Geometry of Higher Yang-Mills Fields

Christian Sämann

School of Mathematical and Computer Sciences
Heriot-Watt University, Edinburgh

Edinburgh Mathematical Physics Group, 23.1.2013

Based on work with:

- S Palmer, D Harland, C Papageorgakis, F Sala (M-brane models)
- M Wolf (Twistor description)
- R Szabo (Geometric Quantization)

Motivation

There might be an effective descripion of M5-branes.

- Effective description of M2-branes proposed in 2007.
- This created lots of interest:
  BLG-model: >625 citations, ABJM-model: >917 citations

Question: Is there a similar description for M5-branes?

For cautious people:
Is there a a reasonably interesting superconformal field theory of a non-abelian tensor multiplet in six dimensions?
(The mysterious, long-sought $\mathcal{N} = (2, 0)$ SCFT in six dimensions)

A possible way to approach the problem: Look at BPS subsector

- This was how the M2-brane models were derived originally.
- BPS subsector is interesting itself: Integrability
- BPS subsector should be more accessible than full theory.

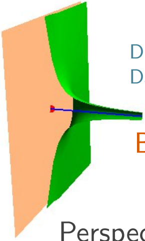
Results so far/Outline

Things do look very promising.

- Integrability found:
  Nahm construction for self-dual strings using loop space
  CS, S Palmer & CS
- Use of loop space justified:
  M-theory suggests this, e.g. Geometric quantization of S^3
  CS & R Szabo
- Integrability reasonable:
  Gauge structure of M2- and M5-brane models the same
  S Palmer & CS
- Integrability works even without loop space:
  Twistor constructions of self-dual strings and non-abelian tensor multiplets work
  CS & M Wolf
- On the way to Geometry of Higher Yang-Mills Fields:
  Explicit solutions to non-abelian tensor multiplet equations
  F Sala, S Palmer & CS

Monopoles and Self-Dual Strings

Lifting monopoles to M-theory yields self-dual strings.



	0	1	2	3	4	5	6
D1	×						×
D3	×	×	×	×			

BPS configuration

Perspective of D1:

Nahm eqn.

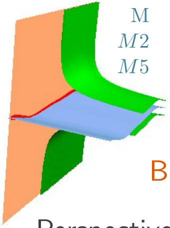
$$\frac{\mathrm{d}}{\mathrm{d}x^6}X^i + \varepsilon^{ijk}[X^j, X^k] = 0$$

$\updownarrow$ Nahm transform $\updownarrow$

Perspective of D3:

Bogomolny monopole eqn.

$$F_{ij} = \varepsilon_{ijk}\nabla_k\Phi$$



M	0	1	2	3	4	5	6
$M2$	×					×	×
$M5$	×	×	×	×	×	×	

BPS configuration

Perspective of M2:

Basu-Harvey eqn.

$$\frac{\mathrm{d}}{\mathrm{d}x^6}X^\mu + \varepsilon^{\mu\nu\rho\sigma}[X^\nu, X^\rho, X^\sigma] = 0$$

$\updownarrow$ generalized Nahm transform $\updownarrow$

Perspective of M5:

Self-dual string eqn.

$$H_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\sigma}\partial_\sigma\Phi$$

3-Lie Algebras

In analogy with Lie algebras, we can introduce 3-Lie algebras.

$$\text{BH:}\quad \frac{\mathrm{d}}{\mathrm{d}s}X^\mu + [A_s, X^\mu] + \varepsilon^{\mu\nu\rho\sigma}[X^\nu, X^\rho, X^\sigma] = 0\ , \quad X^\mu \in \mathcal{A}$$

3-Lie algebra

Obviously: $\mathcal{A}$ is a vector space, $[\cdot,\cdot,\cdot]$ trilinear+antisymmetric.

Satisfies a "3-Jacobi identity," the fundamental identity:

$$[A,B,[C,D,E]] = [[A,B,C],D,E] + [C,[A,B,D],E] + [C,D,[A,B,E]]$$

Filippov (1985)

Gauge transformations from Lie algebra of inner derivations:

$$D : \mathcal{A} \wedge \mathcal{A} \to \mathsf{Der}(\mathcal{A}) =: \mathfrak{g}_{\mathcal{A}} \qquad D(A,B) \rhd C := [A,B,C]$$

Algebra of inner derivations closes due to fundamental identity.

Brief Remarks on 3-Lie Algebras

In analogy with Lie algebras, we can introduce 3-Lie algebras.

