Bounds on Boundary Entropy

Anatoly Konechny
Heriot-Watt University

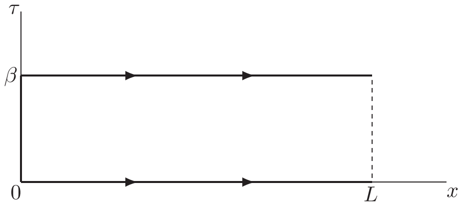
November 28, 2012, EMPG seminar, Edinburgh

Based on a joint work with Daniel Friedan and Cornelius Schmidt-Colenet (PRL, Oct. 2012)

Outline

- Boundary Entropy
- The existence of a lower and upper bounds
- Numerical results
- Discussion

Boundary entropy of critical 1D quantum systems was defined by I. Affleck and A. Ludwig in 1991. It is not hard to generalize it to non-critical boundary conditions



For $L \to \infty$ $\quad \ln Z = \ln g(\beta) + \dfrac{\pi c L}{6\beta} + \ldots$

$$S = (1 - \beta\frac{\partial}{\partial\beta}) \ln Z = s(\beta) + \frac{c\pi L}{3\beta} + \ldots$$

For conformal boundary conditions $s(\beta) = \ln g$ is a number independent of β. If $|B\rangle$ is the boundary state representing a conformal boundary condition in the bulk CFT Hilbert space then $g = \langle B|0\rangle$.

Ordinary entropy$S$ satisfies

- S > 0
- S satisfies the second law of TD - it monotonically decreases with temperature:

$$T\frac{\partial S}{\partial T} = \beta^2 \langle (H - \langle H \rangle)^2 \rangle \geq 0$$

- S satisfies the third law of TD: $S(T) \geq S_0 > 0$

Because of the subtraction of $\dfrac{c\pi L}{3\beta}$ the boundary entropy does not obviously satisfy any of the above 3 properties. In fact the first one is violated. In the $c = 1$ Gaussian model with radius R:

$$g_{Dir} = 2^{-1/4}R^{-1/2}\,, \qquad g_{Neum} = 2^{-1/4}R^{1/2}$$

We see that s can be negative and the lower bound over all conformal boundary conditions for a fixed bulk theory, if exists, cannot depend on c alone, but may depend on moduli such as R.

- Despite these oddities the boundary entropy still merits to be called entropy because it can be proven that it satisfies the second law of thermodynamics.This is a consequence of the so called g-theorem conjectured by I.Affleck, A. Ludwig, 1991 and proved by Daniel Friedan, AK, 2003).
- The existence of an analogue of the 3rd law of thermodynamics (a lower bound which depends on bulk theory) has not so far been established despite some (modest) attempts, D.F, A.K., 2006.
- The existence of a lower bound is important for gaining control over RG flows. Temperature can be traded for RG scale. For bulk flows in unitary theories $c \geq 0$ can flow to a trivial theory $c = 0$. For the boundary flows there is no obvious candidate for a "trivial" boundary condition, or a b.c. with minimal s.

A lower bound for critical boundaries

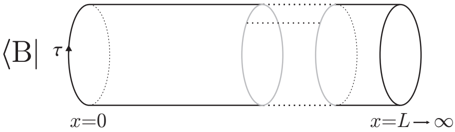
One can study a simpler problem - a lower bound for boundary entropy for all *conformal* boundary conditions with a fixed bulk theory. Such a general bound was found to hold under certain conditions D.Friedan, C. Schmidt-Colinet, AK, 2012. Namely, we showed that assuming $c \geq 1$ and $\Delta_1 \geq \frac{c-1}{12}$ where Δ_1 is the lowest dimension of spin zero bulk primary

$$g \geq g_B = g_B(c, \Delta_1)$$

This result is a general restriction on the spaces of conformal boundary conditions. If during a boundary RG flow s gets below the bound the flow never stops.

