Unification of Type IIA and IIB Supergravities : $\mathcal{N} = 2$ $D = 10$ *Supersymmetric Double Field Theory*

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Edinburgh 25th September 2013

Prologue

- Without vector notation, Maxwell's original equations consisted of eight (or twenty) formulas.
- It was the rotational $\mathbf{SO}(3)$ or Lorentz $\mathbf{SO}(1,3)$ symmetry that reorganized them into four or two compact equations.
- This talk aims to show that Type IIA & IIB supergravities may undergo an analogous 'simplification' and 'unification' with renewed understanding of T-duality, in the name of $\mathcal{N}=2$ $D=10$ **Supersymmetric Double Field Theory.**

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Talk based on works with Imtak Jeon & Kanghoon Lee

- Differential geometry with a projection: Application to double field theory
  JHEP 1104:014 arXiv:1011.1324
- Double field formulation of Yang-Mills theory PLB 701:260(2011) arXiv:1102.0419
- Stringy differential geometry, beyond Riemann PRD 84:044022(2011) arXiv:1105.6294
- Incorporation of fermions into double field theory JHEP 1111:025 arXiv:1109.2035
- Supersymmetric Double Field Theory: Stringy Reformulation of Supergravity
  PRD Rapid Comm. 85:081501 (2012) arXiv:1112.0069
- Ramond-Ramond Cohomology and O(D,D) T-duality JHEP 1209:079 arXiv:1206.3478
- **Stringy Unification of Type IIA and IIB Supergravities under $\mathcal{N} = 2$ $D = 10$ Supersymmetric Double Field Theory** PLB 723:245(2013) arXiv:1210.5078
- U-geometry : SL(5) with Yoonji Suh JHEP 1304:147 arXiv:1302.1652
- Comments on double field theory and diffeomorphisms JHEP 1306:098 arXiv:1304.5946
- Covariant action for a string in doubled yet gauged spacetime arXiv:1307.8377

- In Riemannian geometry, the fundamental object is the metric, $g_{\mu\nu}$.
  - Diffeomorphism: $\partial_\mu \longrightarrow \nabla_\mu = \partial_\mu + \Gamma_\mu$
  - $\nabla_\lambda g_{\mu\nu} = 0,\ \Gamma^\lambda_{[\mu\nu]} = 0 \longrightarrow \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$
  - Curvature: $[\nabla_\mu, \nabla_\nu] \longrightarrow R_{\kappa\lambda\mu\nu} \longrightarrow R$
- On the other hand, string theory puts $g_{\mu\nu}$, $B_{\mu\nu}$ and ϕ on an equal footing, as they form a multiplet of T-duality.
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Closed string

- The low energy effective action of $g_{\mu\nu}$, $B_{\mu\nu}$, ϕ is well known in terms of Riemannian geometry

$$S_{\text{eff.}} = \int_{\Sigma_D} \sqrt{-g}e^{-2\phi}\left(R_g + 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu}\right).$$

- Diffeomorphism and B-field gauge symmetry are manifest,

$$x^\mu \rightarrow x^\mu + \delta x^\mu\,, \qquad B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu\,.$$

- Though not manifest, this enjoys T-duality which mixes $\{g_{\mu\nu}, B_{\mu\nu}, \phi\}$. Buscher

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T-duality

- Redefine the dilaton,

$$e^{-2d} = \sqrt{-g}e^{-2\phi}$$

- Set a $(D+D)\times(D+D)$ symmetric matrix, Duff

$$\mathcal{H}_{AB} = \begin{pmatrix} g^{-1} & -g^{-1}B \\ Bg^{-1} & g - Bg^{-1}B \end{pmatrix}$$

- Hereafter, $A, B, ...$: 'doubled' $(D+D)$-dimensional vector indices, with $D = 10$ for SUSY.

- T-duality is realized by an $\mathbf{O}(D,D)$ rotation, Tseytlin, Siegel

$$\mathcal{H}_{AB} \longrightarrow M_A{}^C M_B{}^D \mathcal{H}_{CD}\,, \qquad\qquad d \longrightarrow d\,,$$

where

$$M \in \mathbf{O}(D,D)\,.$$

T-duality

- $\mathbf{O}(D, D)$ metric,

$$\mathcal{J}_{AB} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

freely raises or lowers the $(D + D)$-dimensional vector indices.

Double Field Theory (DFT)

- Hull and Zwiebach , later with Hohm

$$S_{\mathrm{DFT}} = \int_{\Sigma_D} e^{-2d}\, L_{\mathrm{DFT}}(\mathcal{H}, d)\,,$$

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$$\begin{aligned} L_{\mathrm{DFT}}(\mathcal{H}, d) = \quad & \mathcal{H}^{AB}\left(4\partial_A\partial_B d - 4\partial_A d\,\partial_B d + \tfrac{1}{8}\partial_A\mathcal{H}^{CD}\partial_B\mathcal{H}_{CD} - \tfrac{1}{2}\partial_A\mathcal{H}^{CD}\partial_C\mathcal{H}_{BD}\right) \\ & +4\partial_A\mathcal{H}^{AB}\partial_B d - \partial_A\partial_B\mathcal{H}^{AB}\,. \end{aligned}$$

- Spacetime is formally doubled, $y^A = (\tilde{x}_\mu, x^\nu)$, $A = 1, 2, \cdots, D{+}D$.
- Yet, Double Field Theory (for NS-NS sector) is a D-dimensional theory written in terms of $(D+D)$-dimensional language, i.e. tensors.
- All the fields MUST live on a D-dimensional null hyperplane or 'section', Σ_D.