Examples:

Lie algebra	3-Lie algebra
Heisenberg-algebra: $[\tau_a, \tau_b] = \varepsilon_{ab}\mathbb{1}, \quad [\mathbb{1}, \cdot] = 0$	Nambu-Heisenberg 3-Lie Algebra: $[\tau_i, \tau_j, \tau_k] = \varepsilon_{ijk}\mathbb{1}, \quad [\mathbb{1}, \cdot, \cdot] = 0$
$\mathfrak{su}(2) \simeq \mathbb{R}^3$: $[\tau_i, \tau_j] = \varepsilon_{ijk}\tau_k$	$A_4 \simeq \mathbb{R}^4$: $[\tau_\mu, \tau_\nu, \tau_\kappa] = \varepsilon_{\mu\nu\kappa\lambda}\tau_\lambda$

Generalizations:

- Real 3-algebras: $[\cdot, \cdot, \cdot]$ antisymmetric only in first two slots
  S. Cherkis & CS, 0807.0808
- Hermitian 3-algebras: complex vector spaces, $\rightarrow$ ABJM
  Bagger & Lambert, 0807.0163

Generalizing the ADHMN construction to M-branes

That is, find solutions to $H = \star \mathrm{d}\Phi$
from solutions to the Basu-Harvey equation.

As M5-branes seem to require gerbes, let's start with them.

Manifold M with cover $(U_i)_i$. Principal U(1)-bundle over M:

$$F \in \Omega^2(M, \mathfrak{u}(1)) \text{ with } \mathrm{d}F = 0$$
$$A_{(i)} \in \Omega^1(U_i, \mathfrak{u}(1)) \text{ with } F = \mathrm{d}A_{(i)}$$
$$g_{ij} \in \Omega^0(U_i \cap U_j, \mathsf{U}(1)) \text{ with } A_{(i)} - A_{(j)} = \mathrm{d}\log g_{ij}$$

Consider monopole in $\mathbb{R}^3$, but describe it on S^2 around monopole:

S^2 with patches U_+, U_-, $U_+ \cap U_- \sim S^1$: $g_{+-} = \mathrm{e}^{-\mathrm{i}k\phi},\ k \in \mathbb{Z}$

$$c_1 = \frac{\mathrm{i}}{2\pi}\int_{S^2} F = \frac{\mathrm{i}}{2\pi}\int_{S^1} A^+ - A^- = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi\, k = k$$

Monopole charge: k

Manifold M with cover $(U_i)_i$. Abelian (local) gerbe over M:

$$H \in \Omega^3(M, \mathfrak{u}(1)) \text{ with } \mathrm{d}H = 0$$
$$B_{(i)} \in \Omega^2(U_i, \mathfrak{u}(1)) \text{ with } H = \mathrm{d}B_{(i)}$$
$$A_{(ij)} \in \Omega^1(U_i \cap U_j, \mathfrak{u}(1)) \text{ with } B_{(i)} - B_{(j)} = \mathrm{d}A_{ij}$$
$$h_{ijk} \in \Omega^0(U_i \cap U_j \cap U_k, \mathfrak{u}(1)) \text{ with } A_{(ij)} - A_{(ik)} + A_{(jk)} = \mathrm{d}h_{ijk}$$

Note: Local gerbe: principal $\mathsf{U}(1)$-bundles on intersections $U_i \cap U_j$.

Consider S^3, patches U_+, U_-, $U_+ \cap U_- \sim S^2$: bundle over S^2
Reflected in: $H^2(S^2, \mathbb{Z}) \cong H^3(S^3, \mathbb{Z}) \cong \mathbb{Z}$

$$\frac{\mathrm{i}}{2\pi} \int_{S^3} H = \frac{\mathrm{i}}{2\pi} \int_{S^2} B_+ - B_- = \ldots = k$$

Charge of self-dual string: k

Describe p-gerbes + connective structure $\to$ Deligne cohomology.

Gerbes are somewhat unfamiliar, difficult to work with.

Can we somehow avoid using gerbes?

Consider the following double fibration:

$$\begin{tikzcd} & \mathcal{L}M \times S^1 \arrow[dl, "ev"'] \arrow[dr, "pr"] & \\ M & & \mathcal{L}M \end{tikzcd}$$

Identify $T\mathcal{L}M = \mathcal{L}TM$, then: $x \in \mathcal{L}M \Rightarrow \dot{x}(\tau) \in T\mathcal{L}M$

Transgression

$$\mathcal{T} : \Omega^{k+1}(M) \to \Omega^k(\mathcal{L}M)~, \quad v_i = \oint \mathrm{d}\tau\, v_i^\mu(\tau) \frac{\delta}{\delta x^\mu(\tau)} \in T\mathcal{L}M$$

$$(\mathcal{T}\omega)_x(v_1(\tau), \ldots, v_k(\tau)) := \oint_{S^1} \mathrm{d}\tau\, \omega(x(\tau))(v_1(\tau), \ldots, v_k(\tau), \dot{x}(\tau))$$

Nice properties: reparameterization invariant, chain map, ...