The crucial ingredient (Cardy constraint) and the main idea of deriving the bound go back to J. Cardy, 1986, 1989, 1991. More recently general bounds for bulk quantities were derived by S.Hellerman, 2009; S.Hellerman, C. Schmidt-Colinet, 2010. For the boundary our starting point is Cardy's modular duality formula

$$\mathrm{Tr} e^{-\beta H_{\mathrm{bdry}}} = \langle B| e^{-2\pi H_{\mathrm{bulk}}/\beta} |B\rangle$$



Each side can be expanded in Virasoro characters. For $c > 1$ we have

$$\chi_0(i\beta) + \sum_j \chi_{h_j}(i\beta) = g^2 \chi_0\left(i/\beta\right) + \sum_k b_k^2 \chi_{\Delta_k/2}\left(i/\beta\right)$$

$$\chi_h(i\beta) = \frac{e^{2\pi\beta\left(\frac{c-1}{24} - h\right)}}{\eta(i\beta)}, \quad \chi_0(i\beta) = e^{\pi\beta\left(\frac{c-1}{12}\right)} \frac{\left(1 - e^{-2\pi\beta}\right)}{\eta(i\beta)}$$

Boundary spectrum of primaries:

$$0 < h_1 \leq h_2 \leq \dots ,$$

Bulk spectrum of spin zero primaries:

$$0 < \Delta_1 \leq \Delta_2 \leq \dots$$

We can use the modular transformation formula:
$\eta(i\beta) = \beta^{-1/2}\eta(i/\beta)$ to get rid of all descendant contributions and obtain an equation relating the spectra of primaries

$$e^{\pi\beta(\frac{c-1}{12})}(1 - e^{-2\pi\beta}) + \sum_j e^{2\pi\beta\left(\frac{c-1}{24} - h_j\right)}$$
$$= \beta^{-1/2}\left[g^2 e^{\frac{\pi(c-1)}{12\beta}}(1 - e^{-\frac{2\pi}{\beta}}) + \sum_k e^{\frac{\pi}{\beta}\left(\frac{c-1}{12} - \Delta_k\right)}\right]$$

More succinctly

$$f_0 + \sum_j f_{h_j} = g^2 \tilde{f}_0 + \sum_k b_k^2 \tilde{f}_{\Delta_k}$$

Derivation of the bound

Apply to both sides of this equation a linear functional (a distribution) $\rho(\beta)$:

$$(\rho, f_0) + \sum_j (\rho, f_{h_j}) = g^2(\rho, \tilde{f}_0) + \sum_k b_k^2(\rho, \tilde{f}_{\Delta_k})$$

where

$$(\rho, F) = \int_0^\infty d\beta \, \rho(\beta) F(\beta) \, .$$

If we can choose $\rho(\beta)$ so that

$$(\rho, f_h) \geq 0, \;\; \forall h > 0 \, , \quad (\rho, \tilde{f}_\Delta) \leq 0, \;\; \forall \Delta \geq \Delta_1$$

we get an inequality

$$g^2(\rho, \tilde{f}_0) \geq (\rho, f_0)$$

It is easy to show that under the above assumptions on ρ, $(\rho, \tilde{f}_0) > 0$ so that we get a lower bound on g

$$g^2 \geq g_B^2[\rho] = \frac{(\rho, f_0)}{(\rho, \tilde{f}_0)}\,.$$

These bounds can be maximized over all distributions ρ satisfying the above constraints:

$$g^2 \geq g_B^2(c, \Delta_1) = \max_\rho g_B^2[\rho]$$

To demonstrate the existence of such a bound one can find ρ given by a suitable first order differential operator

$$\mathcal{D} = a_0 + \left(-\frac{1}{2\pi}\frac{\partial}{\partial \beta} + \frac{c-1}{24} \right)$$

The constraint $(\rho, f_h) \geq 0, \ \ \forall h > 0$ is equivalent to $a_0 \geq 0$ and the constraint $(\rho, \tilde{f}_\Delta) \leq 0, \ \ \forall \Delta \geq \Delta_1$ translates into an equation

$$a_0 \leq \frac{\Delta_1 - \left(\frac{c-1}{12}\right)}{2\beta^2} - \frac{1}{4\pi\beta} - \frac{c-1}{24}$$