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Section condition in Double Field Theory

- By stating DFT lives on a D-dimensional null hyperplane, we mean that, the $\mathbf{O}(D,D)$ d'Alembert operator is trivial, acting on arbitrary fields as well as their products:

$$\partial_A\partial^A\Phi = 2\frac{\partial^2}{\partial\tilde{x}_\mu\partial x^\mu}\Phi \equiv 0\,, \qquad \partial_A\Phi_1\partial^A\Phi_2 \equiv 0 \quad : \quad \text{section condition}$$

What does $\mathbf{O}(D, D)$ do in Double Field Theory?

- $\mathrm{O}(D, D)$ rotates the D-dimensional null hyperplane where DFT lives.
- *A priori*, the $\mathrm{O}(D, D)$ structure in DFT is a 'meta-symmetry' or 'hidden symmetry' rather than a Noether symmetry.
- Only after dimensional reductions,

$$D = d + n \Longrightarrow d\,,$$

it can generate a Noether symmetry,

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String worldsheet origin of the section condition

- Closed string

$$X_L(\sigma^+) = \tfrac{1}{2}(x+\tilde{x}) + \tfrac{1}{2}(p+w)\sigma^+ + \cdots\,,$$

$$X_R(\sigma^-) = \tfrac{1}{2}(x-\tilde{x}) + \tfrac{1}{2}(p-w)\sigma^- + \cdots\,.$$

- Under T-duality,

$$X_L + X_R \quad\longrightarrow\quad X_L - X_R\,,$$

such that

$$(x,\tilde{x},p,w) \quad\longrightarrow\quad (\tilde{x},x,w,p)\,.$$

- Level matching condition for the massless sector,

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- Up to $\mathbf{O}(D,D)$ rotation, we may further choose to set

$$\frac{\partial}{\partial \tilde{x}_\mu} \equiv 0\,.$$

- Then DFT reduces to the effective action:

$$S_{\rm DFT} \Longrightarrow S_{\rm eff.} = \int_{\Sigma_D} \sqrt{-g}e^{-2\phi}\left(R_g + 4(\partial\phi)^2 - \tfrac{1}{12}H^2\right).$$

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Diffeomorphism & B-field gauge symmetry

- Introducing a unifying $(D+D)$-dimensional parameter,

$$X^{A}=(\Lambda_{\mu},\delta x^{\nu})$$

it is possible to spell a unifying transformation rule, up to the section condition,

$$\delta_{X}\mathcal{H}_{AB}\equiv X^{C}\partial_{C}\mathcal{H}_{AB}+2\partial_{[A}X_{C]}\mathcal{H}^{C}{}_{B}+2\partial_{[B}X_{C]}\mathcal{H}_{A}{}^{C}\,,$$

$$\delta_{X}\left(e^{-2d}\right)\equiv\partial_{A}\left(X^{A}e^{-2d}\right)\,.$$

- In fact, these coincide with the generalized Lie derivative,

$$\delta_{X}\mathcal{H}_{AB}=\hat{\mathcal{L}}_{X}\mathcal{H}_{AB}\,,\qquad\qquad \delta_{X}(e^{-2d})=\hat{\mathcal{L}}_{X}(e^{-2d})=-2(\hat{\mathcal{L}}_{X}d)e^{-2d}\,.$$

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Generalized Lie derivative

- **Definition** Siegel, Courant, Grana ...

$$\hat{\mathcal{L}}_X T_{A_1\cdots A_n} := X^B\partial_B T_{A_1\cdots A_n} + \omega\partial_B X^B T_{A_1\cdots A_n} + \sum_{i=1}^{n}(\partial_{A_i}X_B - \partial_B X_{A_i})T_{A_1\cdots A_{i-1}}{}^{B}{}_{A_{i+1}\cdots A_n}.$$

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Diffeomorphism & B-field gauge symmetry

- Brute force computation can show that

$$\hat{\mathcal{L}}_X \mathcal{H}_{AB} \equiv X^C \partial_C \mathcal{H}_{AB} + 2\partial_{[A} X_{C]} \mathcal{H}^C{}_B + 2\partial_{[B} X_{C]} \mathcal{H}_A{}^C \,,$$

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$$\hat{\mathcal{L}}_X \mathcal{H}_{AB} \equiv X^C \partial_C \mathcal{H}_{AB} + 2\partial_{[A} X_{C]} \mathcal{H}^C{}_B + 2\partial_{[B} X_{C]} \mathcal{H}_A{}^C \,,$$

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are symmetry of the action by Hull, Zwiebach & Hohm

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- **Proposal of a underlying stringy differential geometry for DFT**
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- **O**(D,D) **T-duality:** ***Meta-symmetry***

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  2. **A *pair* of local Lorentz symmetries, Spin**$(1,D-1)_L \times$ **Spin**$(D-1,1)_R$
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which implies an invariance under a shift set by a 'derivative-index-valued' vector,

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- **The section condition implies, and in fact can be shown to be equivalent to, an equivalence relation for the coordinates ,**

$$x^A \sim x^A + \varphi\partial^A\varphi' \quad : \quad \text{'Coordinate Gauge Symmetry'}$$