An abelian local gerbe over M is a principal $\mathsf{U}(1)$-bundle over $\mathcal{L}M$.

Recall the self-dual string equation on $\mathbb{R}^4$: $H_{\mu\nu\kappa} = \varepsilon_{\mu\nu\kappa\lambda}\frac{\partial}{\partial x^\lambda}\Phi$

Its transgressed form is an equation for a 2-form F on $\mathcal{L}\mathbb{R}^4$:

$$F_{(\mu\sigma)(\nu\rho)} = \delta(\sigma-\rho)\varepsilon_{\mu\nu\kappa\lambda}\dot{x}^\kappa(\tau)\,\frac{\partial}{\partial y^\lambda}\Phi(y)\Big|_{y=x(\tau)}$$

Extend to full non-abelian loop space curvature:

$$\begin{aligned}F^{\pm}_{(\mu\sigma)(\nu\tau)} =& \big(\varepsilon_{\mu\nu\kappa\lambda}\dot{x}^\kappa(\sigma)\nabla_{(\lambda\tau)}\Phi\big)_{(\sigma\tau)}\\ &\mp\big(\dot{x}_\mu(\sigma)\nabla_{(\nu\tau)}\Phi + \dot{x}_\nu(\sigma)\nabla_{(\mu\tau)}\Phi - \delta_{\mu\nu}\dot{x}^\kappa(\sigma)\nabla_{(\kappa\tau)}\Phi\big)_{[\sigma\tau]}\end{aligned}$$

where $\nabla_{(\mu\sigma)} := \oint \mathrm{d}\tau\,\delta x^\mu(\tau)\wedge\left(\frac{\delta}{\delta x^\mu(\tau)} + A_{(\mu\tau)}\right)$

Goal: Construct solutions to this equation.

The ADHMN Construction

The ADHMN construction nicely translates to self-dual strings on loop space.

Nahm transform: Instantons on $T^4 \mapsto$ instantons on $(T^4)^*$

Roughly here:

$$T^4\colon \begin{cases} 3 \text{ rad. } 0 \\ 1 \text{ rad. } \infty & : \text{ D1 WV} \end{cases} \quad \text{and} \quad (T^4)^*\colon \begin{cases} 3 \text{ rad. } \infty & : \text{ D3 WV} \\ 1 \text{ rad. } 0 \end{cases}$$

Dirac operators: X^i solve Nahm eqn., X^μ solve Basu-Harvey eqn.

$$\text{IIB}: \quad \nabla\!\!\!/ = -\mathbb{1}\frac{\mathrm{d}}{\mathrm{d}x^6} + \sigma^i(\mathrm{i}X^i + x^i\mathbb{1}_k)$$

$$\text{M}: \quad \nabla\!\!\!/ = -\gamma_5\frac{\mathrm{d}}{\mathrm{d}x^6} + \tfrac{1}{2}\gamma^{\mu\nu}\left(D(X^\mu, X^\nu) - \mathrm{i}\oint \mathrm{d}\tau\, x^\mu(\tau)\dot{x}^\nu(\tau)\right)$$

normalized zero modes: $\bar{\nabla\!\!\!/}\psi = 0$ and $\mathbb{1} = \int_{\mathcal{I}} \mathrm{d}s\, \bar{\psi}\psi$

Solution to Bogomolny/self-dual string equations:

$$A := \int_{\mathcal{I}} \mathrm{d}s\, \bar{\psi}\,\mathrm{d}\,\psi \quad \text{and} \quad \Phi := -\mathrm{i}\int_{\mathcal{I}} \mathrm{d}s\, \bar{\psi}\, s\, \psi$$

- Nahm eqn. and Basu-Harvey eqn. play analogous roles.
- Construction extends to general. Basu-Harvey eqn. (ABJM).
- One can construct many examples explicitly.
- It reduces nicely to ADHMN via the M2-Higgs mechanism.

