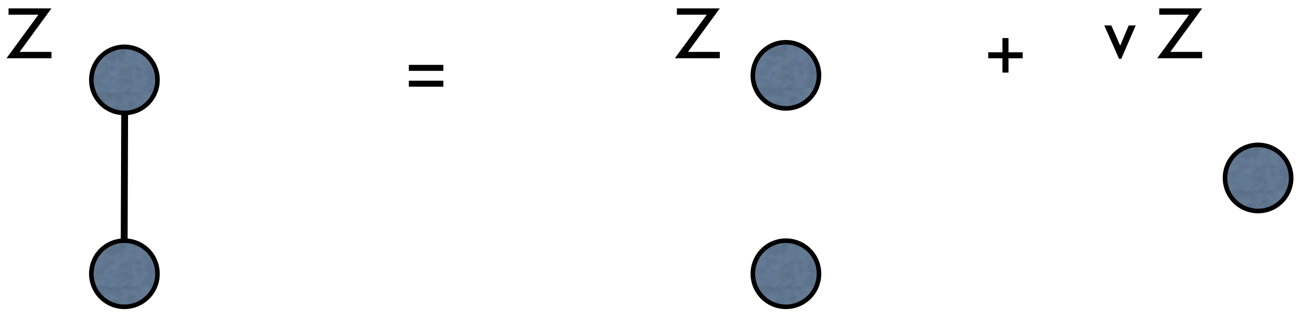
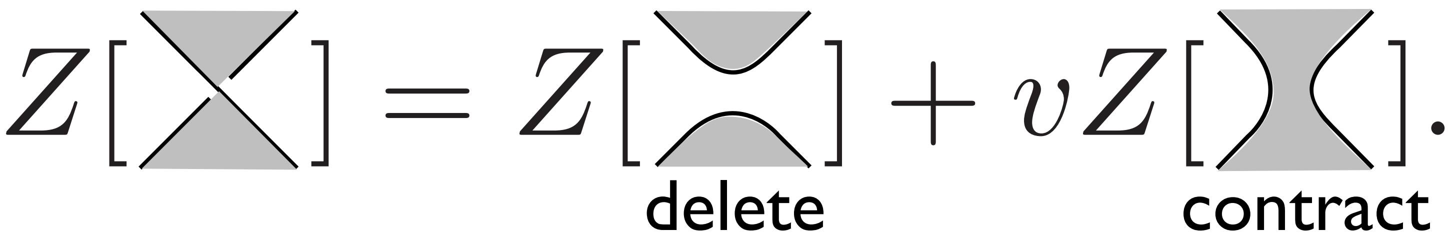


Figure 4: **Medial Graph, Checkerboard Graph and K(G)**

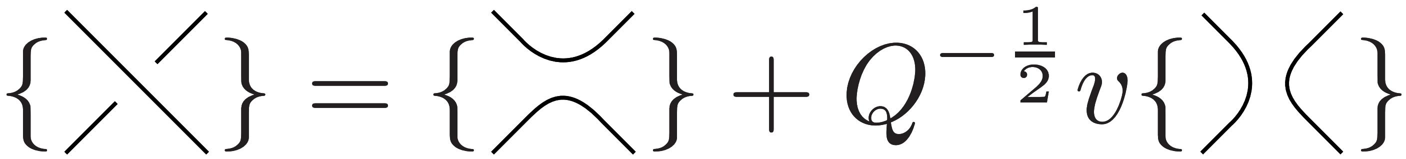




$$Z[R \sqcup K] = Q Z[K].$$

Theorem: $Z[G](v,Q) = Q^{N/2}\{K(G)\}$

where K(G) is an alternating link associated with the medial graph of G and

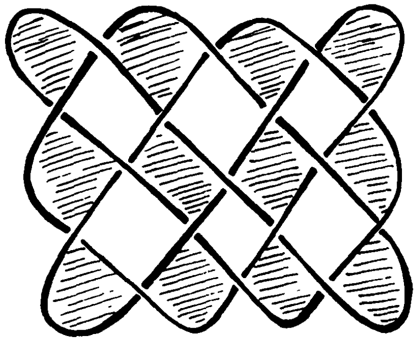


$$\{\bigcirc\} = Q^{\frac{1}{2}}$$

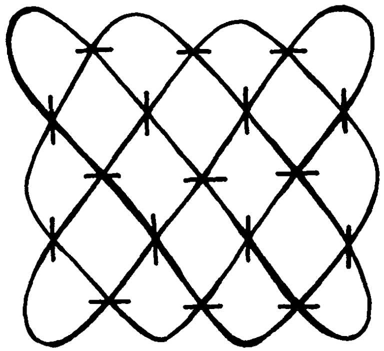
For the Potts Model the critical point is at