- A 'physical point' is one-to-one identified with a 'gauge orbit' in the coordinate space.
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- **Bosons**
  - **NS-NS sector** $\begin{cases} \textbf{DFT-dilaton:} & d \\ \textbf{DFT-vielbeins:} & V_{Ap}\,, \quad \bar{V}_{A\bar{p}} \end{cases}$
  - **R-R potential:** $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$
- **Fermions**
  - **DFT-dilatinos:** $\rho^{\alpha}\,, \quad \rho'^{\bar{\alpha}}$
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Index	Representation	Metric (raising/lowering indices)
$A, B, \cdots$	**O**(D, D) **& DFT-diffeom. vector**	$\mathcal{J}_{AB}$
$p, q, \cdots$	**Spin**$(1, D-1)_L$ **vector**	$\eta_{pq} = \textbf{diag}(-++\cdots+)$
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R-R potential and Fermions carry NOT $(D+D)$-dimensional BUT undoubled D-dimensional indices.

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A priori, **O**(D,D) **rotates only the O**(D,D) **vector indices (capital Roman), and the R-R sector and all the fermions are O**(D,D) **T-duality singlet.**

The usual IIA $\Leftrightarrow$ IIB exchange will follow only after fixing a gauge.

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- Hence, without loss of generality, we may safely set

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- The DFT-vielbeins generate **a pair of rank-two projectors,**

$$P_{AB} := V_{A}{}^{p} V_{Bp}\,, \qquad P_{A}{}^{B} P_{B}{}^{C} = P_{A}{}^{C}\,, \qquad \bar{P}_{AB} := \bar{V}_{A}{}^{\bar{p}} \bar{V}_{B\bar{p}}\,, \qquad \bar{P}_{A}{}^{B} \bar{P}_{B}{}^{C} = \bar{P}_{A}{}^{C}\,,$$

which are symmetric, orthogonal and complementary to each other,

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$$P_{A}{}^{B} V_{Bp} = V_{Ap}\,, \qquad \bar{P}_{A}{}^{B} \bar{V}_{B\bar{p}} = \bar{V}_{A\bar{p}}\,, \qquad \bar{P}_{A}{}^{B} V_{Bp} = 0\,, \qquad P_{A}{}^{B} \bar{V}_{B\bar{p}} = 0\,.$$

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- Further, we construct **a pair of rank-six projectors,**

$$\mathcal{P}_{CAB}{}^{DEF} := P_C{}^D P_{[A}{}^{[E} P_{B]}{}^{F]} + \tfrac{2}{D-1} P_{C[A} P_{B]}{}^{[E} P^{F]D} \,, \qquad \mathcal{P}_{CAB}{}^{DEF} \mathcal{P}_{DEF}{}^{GHI} = \mathcal{P}_{CAB}{}^{GHI} \,,$$

$$\bar{\mathcal{P}}_{CAB}{}^{DEF} := \bar{P}_C{}^D \bar{P}_{[A}{}^{[E} \bar{P}_{B]}{}^{F]} + \tfrac{2}{D-1} \bar{P}_{C[A} \bar{P}_{B]}{}^{[E} \bar{P}^{F]D} \,, \qquad \bar{\mathcal{P}}_{CAB}{}^{DEF} \bar{\mathcal{P}}_{DEF}{}^{GHI} = \bar{\mathcal{P}}_{CAB}{}^{GHI} \,,$$

which are symmetric and traceless,

$$\mathcal{P}_{CABDEF} = \mathcal{P}_{DEFCAB} = \mathcal{P}_{C[AB]D[EF]} \,, \qquad \bar{\mathcal{P}}_{CABDEF} = \bar{\mathcal{P}}_{DEFCAB} = \bar{\mathcal{P}}_{C[AB]D[EF]} \,,$$

$$\mathcal{P}^A{}_{ABDEF} = 0 \,, \quad P^{AB} \mathcal{P}_{ABCDEF} = 0 \,, \qquad \bar{\mathcal{P}}^A{}_{ABDEF} = 0 \,, \quad \bar{P}^{AB} \bar{\mathcal{P}}_{ABCDEF} = 0 \,.$$

- **Having all the 'right' field-variables prepared, we now discuss their derivatives or what we call, 'semi-covariant derivative'.**

Semi-covariant derivatives

- For each gauge symmetry we assign a corresponding connection,
  - Γ_A for the DFT-diffeomorphism (generalized Lie derivative),
  - Φ_A for the 'unbarred' local Lorentz symmetry, $\mathbf{Spin}(1,D{-}1)_L$,
  - $\bar{\Phi}_A$ for the 'barred' local Lorentz symmetry, $\mathbf{Spin}(D{-}1,1)_R$.
- Combining all of them, we introduce **master 'semi-covariant' derivative**

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- It is also useful to set

$$\nabla_A = \partial_A + \Gamma_A \,, \qquad\qquad D_A = \partial_A + \Phi_A + \bar{\Phi}_A \,.$$

- The former is the 'semi-covariant' derivative for the DFT-diffeomorphism (set by the generalized Lie derivative),

$$\nabla_C T_{A_1 A_2 \cdots A_n} := \partial_C T_{A_1 A_2 \cdots A_n} - \omega \Gamma^B{}_{BC} T_{A_1 A_2 \cdots A_n} + \sum_{i=1}^{n} \Gamma_{CA_i}{}^B T_{A_1 \cdots A_{i-1} B A_{i+1} \cdots A_n} \,.$$

- And the latter is the covariant derivative for the $\mathbf{Spin}(1, D-1)_L \times \mathbf{Spin}(D-1, 1)_R$ local Lorenz symmetries.