CS, 1007.3301, S Palmer & CS, 1105.3904

More Motivation for Loop Spaces

3-Lie algebra valued tensor multiplet equations:

$$\nabla^2 X^I - \tfrac{\mathrm{i}}{2}[\bar{\Psi}, \Gamma_\nu \Gamma^I \Psi, C^\nu] - [X^J, C^\nu, [X^J, C_\nu, X^I]] = 0$$

$$\Gamma^\mu \nabla_\mu \Psi - [X^I, C^\nu, \Gamma_\nu \Gamma^I \Psi] = 0$$

$$\nabla_{[\mu} H_{\nu\lambda\rho]} + \tfrac{1}{4}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[X^I, \nabla^\tau X^I, C^\sigma] + \tfrac{\mathrm{i}}{8}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[\bar{\Psi}, \Gamma^\tau \Psi, C^\sigma] = 0$$

$$F_{\mu\nu} - D(C^\lambda, H_{\mu\nu\lambda}) = 0$$

$$\nabla_\mu C^\nu = D(C^\mu, C^\nu) = 0$$

$$D(C^\rho, \nabla_\rho X^I) = D(C^\rho, \nabla_\rho \Psi) = D(C^\rho, \nabla_\rho H_{\mu\nu\lambda}) = 0$$

N Lambert & C Papageorgakis, 1007.2982

Factorization of $C^\rho = C\dot{x}^\rho$. Here, 3-Lie algebra transgression:

$$(\mathcal{T}\omega)_x(v_1(\tau), \ldots, v_k(\tau)) := \int_{S^1} \mathrm{d}\tau\, D(\omega(v_1(\tau), \ldots, v_k(\tau), \dot{x}(\tau)), C)$$

C Papageorgakis & CS, 1103.6192

Often: A vector short of happiness. Loop space has this vector.

Side Remark: Quantization of $\mathbb{R}^3$

In the quantization problem, one is naturally led to loop space.

Geometric quantization prescription: (e.g. fuzzy sphere)

Special symplectic manifold (M,ω) $\rightarrow$ line bundle L with (h,∇) over M $\rightarrow$ Hilbert space $\mathscr{H}$: global holomorphic sections of L

Quantization map: $[\hat{f},\hat{g}] = \mathrm{i}\hbar\widehat{\{f,g\}} + \mathcal{O}(\hbar^2)$

M-theory: 2-plectic manifold (M,ϖ), $\varpi \in \Omega^3(M)$

- hol. secs. of gerbe?, quantization of one-forms? Rogers, ...
- Solution: ω on $\mathcal{L}M$ as $\omega := \mathcal{T}\varpi$, then proceed as above
- Example: $\mathbb{R}^3$ with 2-plectic form $\varpi = \varepsilon_{ijk}\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$:

$$[x^i(\tau), x^j(\sigma)] = \varepsilon^{ijk}\frac{\dot{x}_k(\tau)}{|\dot{x}(\tau)|^2}\delta(\tau-\sigma) + \mathcal{O}(\theta^2)$$

CS & R Szabo, 1211.0395

- Cf. Kawamoto & Sasakura, Bergshoeff, Berman et al. [2000]

The duality D1 $\leftrightarrow$ D3 is a duality between Yang-Mills theories.

Question: In what sense are M2- and M5-brane models related?

Start by looking at gauge structure

Parallel transport of particles in representation of gauge group G:

- holonomy functor: $\mathsf{hol} : \text{path } p \mapsto \mathsf{hol}(p) \in G$
- $\mathsf{hol}(p) = P\exp(\int_p A)$, P: path ordering, trivial for $\mathsf{U}(1)$.

Parallel transport of strings with gauge group $\mathsf{U}(1)$:

- 2-holonomy functor: $\mathsf{hol}_2 : \text{surface } s \mapsto \mathsf{hol}_2(s) \in \mathsf{U}(1)$
- $\mathsf{hol}_2(s) = \exp(\int_s B)$, B: connective structure on gerbe.

Nonabelian case:

- much more involved!
- no straightforward definition of surface ordering
- solution: Categorification!

see Baez, Huerta, 1003.4485

Warning: Categorification neither unique nor straightforward.

Lie 2-group

A Lie 2-group is a

- monoidal category, morph. invertible, obj. weakly invertible.
- Lie groupoid + product $\otimes$ obeying weakly the group axioms.

Simplification: use strict Lie 2-groups $\xleftrightarrow{1:1}$ Lie crossed modules

Lie crossed modules

Pair of Lie groups (G,H), written as $(\mathsf{H}\xrightarrow{\mathsf{t}}\mathsf{G})$ with:

- left automorphism action $\rhd : \mathsf{G}\times\mathsf{H}\to\mathsf{H}$
- group homomorphism $\mathsf{t}:\mathsf{H}\to\mathsf{G}$ such that

$$\mathsf{t}(g\rhd h)=g\mathsf{t}(h)g^{-1}\quad\text{and}\quad \mathsf{t}(h_1)\rhd h_2=h_1h_2h_1^{-1}$$

Also: strict Lie 2-algebras $\xleftrightarrow{1:1}$ differential crossed modules

Lie crossed modules

Pair of Lie groups (G, H), written as $(\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G})$ with:

- left automorphism action $\rhd\colon \mathsf{G} \times \mathsf{H} \to \mathsf{H}$
- group homomorphism $\mathsf{t} : \mathsf{H} \to \mathsf{G}$

$$\mathsf{t}(g \rhd h) = g\mathsf{t}(h)g^{-1} \quad \text{and} \quad \mathsf{t}(h_1) \rhd h_2 = h_1 h_2 h_1^{-1}$$

Simplest examples:

- Lie group G, Lie crossed module: $(1 \xrightarrow{\mathsf{t}} G)$.
- Abelian Lie group G, Lie crossed module: $BG = (G \xrightarrow{\mathsf{t}} 1)$.