The two constraints thus imply

$$\frac{\Delta_1 - \left(\frac{c-1}{12}\right)}{2\beta^2} - \frac{1}{4\pi\beta} - \frac{c-1}{24} \geq 0$$

which cannot be satisfied for any value of β if $\Delta_1 \leq \frac{c-1}{12}$. For

$$\Delta_1 > \frac{c-1}{12}$$

both constraints are satisfied for appropriate a_0 and β and we get a non-trivial bound.

$$g^2 \geq g_B^2(c, \Delta_1, 1) = \max_{0<\beta<\beta_1} A(c, \beta, \Delta_1)$$

The above can be generalized to $c = 1$ theories. In this case there is no condition on the bulk gap Δ_1, but one needs to take into account degenrate representations:

$$\chi_n = e^{2\pi\beta\left(\frac{c-1}{24} - n^2\right)}\left(1 - e^{-2\pi\beta(2n+1)}\right).$$

We constructed an appropriate first order differential operator, got a bound and maximized it over β. In the $c = 1$ Gaussian model of radius R,

$$\Delta_1 = \min(R^2/2, 1/2R^2) \leq 1/2$$

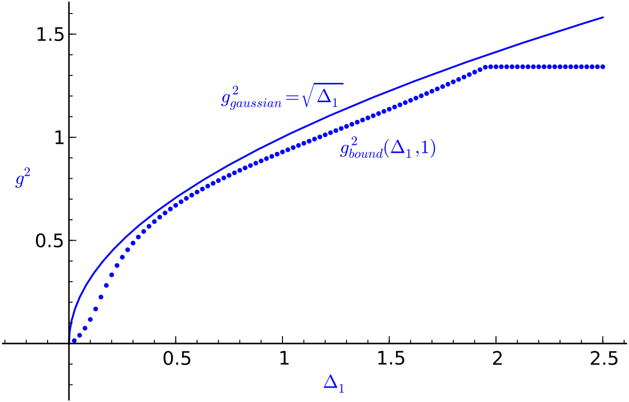


Figure: The bound for $c = 1$ compared to the minimum value of g^2 for the $c = 1$ gaussian model. The comparison can be extended past $\Delta_1 = \frac{1}{2}$ if, for purposes of the bound, Δ_1 is interpreted as the lowest dimension of the spin-0 primaries *occurring in the boundary state*.

Other bounds

The same idea can be turned around to derive an upper bound

$$g^2 \leq g^2_{\rm UB}(h_1, c)\,.$$

We derive such a bound under the assumption

$$h_1 > \frac{c-1}{24}$$

The upper bound depends on the boundary lowest primary dimension h_1 and c. We find that in the limit $h_1 \to \infty$ the upper bound tends to zero. Thus, there exists an upper bound on h_1:

$$h_1 \leq h_B(\Delta_1, c)\,.$$

Moreover we also found that the upper bound becomes zero for sufficiently high multiplicity N_1 and thus there is also a bound

$$N_1 \leq N_B(h_1, c, \Delta_1)\,.$$

Numerical calculations

The best linear functional bounds can be calculated numerically for particular models. The problem of optimizing over the functionals ρ can be translated into a semi-definite programming (SDP) problem. The constraints for a general differential operator can be represented in terms of two non-negative polynomials

$$p(h) \geq 0\,, \ \ \forall h \geq 0 \quad q(x) \geq 0, \ \ \forall x \geq x_1\,, \ \ x_1 = 2\pi^2(\Delta_1 - \frac{c-1}{12})$$

related by

$$q(x) = -p(-\partial_s + \frac{c-1}{24} + \frac{1}{2s} - \frac{x}{s^2})1\,, \quad s = 2\pi\beta$$