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- ***By definition,*** **the master derivative annihilates all the 'constants',**

$$\mathcal{D}_A\mathcal{J}_{BC}=\nabla_A\mathcal{J}_{BC}=\Gamma_{AB}{}^D\mathcal{J}_{DC}+\Gamma_{AC}{}^D\mathcal{J}_{BD}=0\,,$$

$$\mathcal{D}_A\eta_{pq}=D_A\eta_{pq}=\Phi_{Ap}{}^r\eta_{rq}+\Phi_{Aq}{}^r\eta_{pr}=0\,,$$

$$\mathcal{D}_A\bar{\eta}_{\bar{p}\bar{q}}=D_A\bar{\eta}_{\bar{p}\bar{q}}=\bar{\Phi}_{A\bar{p}}{}^{\bar{r}}\bar{\eta}_{\bar{r}\bar{q}}+\bar{\Phi}_{A\bar{q}}{}^{\bar{r}}\bar{\eta}_{\bar{p}\bar{r}}=0\,,$$

$$\mathcal{D}_AC_{+\alpha\beta}=D_AC_{+\alpha\beta}=\Phi_{A\alpha}{}^{\delta}C_{+\delta\beta}+\Phi_{A\beta}{}^{\delta}C_{+\alpha\delta}=0\,,$$

$$\mathcal{D}_A\bar{C}_{+\bar{\alpha}\bar{\beta}}=D_A\bar{C}_{+\bar{\alpha}\bar{\beta}}=\bar{\Phi}_{A\bar{\alpha}}{}^{\bar{\delta}}\bar{C}_{+\bar{\delta}\bar{\beta}}+\bar{\Phi}_{A\bar{\beta}}{}^{\bar{\delta}}\bar{C}_{+\bar{\alpha}\bar{\delta}}=0\,,$$

including the gamma matrices,

$$\mathcal{D}_A(\gamma^p)^\alpha{}_\beta=D_A(\gamma^p)^\alpha{}_\beta=\Phi_A{}^p{}_q(\gamma^q)^\alpha{}_\beta+\Phi_A{}^\alpha{}_\delta(\gamma^p)^\delta{}_\beta-(\gamma^p)^\alpha{}_\delta\Phi_A{}^\delta{}_\beta=0\,,$$

$$\mathcal{D}_A(\bar{\gamma}^{\bar{p}})^{\bar{\alpha}}{}_{\bar{\beta}}=D_A(\bar{\gamma}^{\bar{p}})^{\bar{\alpha}}{}_{\bar{\beta}}=\bar{\Phi}_A{}^{\bar{p}}{}_{\bar{q}}(\bar{\gamma}^{\bar{q}})^{\bar{\alpha}}{}_{\bar{\beta}}+\bar{\Phi}_A{}^{\bar{\alpha}}{}_{\bar{\delta}}(\bar{\gamma}^{\bar{p}})^{\bar{\delta}}{}_{\bar{\beta}}-(\bar{\gamma}^{\bar{p}})^{\bar{\alpha}}{}_{\bar{\delta}}\bar{\Phi}_A{}^{\bar{\delta}}{}_{\bar{\beta}}=0\,.$$

- It follows then that the connections are all anti-symmetric,

$$\Gamma_{ABC} = -\Gamma_{ACB}\,,$$

$$\Phi_{Apq} = -\Phi_{Aqp}\,, \qquad \Phi_{A\alpha\beta} = -\Phi_{A\beta\alpha}\,,$$

$$\bar{\Phi}_{A\bar{p}\bar{q}} = -\bar{\Phi}_{A\bar{q}\bar{p}}\,, \qquad \bar{\Phi}_{A\bar{\alpha}\bar{\beta}} = -\bar{\Phi}_{A\bar{\beta}\bar{\alpha}}\,,$$

and as usual,

$$\Phi_{A}{}^{\alpha}{}_{\beta} = \tfrac{1}{4}\Phi_{Apq}(\gamma^{pq})^{\alpha}{}_{\beta}\,, \qquad \bar{\Phi}_{A}{}^{\bar{\alpha}}{}_{\bar{\beta}} = \tfrac{1}{4}\bar{\Phi}_{A\bar{p}\bar{q}}(\bar{\gamma}^{\bar{p}\bar{q}})^{\bar{\alpha}}{}_{\bar{\beta}}\,.$$

- ***Further,*** **the master derivative is compatible with the whole NS-NS sector,**

$$\mathcal{D}_A d = \nabla_A d := -\tfrac{1}{2}e^{2d}\nabla_A(e^{-2d}) = \partial_A d + \tfrac{1}{2}\Gamma^B{}_{BA} = 0\,,$$

$$\mathcal{D}_A V_{Bp} = \partial_A V_{Bp} + \Gamma_{AB}{}^C V_{Cp} + \Phi_{Ap}{}^q V_{Bq} = 0\,,$$

$$\mathcal{D}_A \bar{V}_{B\bar{p}} = \partial_A \bar{V}_{B\bar{p}} + \Gamma_{AB}{}^C \bar{V}_{C\bar{p}} + \bar{\Phi}_{A\bar{p}}{}^{\bar{q}} \bar{V}_{B\bar{q}} = 0\,.$$