More involved:

- Automorphism 2-group of Lie group G: $(G \xrightarrow{\mathsf{t}} \mathsf{Aut}(G))$

Principal 2-Bundles

Higher gauge theory is the dynamical theory of principal 2-bundles.

Consider a manifold M with cover (U_a)

Object	Principal G-bundle	Principal $(\mathsf{H} \xrightarrow{\mathsf{t}} \mathsf{G})$-bundle
Cochains	(g_{ab}) valued in G	(g_{ab}) valued in G, (h_{abc}) valued in H
Cocycle	$g_{ab}g_{bc} = g_{ac}$	$\mathsf{t}(h_{abc})g_{ab}g_{bc} = g_{ac}$ $h_{acd}h_{abc} = h_{abd}(g_{ab} \rhd h_{bcd})$
Coboundary	$g_a g'_{ab} = g_{ab}g_b$	$g_a g'_{ab} = \mathsf{t}(h_{ab})g_{ab}g_b$ $h_{ac}h_{abc} = (g_a \rhd h'_{abc})h_{ab}(g_{ab} \rhd h_{bc})$
gauge pot.	$A_a \in \Omega^1(U_a) \otimes \mathfrak{g}$	$A_a \in \Omega^1(U_a) \otimes \mathfrak{g}$, $B_a \in \Omega^2(U_a) \otimes \mathfrak{h}$
Curvature	$F_a = \mathrm{d}A_a + A_a \wedge A_a$	$F_a = \mathrm{d}A_a + A_a \wedge A_a$, $\quad F_a = \mathsf{t}(B_a)$ $H_a = \mathrm{d}B_a + A_a \rhd B_a$
Gauge trafos	$\tilde{A}_a := g_a^{-1}A_a g_a + g_a^{-1}\mathrm{d}g_a$	$\tilde{A}_a := g_a^{-1}A_a g_a + g_a^{-1}\mathrm{d}g_a + \mathsf{t}(\Lambda_a)$ $\tilde{B}_a := g_a^{-1} \rhd B_a + \tilde{A}_a \rhd \Lambda_a + \mathrm{d}\Lambda_a - \Lambda_a \wedge \Lambda_a$

Remarks:

- A principal $(1 \xrightarrow{\mathsf{t}} \mathsf{G})$-bundle is a principal G-bundle.
- A principal $(\mathsf{U}(1) \xrightarrow{\mathsf{t}} 1) = B\mathsf{U}(1)$-bundle is an abelian gerbe.
- Gauge part of (2,0)-theory: $H = \star H$, $F = \mathsf{t}(B)$.

Is all this machinery really useful/necessary?

Differential Crossed Modules from 3-Algebras

3-algebras are merely special classes of differential crossed modules.

Recall the definition of a 3-algebra $\mathcal{A}$:

- $[\cdot,\cdot,\cdot] : \mathcal{A}^{\otimes 3} \to \mathcal{A}$
- Fundamental identity says that $[a,b,\cdot] \in \mathsf{Der}(\mathcal{A})$, $a,b \in \mathcal{A}$.

Theorem

3-algebras $\overset{1:1}{\longleftrightarrow}$ metric Lie algebras $\mathfrak{g} \cong \mathsf{Der}(\mathcal{A})$
faithful orthog. representations $V \cong \mathcal{A}$

J Figueroa-O'Farrill et al., 0809.1086

Observations

- $\mathfrak{g} \xrightarrow{\mathrm{t}} V$ is a simple differential crossed modules
- M2- and M5-brane models have the same gauge structure.
- Via Faulkner construction, all DCMs come with $[\cdot,\cdot,\cdot]$
- Application of this to M2- and M5-models looks promising.