$$Q^{-\frac{1}{2}} v = 1$$

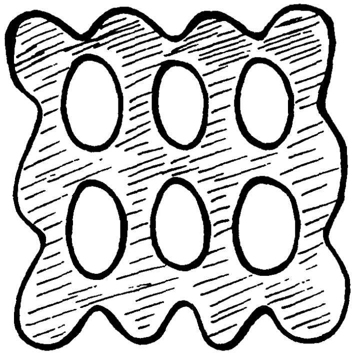
In the corresponding loop expansion for the Potts model the loops are the boundaries of regions of constant spin.



K



S



S (split)

$V = 17$

$W = 7, \ B = 12$

$R = 7 + 12 = 19 = V + 2$

$|S| = 7 = W$

To analyze Khovanov homology, we adopt a new bracket

$$[\times] = [\asymp] - q\rho[)(]$$

$$[\bigcirc] = q + q^{-1}$$

When rho = 1, we have the topological bracket in Khovanov form.

When $$-q\rho = Q^{-\frac{1}{2}}v$$

$$q + q^{-1} = Q^{\frac{1}{2}}$$

we have the Potts model.

$$[K] = \sum_s (-\rho)^{n_B(s)} q^{j(s)}$$

$$[K] = \sum_{i\,,j} (-\rho)^i q^j dim(\mathcal{C}^{ij})$$

$$= \sum_j q^j \sum_i (-\rho)^i dim(\mathcal{C}^{ij}) = \sum_j q^j \chi_\rho(\mathcal{C}^{\bullet\, j})$$

where

$$\chi_\rho(\mathcal{C}^{\bullet\, j}) = \sum_i (-\rho)^i dim(\mathcal{C}^{ij}).$$

$$[K](q, \rho = 1) = \sum_j q^j \chi_\rho(\mathcal{C}^{\bullet\, j}) = \sum_j q^j \chi(H(\mathcal{C}^{\bullet\, j}))$$

Away from rho=1, one can ask what is the influence of the Khovanov homology on the coefficients in the expansion of

$$[K](q, \rho)$$

and corresponding questions about the Potts model.

Tracking Potts

$$-q\rho = Q^{-\frac{1}{2}}v$$

$$q + q^{-1} = Q^{\frac{1}{2}}$$

whence

$$q^2 - \sqrt{Q}q + 1 = 0.$$

$$q = \frac{\sqrt{Q} \pm \sqrt{Q-4}}{2}$$

At criticality Potts meets Khovanov at four colors and imaginary temperature!

$$\frac{1}{q} = \frac{\sqrt{Q} \mp \sqrt{Q-4}}{2}.$$

Criticality: $-\rho q = 1$

$$\rho = -\frac{1}{q} = \frac{-\sqrt{Q} \pm \sqrt{Q-4}}{2}.$$

Suppose that $\rho = 1.$

Then $2 = -\sqrt{Q} \pm \sqrt{Q-4}.$

So $4 - Q = \mp\sqrt{Q}\sqrt{Q-4}.$

And need Q = 4 and $e^K = -1$.

Now consider rho = 1 without insisting on criticality.

$$1 = -v/(q\sqrt{Q})$$

$$\rho = -\frac{v}{\sqrt{Q}q} = v(\frac{-1 \pm \sqrt{1 - 4/Q}}{2})$$

$$v = -q\sqrt{Q} = \frac{-Q \mp \sqrt{Q}\sqrt{Q-4}}{2}.$$

$$e^K = 1 + v = \frac{2 - Q \mp \sqrt{Q}\sqrt{Q-4}}{2}.$$

For $Q = 2$ we have $e^K = \pm i$.

For Q = 3, $e^K = \frac{-1 \pm \sqrt{3}i}{2}$.

For $Q = 4$ we have $e^K = -1$.

For Q >4, e^K is real and negative.

Thus we get complex temperature values in all cases where the coefficients of the Potts model are given directly in terms of Euler characteristics from Khovanov homology.