- It follows that

$$\mathcal{D}_A P_{BC} = \nabla_A P_{BC} = 0\,, \qquad \mathcal{D}_A \bar{P}_{BC} = \nabla_A \bar{P}_{BC} = 0\,,$$

and the connections are related to each other,

$$\Gamma_{ABC} = V_B{}^p D_A V_{Cp} + \bar{V}_B{}^{\bar{p}} D_A \bar{V}_{C\bar{p}}\,,$$

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- **The connections assume the following most general forms:**

$$\Gamma_{CAB} = \Gamma^0_{CAB} + \Delta_{Cpq} V_A{}^p V_B{}^q + \bar{\Delta}_{C\bar{p}\bar{q}} \bar{V}_A{}^{\bar{p}} \bar{V}_B{}^{\bar{q}} ,$$

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Here

$$\begin{aligned}\Gamma^0_{CAB} = \;& 2\,(P\partial_C P\bar{P})_{[AB]} + 2\,(\bar{P}_{[A}{}^D \bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E)\,\partial_D P_{EC} \\ & - \tfrac{4}{D-1}\,(\bar{P}_{C[A}\bar{P}_{B]}{}^D + P_{C[A}P_{B]}{}^D)\,(\partial_D d + (P\partial^E P\bar{P})_{[ED]}) \,,\end{aligned}$$

and, with the corresponding derivative, $\nabla^0_A = \partial_A + \Gamma^0_A$,

$$\Phi^0_{Apq} = V^B{}_p \nabla^0_A V_{Bq} = V^B{}_p \partial_A V_{Bq} + \Gamma^0_{ABC} V^B{}_p V^C{}_q ,$$

$$\bar{\Phi}^0_{A\bar{p}\bar{q}} = \bar{V}^B{}_{\bar{p}} \nabla^0_A \bar{V}_{B\bar{q}} = \bar{V}^B{}_{\bar{p}} \partial_A \bar{V}_{B\bar{q}} + \Gamma^0_{ABC} \bar{V}^B{}_{\bar{p}} \bar{V}^C{}_{\bar{q}} .$$

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- Further, the extra pieces, Δ_{Apq} and $\bar{\Delta}_{A\bar{p}\bar{q}}$, correspond to the **torsion** of SDFT, which must be covariant and, in order to maintain $\mathcal{D}_{A}d = 0$, must satisfy

$$\Delta_{Apq}V^{Ap} = 0\,, \qquad \bar{\Delta}_{A\bar{p}\bar{q}}\bar{V}^{A\bar{p}} = 0\,.$$

  Otherwise they are arbitrary.

- As in SUGRA, the torsion can be constructed from the bi-spinorial objects, e.g.

$$\bar{\rho}\gamma_{pq}\psi_{A}\,, \qquad \bar{\psi}_{\bar{p}}\gamma_{A}\psi_{\bar{q}}\,, \qquad \bar{\rho}\gamma_{Apq}\rho\,, \qquad \bar{\psi}_{\bar{p}}\gamma_{Apq}\psi^{\bar{p}}\,,$$

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  where we set $\psi_A = \bar{V}_A{}^{\bar{p}}\psi_{\bar{p}}$, $\gamma_A = V_A{}^p\gamma_p$.

- The 'torsionless' connection,

$$\begin{aligned}\Gamma^0_{CAB} = \quad & 2\left(P\partial_C P\bar{P}\right)_{[AB]} + 2\left(\bar{P}_{[A}{}^D\bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E\right)\partial_D P_{EC} \\ & -\tfrac{4}{D-1}\left(\bar{P}_{C[A}\bar{P}_{B]}{}^D + P_{C[A}P_{B]}{}^D\right)\left(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\right) ,\end{aligned}$$

further obeys

$$\Gamma^0_{ABC} + \Gamma^0_{BCA} + \Gamma^0_{CAB} = 0 ,$$

and

$$\mathcal{P}_{CAB}{}^{DEF}\Gamma^0_{DEF} = 0 , \qquad \bar{\mathcal{P}}_{CAB}{}^{DEF}\Gamma^0_{DEF} = 0 .$$

- **In fact, the torsionless connection,**

$$\begin{aligned}\Gamma^0_{CAB} = \quad & 2\left(P\partial_C P\bar{P}\right)_{[AB]} + 2\left(\bar{P}_{[A}{}^D\bar{P}_{B]}{}^E - P_{[A}{}^D P_{B]}{}^E\right)\partial_D P_{EC} \\ & -\tfrac{4}{D-1}\left(\bar{P}_{C[A}\bar{P}_{B]}{}^D + P_{C[A}P_{B]}{}^D\right)\left(\partial_D d + (P\partial^E P\bar{P})_{[ED]}\right)\,,\end{aligned}$$