S Palmer & CS, 1203.5757

3-Lie algebra valued tensor multiplet equations:

$$\nabla^2 X^I - \tfrac{\mathrm{i}}{2}[\bar\Psi, \Gamma_\nu \Gamma^I \Psi, C^\nu] - [X^J, C^\nu, [X^J, C_\nu, X^I]] = 0$$

$$\Gamma^\mu \nabla_\mu \Psi - [X^I, C^\nu, \Gamma_\nu \Gamma^I \Psi] = 0$$

$$\nabla_{[\mu} H_{\nu\lambda\rho]} + \tfrac{1}{4}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[X^I, \nabla^\tau X^I, C^\sigma] + \tfrac{\mathrm{i}}{8}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}[\bar\Psi, \Gamma^\tau \Psi, C^\sigma] = 0$$

$$F_{\mu\nu} - D(C^\lambda, H_{\mu\nu\lambda}) = 0$$

$$\nabla_\mu C^\nu = D(C^\mu, C^\nu) = 0$$

$$D(C^\rho, \nabla_\rho X^I) = D(C^\rho, \nabla_\rho \Psi) = D(C^\rho, \nabla_\rho H_{\mu\nu\lambda}) = 0$$

N Lambert & C Papageorgakis, 1007.2982

Factorization of $C^\rho = C\dot{x}^\rho$. Here, fake curvature equation:

$$\mathsf{t} : \mathcal{A} \to \mathsf{Der}(\mathcal{A}) \ , \ a \mapsto D(C, a) \ , \quad F_{\mu\nu} = \mathsf{t}(H_{\mu\nu\lambda} x^\lambda) =: \mathsf{t}(B)$$

$\Rightarrow$ More natural interpretation as higher gauge theory.

S Palmer & CS, 1203.5757

Division algebras, spheres and groups:

$\mathcal{A}$	$\mathcal{A}P^1$	$\lvert a\rvert = 1$	$\mathsf{Aut}(\mathcal{A})$	Physics
$\mathbb{R}$	$\mathbb{R}P^1 \cong S^1$	$\mathbb{Z}_2 \cong S^0$	$\mathsf{Aut}(\mathbb{R}) \cong 1$	Vortex?
$\mathbb{C}$	$\mathbb{C}P^1 \cong S^2$	$\mathsf{U}(1) \cong S^1$	$\mathsf{Aut}(\mathsf{U}(1)) \cong \mathbb{Z}_2$	Monopole
$\mathbb{H}$	$\mathbb{H}P^1 \cong S^4$	$\mathsf{SU}(2) \cong S^3$	$\mathsf{Aut}(\mathsf{SU}(2)) \cong \mathsf{SU}(2)$	Instanton
$\mathbb{O}$	$\mathbb{O}P^1 \cong S^8$	S^7	$\mathsf{Aut}(\mathbb{O}) \cong \mathsf{G}_2$	?

How should we regard the unit octonions?

- By themselves, they form a Moufang loop ☹
- Better: Use Faulkner construction to get a 3-algebra
  Nambu, Yamazaki, Figueroa-O'Farrill et al.
- Therefore, we have a DCM $(\mathfrak{g}_2 \xrightarrow{\mathsf{t}} \mathbb{R}^8 \cong \mathbb{O})$
- This suggests sequence: $\mathbb{Z}_2$, $\mathsf{U}(1)$, $\mathsf{SU}(2)$, a Lie 2-group ☺
- Not (yet) clear how useful this actually is.

Drop loop spaces: Principal 2-bundles over Twistor Spaces

Now that we saw the power of non-abelian gerbes, let's use them!

Recall the principle of the Penrose-Ward transform:

- Interested in field equations that are equivalent to integrability of connections along subspaces of spacetime M
- Establish a double fibration

$$\begin{tikzcd} & F \arrow[dl] \arrow[dr] & \\ P & & M \end{tikzcd}$$

  P: twistor space, moduli space of subspaces in M
  F: correspondence space

- $H^n(P, \mathfrak{S})$ (e.g. vector bundles) $\xleftrightarrow{1:1}$ sols. to field equations.
- Explicitly appearing: gauge transformations, moduli, symmetries of the equations, etc.
- BTW: here, $\xleftrightarrow{1:1}$ is actually a "holomorphic transgression".

Known Examples of Twistor Descriptions

For Yang-Mills theories and its BPS subsectors, there is a wealth of twistor descriptions.

$$\begin{tikzcd} & \mathbb{C}^4 \times \mathbb{C}P^1 \arrow[dl] \arrow[dr] & \\ \mathbb{C}P^3_\circ & & \mathbb{C}^4 \end{tikzcd}$$

Instantons
hol. vector bundle

$$\begin{tikzcd} & \mathbb{C}^3 \times \mathbb{C}P^1 \arrow[dl] \arrow[dr] & \\ T\mathbb{C}P^1 & & \mathbb{C}^3 \end{tikzcd}$$

Monopoles
hol. vector bundle

$$\begin{tikzcd} & \mathbb{C}^{4|12} \times \mathbb{C}P^1 \times \mathbb{C}P^1 \arrow[dl] \arrow[dr] & \\ P^{5|6} & & \mathbb{C}^{4|12} \end{tikzcd}$$

(Super) Yang-Mills
hol. vector bundle

$$\begin{tikzcd} & \mathbb{C}^6 \times \mathbb{C}P^3 \arrow[dl] \arrow[dr] & \\ P^6 & & \mathbb{C}^6 \end{tikzcd}$$

abelian $H = \star H$
hol. gerbe

Hughston, Murray, Eastwood, CS & M.Wolf, Mason et al.