Each of$p(x), q(x)$is decomposed in terms of a pair of symmetric positive semidefinite matrices $P_{1,2}, Q_{1,2}$ (D. Hilbert)

$$p(x) = \mathbf{u}^t P_1 \mathbf{u} + x(\mathbf{u}^t P_2 \mathbf{u})\,, \;\; \mathbf{u}_k = x^k\,, 0 \le k \le N$$

$$q(x) = \mathbf{u}^t Q_1 \mathbf{u} + x(\mathbf{u}^t Q_2 \mathbf{u})$$

With an additional normalization constraint on $q(x)$ the lower bound is just $g_B^2 = p(0) - p(1)$ and the SDP problem is to maximize $p(0) - p(1)$ over all symmetric positive semidefinite matrices P_i, Q_j subject to ordinary (equation) constraints. We wrote a SAGE code which uses a free SDP solver called SDPA http://sdpa.sourceforge.net/

For concrete CFT's one can also benefit from putting more details of the spectrum restricting the positivity constraints to the points of the bulk spectrum. For $c = 24$ Monster CFT. (constructed from 24 free bosons compactified on a torus induced by Leech lattice) we calculated

$$g^2 > 1 \pm 6.03 \times 10^{-19}$$

For the known conformal boundary conditions in this CFT (B.Craps, M.R. Gaberdiel, J.A. Harvey, 2003) $g = 1$. Moreover, from the extremal functional ρ we get information on the spectrum:

$$g^2 = g_B^2 + \sum_j f_\rho(h_j) + \sum_k b_k^2 \tilde{f}_\rho(\Delta_k/2)$$

So if the minimal boundary condition exists the boundary spectrum h_j is given by the zeroes of f_ρ function and if $b_k \neq 0$ then Δ_k is a zero of $\tilde{f}_\rho$.

For the Monster CFT we get the boundary spectrum of the known branes. This is not always the situation. It may happen that the minimal functional ρ does not correspond to any conformal boundary condition at all. We found that this is the case for a free boson $c = 2$ CFT on a square torus with radii $R_1 = R_2 = \sqrt{2}R_{s.d.}$. We found a minimal point ρ with

$$g_B^2 \approx 0.1008$$

Using the emerging spectrum we found bounds on the degeneracy of the lowest boundary state $h_1 \approx 2.527$

$$6.30974556956841 < deg_1 < 6.30977160576788$$

So that the minimum does not correspond to any boundary condition. N.B.: all known conformal boundary conditions have $g^2 \geq 0.25$. An improved algorithm is needed which ensures the integrality of the state degeneracies.

Genralizations to include extended symmetries

The linear functional bounds can be generalized to branes respecting chiral algebras, e.g. supersymmetry. This might be of interest in string theory. Another example is branes on N-dimensional tori which respect $U(1)^N$symmetry. There is no bulk gap restriction for such branes. The first order differential operator gives the following compact analytic bound

$$g_B^2 \geq \frac{(\pi\Delta_1)^{N/2}}{(1+N/2)^{1+N/2}}\,.$$

Discussion

The most pressing issue is to overcome the limitation of the constraint $\Delta_1 > \frac{c-1}{12}$. We have a no-go theorem which says that the linear functional method cannot overcome this bound. Use the identity

$$\beta^{-1/2} e^{\frac{\pi}{\beta}\left(\frac{c-1}{12}-\Delta\right)} = \int_{-\infty}^{+\infty} dy\, e^{-\pi\beta y^2 + 2\pi i y\sqrt{\Delta-(c-1)/12}}$$

we see that the condition

$$(\rho, \tilde{f}_\Delta) \geq 0 \quad \forall \Delta \geq \Delta_1$$

requires

$$\int dy\, (\rho, f_{\gamma+y^2/2}) \cos(2\pi y\sqrt{\Delta_1 - 2\gamma}) \leq 0$$

where $\gamma = (c-1)/24$.

To be compatible with the condition

$$(\rho, f_h) \geq 0\,, \ \ \forall h > 0$$

the inequality

$$\Delta_1 \geq \frac{c-1}{12}$$

must be satisfied.

Intuition. The states below the threshold are "false vacua". One may invoke other CFT sawing constraints to deal with them.