is uniquely determined by requiring

$$\nabla_A \mathcal{J}_{BC} = 0 \iff \Gamma_{CAB} + \Gamma_{CBA} = 0\,,$$

$$\nabla_A P_{BC} = 0\,,$$

$$\nabla_A d = 0\,,$$

$$\Gamma_{ABC} + \Gamma_{CAB} + \Gamma_{BCA} = 0\,,$$

$$(\mathcal{P} + \bar{\mathcal{P}})_{CAB}{}^{DEF}\,\Gamma_{DEF} = 0\,.$$

- Having the two symmetric properties, $\Gamma_{A(BC)}=0$, $\Gamma_{[ABC]}=0$, we may safely replace ∂_A by $\nabla^0_A=\partial_A+\Gamma^0_A$ in $\hat{\mathcal{L}}_X$ and also in $[X,Y]^A_{\mathbf{C}}$,

$$\hat{\mathcal{L}}_X T_{A_1\cdots A_n}=X^B\nabla^0_B T_{A_1\cdots A_n}+\omega\nabla^0_B X^B T_{A_1\cdots A_n}+\textstyle\sum_{i=1}^n(\nabla^0_{A_i}X_B-\nabla^0_B X_{A_i})T_{A_1\cdots A_{i-1}}{}^B{}_{A_{i+1}\cdots A_n}\,,$$

$$[X,Y]^A_{\mathbf{C}}=X^B\nabla^0_B Y^A-Y^B\nabla^0_B X^A+\tfrac{1}{2}Y^B\nabla^{0A}X_B-\tfrac{1}{2}X^B\nabla^{0A}Y_B\,,$$

just like in Riemannian geometry.

- In this way, Γ^0_{ABC} is the **DFT analogy of the Christoffel connection.**

- Precisely the same expression was later re-derived by Hohm & Zwiebach.

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Semi-covariant curvature

- The usual curvatures for the three connections,

$$R_{CDAB} = \partial_A \Gamma_{BCD} - \partial_B \Gamma_{ACD} + \Gamma_{AC}{}^E \Gamma_{BED} - \Gamma_{BC}{}^E \Gamma_{AED} \,,$$

$$F_{ABpq} = \partial_A \Phi_{Bpq} - \partial_B \Phi_{Apq} + \Phi_{Apr} \Phi_B{}^r{}_q - \Phi_{Bpr} \Phi_A{}^r{}_q \,,$$

$$\bar{F}_{AB\bar{p}\bar{q}} = \partial_A \bar{\Phi}_{B\bar{p}\bar{q}} - \partial_B \bar{\Phi}_{A\bar{p}\bar{q}} + \bar{\Phi}_{A\bar{p}\bar{r}} \bar{\Phi}_B{}^{\bar{r}}{}_{\bar{q}} - \bar{\Phi}_{B\bar{p}\bar{r}} \bar{\Phi}_A{}^{\bar{r}}{}_{\bar{q}} \,,$$

are, from $[\mathcal{D}_A, \mathcal{D}_B] V_{Cp} = 0$ and $[\mathcal{D}_A, \mathcal{D}_B] \bar{V}_{C\bar{p}} = 0$, related to each other,

$$R_{ABCD} = F_{CDpq} V_A{}^p V_B{}^q + \bar{F}_{CD\bar{p}\bar{q}} \bar{V}_A{}^{\bar{p}} \bar{V}_B{}^{\bar{q}} \,.$$

- **However, the crucial object in DFT turns out to be**

$$S_{ABCD} := \tfrac{1}{2} \Big(R_{ABCD} + R_{CDAB} - \Gamma^E{}_{AB} \Gamma_{ECD} \Big) \,,$$

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Properties of the semi-covariant curvature

- Precisely the same symmetric property as the Riemann curvature,

$$S_{ABCD}=\tfrac{1}{2}\left(S_{[AB][CD]}+S_{[CD][AB]}\right)\,,$$

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$$P_{I}{}^{A}\bar{P}_{J}{}^{B}P_{K}{}^{C}\bar{P}_{L}{}^{D}S_{ABCD}\equiv 0\,.$$

- Under arbitrary variation of the connection, $\delta\Gamma_{ABC}$, it transforms as

$$\delta S_{ABCD}=\mathcal{D}_{[A}\delta\Gamma_{B]CD}+\mathcal{D}_{[C}\delta\Gamma_{D]AB}-\tfrac{3}{2}\Gamma_{[ABE]}\delta\Gamma^{E}{}_{CD}-\tfrac{3}{2}\Gamma_{[CDE]}\delta\Gamma^{E}{}_{AB}\,,$$

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- Generically, under $\delta_X P_{AB} = \hat{\mathcal{L}}_X P_{AB}$, $\delta_X d = \hat{\mathcal{L}}_X d$ (DFT-diffeomorphism), the variation of $\nabla_C T_{A_1 \cdots A_n}$ contains an anomalous non-covariant part,

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- Hence, it is not DFT-diffeomorphism covariant,

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- However, the characteristic property of our 'semi-covariant' derivative is that, **combined with the projectors it can generate various fully covariant quantities,** as listed below.