Note: last twistor space reduces nicely to the above ones.

New Results

New: Penrose-Ward transform for self-dual strings.

New twistor space parameterizing hyperplanes in $\mathbb{C}^4$:

$$\begin{tikzcd} & \mathbb{C}^4 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \arrow[dl] \arrow[dr] & \\ P^3 & & \mathbb{C}^4 \end{tikzcd}$$

self-dual strings
hol. principal 2-bundle

CS & M Wolf, 1111.2539, 1205.3108

Note:

- The Hyperplane twistor space P^3 is the total space of the line bundle $\mathcal{O}(1,1) \to \mathbb{C}P^1 \times \mathbb{C}P^1$.
- The spheres $\mathbb{C}P^1 \times \mathbb{C}P^1$ parameterize an α- and a β-plane.
- The span of both is a hyperplane.
- Nonabelian self-dual string equations: $H = \star \mathrm{d}_A \Phi$, $F = \mathsf{t}(B)$.
- Reduces nicely to the monopole twistor space: $\mathcal{O}(2) \to \mathbb{C}P^1$.

$$
\begin{array}{ccc}
 & \mathbb{C}^{6|16} \times \mathbb{C}P^3 & \\
\swarrow & & \searrow \\
P^{6|4} & & \mathbb{C}^{6|16}
\end{array}
$$

non-abelian self-dual tensor multiplet

hol. principal 2-bundle

CS & M Wolf, 1205.3108

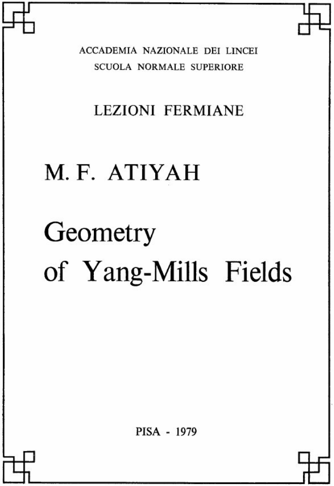
Note:

- $P^{6|4}$ is a straightforward SUSY generalization of P^6
- EOMs, abelian: $H = \star H$, $F = \mathsf{t}(B)$, $\nabla\!\!\!/ \psi = 0$, $\Box\phi = 0$
- $\mathcal{N} = (2,0)$ SC non-abelian tensor multiplet EOMs!
- EOMs on superspace, remain to be boiled down (expected).
- Non-gerby Alternatives: Chu, Samtleben et al., ...

Higher ADHM construction

Recall that the conventional ADHM and ADHMN constructions exist due to a twistor construction in the background.

Thus, there should be a direct ADHM-like construction here, too.



ACCADEMIA NAZIONALE DEI LINCEI
SCUOLA NORMALE SUPERIORE

LEZIONI FERMIANE

M. F. ATIYAH

Geometry
of Yang-Mills Fields

PISA - 1979

Translate this to higher gauge theory:

- Find elementary solutions
- Identify moduli
- Identify topological charges
- Higher Serre-Swan theorem
- Higher ADHM construction

Work in progress

F Sala & S Palmer & CS

Recall the quaternionic form of the elementary instanton on S^4:

Conformal geometry of S^4

Describe S^4 by $\mathbb{H} \cup \{\infty\}$. Coordinates: $x = x^1 + \mathrm{i}x^2 + \mathrm{j}x^3 + \mathrm{k}x^4$.
Conformal transformations:

$$x \mapsto (ax+b)(cx+d)^{-1}\ , \quad a,b,c,d \in \mathbb{H}$$

SU(2)-Instanton:

$$A = \mathrm{im}\left(\frac{\bar{x}\mathrm{d}x}{1+|x|^2}\right) \quad \Rightarrow \quad F = \mathrm{im}\left(\frac{\mathrm{d}\bar{x}\wedge\mathrm{d}x}{(1+|x|^2)^2}\right)$$

SU(2)-Anti-Instanton:

$$A = \mathrm{im}\left(\frac{x\mathrm{d}\bar{x}}{1+|x|^2}\right) \quad \Rightarrow \quad F = \mathrm{im}\left(\frac{\mathrm{d}x\wedge\mathrm{d}\bar{x}}{(1+|x|^2)^2}\right)$$

Belavin et al. 1975, Atiyah 1979

Elementary Solution: The Higher Instanton

The quaternionic form of Belavin et al.'s solution almost translates perfectly.