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- **For $\mathbf{O}(D,D)$ tensors:**

$$P_{C}{}^{D}\bar{P}_{A_1}{}^{B_1}\bar{P}_{A_2}{}^{B_2}\cdots\bar{P}_{A_n}{}^{B_n}\nabla_{D}T_{B_1B_2\cdots B_n}\,,$$

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$$\left.\begin{array}{l}P^{AB}\bar{P}_{C_1}{}^{D_1}\bar{P}_{C_2}{}^{D_2}\cdots\bar{P}_{C_n}{}^{D_n}\nabla_{A}T_{BD_1D_2\cdots D_n}\,,\\[2mm]\bar{P}^{AB}P_{C_1}{}^{D_1}P_{C_2}{}^{D_2}\cdots P_{C_n}{}^{D_n}\nabla_{A}T_{BD_1D_2\cdots D_n}\end{array}\right\}\quad\text{Divergences}\,,$$

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Projector-aided, fully covariant derivatives

- **For Spin$(1,D{-}1)_L \times$ Spin$(D{-}1,1)_R$ tensors:**

$$\mathcal{D}_p T_{\bar{q}_1\bar{q}_2\cdots\bar{q}_n}\,, \qquad \mathcal{D}_{\bar{p}} T_{q_1q_2\cdots q_n}\,,$$
$$\mathcal{D}^p T_{p\bar{q}_1\bar{q}_2\cdots\bar{q}_n}\,, \qquad \mathcal{D}^{\bar{p}} T_{\bar{p}q_1q_2\cdots q_n}\,,$$
$$\mathcal{D}_p\mathcal{D}^p T_{\bar{q}_1\bar{q}_2\cdots\bar{q}_n}\,, \qquad \mathcal{D}_{\bar{p}}\mathcal{D}^{\bar{p}} T_{q_1q_2\cdots q_n}\,,$$

where we set

$$\mathcal{D}_p := V^A{}_p\mathcal{D}_A\,, \qquad \mathcal{D}_{\bar{p}} := \bar{V}^A{}_{\bar{p}}\mathcal{D}_A\,.$$

These are the pull-back of the previous results using the DFT-vielbeins.

Projector-aided, fully covariant derivatives

- **Dirac operators for fermions,** $\rho^{\alpha},\ \psi^{\alpha}_{\bar{p}},\ \rho'^{\bar{\alpha}},\ \psi'^{\bar{\alpha}}_{p}$:

$$\gamma^{p}\mathcal{D}_{p}\rho=\gamma^{A}\mathcal{D}_{A}\rho\,,\qquad\qquad \gamma^{p}\mathcal{D}_{p}\psi_{\bar{p}}=\gamma^{A}\mathcal{D}_{A}\psi_{\bar{p}}\,,$$

$$\mathcal{D}_{\bar{p}}\rho\,,\qquad\qquad \mathcal{D}_{\bar{p}}\psi^{\bar{p}}=\mathcal{D}_{A}\psi^{A}\,,$$

$$\bar{\psi}^{A}\gamma_{p}(\mathcal{D}_{A}\psi_{\bar{q}}-\tfrac{1}{2}\mathcal{D}_{\bar{q}}\psi_{A})\,,$$

$$\bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\rho'=\bar{\gamma}^{A}\mathcal{D}_{A}\rho'\,,\qquad\qquad \bar{\gamma}^{\bar{p}}\mathcal{D}_{\bar{p}}\psi'_{p}=\bar{\gamma}^{A}\mathcal{D}_{A}\psi'_{p}\,,$$

$$\mathcal{D}_{p}\rho'\,,\qquad\qquad \mathcal{D}_{p}\psi'^{p}=\mathcal{D}_{A}\psi'^{A}\,,$$

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Incorporation of fermions into DFT   1109.2035

Projector-aided, fully covariant derivatives

- **For Spin$(1,D{-}1)_L \times$ Spin$(D{-}1,1)_R$ bi-fundamental spinorial fields,** $\mathcal{T}^{\alpha}{}_{\bar{\beta}}$ **:**

$$\mathcal{D}_{+}\mathcal{T} := \gamma^{A}\mathcal{D}_{A}\mathcal{T} + \gamma^{(D+1)}\mathcal{D}_{A}\mathcal{T}\bar{\gamma}^{A}\,,$$

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- Especially for the torsionless case, the corresponding operators are **nilpotent**

$$(\mathcal{D}^{0}_{+})^{2}\mathcal{T} \equiv 0\,, \qquad\qquad (\mathcal{D}^{0}_{-})^{2}\mathcal{T} \equiv 0\,,$$

  and hence, they define **O(D,D) covariant cohomology.**

- **The field strength of the R-R potential, $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$, is then defined by**

$$\mathcal{F} := \mathcal{D}^{0}_{+}\mathcal{C}\,.$$

- Thanks to the nilpotency, the **R-R gauge symmetry** is simply realized

$$\delta\mathcal{C} = \mathcal{D}^{0}_{+}\Delta \qquad \Longrightarrow \qquad \delta\mathcal{F} = \mathcal{D}^{0}_{+}(\delta\mathcal{C}) = (\mathcal{D}^{0}_{+})^{2}\Delta \equiv 0\,.$$

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Projector-aided, fully covariant curvatures

- **Scalar curvature:**

$$(P^{AB}P^{CD} - \bar{P}^{AB}\bar{P}^{CD})S_{ACBD} \,.$$

- **“Ricci” curvature:**

$$S_{p\bar{q}} + \tfrac{1}{2}\mathcal{D}_{\bar{r}}\bar{\Delta}_{p\bar{q}}{}^{\bar{r}} + \tfrac{1}{2}\mathcal{D}_{r}\Delta_{\bar{q}p}{}^{r} \,,$$

where we set

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Combining all the results above, we are now ready to spell