Issue: $H = \pm \star H$ is sensible only on Minkowski space $\mathbb{R}^{1,5}$.

Recall:

- conformally compactify $\mathbb{R}^4$, $\mathbb{R}^{1,3}$ yields S^4, $M^c \cong S^1 \times S^3$.
- Both S^4 and M^c real slices of $G_{2;4}$, a quadric in $\mathbb{C}P^5$.

General pattern:

Conf. compact. of $\mathbb{R}^{i,n-i} \to \mathbb{C}^n$: real slice of quadric in $\mathbb{C}P^{n+1}$
This illuminates also the conformal transformations:

$$x = x^\mu \gamma_\mu \mapsto (ax+b)(cx+d)^{-1}$$

For certain elements $a, d \in \mathcal{C}\ell_{\mathrm{even}}(\mathbb{C}^n)$, $b, c \in \mathcal{C}\ell_{\mathrm{odd}}(\mathbb{C}^n)$.

Solution: Quaternions have to be regarded as blocks of $\mathcal{C}\ell(\mathbb{C}^4)$
Work with blocks of the Clifford algebra $\mathcal{C}\ell(\mathbb{C}^6)$.

Elementary Solution: The Higher Instanton

The quaternionic form of Belavin et al.'s solution almost translates perfectly.

Solution to the higher instanton equations $H = \star H$, $F = \mathsf{t}(B)$:

- Gauge structure: $(\mathbb{C}^3 \otimes \mathfrak{sl}(4,\mathbb{C}) \xrightarrow{\mathsf{t}} \mathfrak{sl}(4,\mathbb{C}) \oplus \mathfrak{sl}(4,\mathbb{C}))$

$$\mathsf{t} : h = \left(\begin{array}{c|c} h_1 & h_3 \\ \hline 0 & h_2 \end{array}\right) \mapsto \left(\begin{array}{c|c} h_1 & 0 \\ \hline 0 & h_2 \end{array}\right) \in \mathfrak{sl}(4,\mathbb{C}) \oplus \mathfrak{sl}(4,\mathbb{C}) \ ,$$

  $h_1, h_2, h_3 \in \mathfrak{sl}(4,\mathbb{C})$, $\rhd$: the usual commutator.

- Solution in coordinates $x = x^M \sigma_M$, $\hat{x} = x^M \bar{\sigma}_M$

$$A = \begin{pmatrix} \frac{\hat{x}\, \mathrm{d}x}{1+|x|^2} & 0 \\ 0 & \frac{\mathrm{d}x\, \hat{x}}{1+|x|^2} \end{pmatrix} \qquad B = F + \begin{pmatrix} 0 & \frac{\hat{x}\, \mathrm{d}x \wedge \mathrm{d}\hat{x}}{(1+|x|^2)^2} \\ 0 & 0 \end{pmatrix}$$

$$F := \mathrm{d}A + A \wedge A = \begin{pmatrix} \frac{\mathrm{d}\hat{x} \wedge \mathrm{d}x}{(1+|x|^2)^2} + \frac{2\, \mathrm{d}\hat{x}\, x \wedge \mathrm{d}\hat{x}\, x}{(1+|x|^2)^2} & 0 \\ 0 & -\frac{\mathrm{d}x \wedge \mathrm{d}\hat{x}}{(1+|x|^2)^2} \end{pmatrix}$$

$$H := \mathrm{d}B + A \rhd B = \begin{pmatrix} 0 & \frac{\mathrm{d}\hat{x} \wedge \mathrm{d}x \wedge \mathrm{d}\hat{x}}{(1+|x|^2)^3} \\ 0 & 0 \end{pmatrix}$$

but: Peiffer violated

F Sala & S Palmer & CS

Summary:

- ✓ Generalized ADHMN-like construction on loop space
- ✓ Geometric quantization using loop space
- ✓ Gauge structures in M2- and M5-brane models similar
- ✓ Twistor construction of self-dual tensor fields
- ✓ 6d superconformal tensor multiplet equations
- ✓ On our way to develop Geometry of Higher Yang-Mills Fields

Future directions:

- ▷ More general higher bundles and twistors — with M Wolf
- ▷ Continue translation of ADHM — with S Palmer, F Sala
- ▷ Geometric Quant. with higher Hilbert spaces — with R Szabo

Geometry of Higher Yang-Mills Fields

Christian Sämann

School of Mathematical and Computer Sciences
Heriot-Watt University, Edinburgh

Edinburgh Mathematical Physics Group, 23.1.2013