- **Type II *i.e.* $\mathcal{N}=2$ $D=10$ Supersymmetric Double Field Theory**

- **Lagrangian** :

$$\mathcal{L}_{\rm Type\,II} = e^{-2d}\Big[\tfrac{1}{8}(P^{AB}P^{CD}-\bar{P}^{AB}\bar{P}^{CD})S_{ACBD}+\tfrac{1}{2}{\rm Tr}(\mathcal{F}\bar{\mathcal{F}})-i\bar{\rho}\mathcal{F}\rho'+i\bar{\psi}_{\bar{p}}\gamma_{q}\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^{q}$$

$$+i\tfrac{1}{2}\bar{\rho}\gamma^{p}\mathcal{D}^{\star}_{p}\rho-i\bar{\psi}^{\bar{p}}\mathcal{D}^{\star}_{\bar{p}}\rho-i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^{q}\mathcal{D}^{\star}_{q}\psi_{\bar{p}}-i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}'^{\star}_{\bar{p}}\rho'+i\bar{\psi}'^{p}\mathcal{D}'^{\star}_{p}\rho'+i\tfrac{1}{2}\bar{\psi}'^{p}\bar{\gamma}^{\bar{q}}\mathcal{D}'^{\star}_{\bar{q}}\psi'_{p}\Big]\,.$$

where $\bar{\mathcal{F}}^{\bar{\alpha}}{}_{\alpha}$ denotes the charge conjugation, $\bar{\mathcal{F}}:=\bar{C}_{+}^{-1}\mathcal{F}^{T}C_{+}$.

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every term in the Lagrangian is fully covariant.

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- **Torsions:** The semi-covariant curvature, S_{ABCD}, is given by the connection,

$$\Gamma_{ABC} = \Gamma^0_{ABC} + i\tfrac{1}{3}\bar{\rho}\gamma_{ABC}\rho - 2i\bar{\rho}\gamma_{BC}\psi_A - i\tfrac{1}{3}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}} + 4i\bar{\psi}_B\gamma_A\psi_C$$
$$+ i\tfrac{1}{3}\bar{\rho}'\bar{\gamma}_{ABC}\rho' - 2i\bar{\rho}'\bar{\gamma}_{BC}\psi'_A - i\tfrac{1}{3}\bar{\psi}'^p\bar{\gamma}_{ABC}\psi'_p + 4i\bar{\psi}'_B\bar{\gamma}_A\psi'_C\,,$$

which corresponds to the solution for 1.5 formalism.

The master derivatives in the fermionic kinetic terms are twofold:
$\mathcal{D}^\star_A$ for the unprimed fermions and $\mathcal{D}'^\star_A$ for the primed fermions, set by

$$\Gamma^\star_{ABC} = \Gamma_{ABC} - i\tfrac{11}{96}\bar{\rho}\gamma_{ABC}\rho + i\tfrac{5}{4}\bar{\rho}\gamma_{BC}\psi_A + i\tfrac{5}{24}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}} - 2i\bar{\psi}_B\gamma_A\psi_C + i\tfrac{5}{2}\bar{\rho}'\bar{\gamma}_{BC}\psi'_A\,,$$
$$\Gamma'^\star_{ABC} = \Gamma_{ABC} - i\tfrac{11}{96}\bar{\rho}'\bar{\gamma}_{ABC}\rho' + i\tfrac{5}{4}\bar{\rho}'\bar{\gamma}_{BC}\psi'_A + i\tfrac{5}{24}\bar{\psi}'^p\bar{\gamma}_{ABC}\psi'_p - 2i\bar{\psi}'_B\bar{\gamma}_A\psi'_C + i\tfrac{5}{2}\bar{\rho}\gamma_{BC}\psi_A\,.$$

Type II $\mathcal{N}=2$ $D=10$ SDFT [1210.5078]

- **Lagrangian** :

$$\mathcal{L}_{\mathrm{Type\,II}} = e^{-2d}\Big[\tfrac{1}{8}(P^{AB}P^{CD}-\bar{P}^{AB}\bar{P}^{CD})S_{ACBD}+\tfrac{1}{2}\mathrm{Tr}(\mathcal{F}\bar{\mathcal{F}})-i\bar{\rho}\mathcal{F}\rho'+i\bar{\psi}_{\bar{p}}\gamma_{q}\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^{q}$$
$$+i\tfrac{1}{2}\bar{\rho}\gamma^{p}\mathcal{D}^{\star}_{p}\rho-i\bar{\psi}^{\bar{p}}\mathcal{D}^{\star}_{\bar{p}}\rho-i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^{q}\mathcal{D}^{\star}_{q}\psi_{\bar{p}}-i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}'^{\star}_{\bar{p}}\rho'+i\bar{\psi}'^{p}\mathcal{D}'^{\star}_{p}\rho'+i\tfrac{1}{2}\bar{\psi}'^{p}\bar{\gamma}^{\bar{q}}\mathcal{D}'^{\star}_{\bar{q}}\psi'_{p}\Big].$$

- **Torsions:** The semi-covariant curvature, S_{ABCD}, is given by the connection,

$$\begin{aligned}\Gamma_{ABC} = \ & \Gamma^{0}_{ABC}+i\tfrac{1}{3}\bar{\rho}\gamma_{ABC}\rho-2i\bar{\rho}\gamma_{BC}\psi_{A}-i\tfrac{1}{3}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}}+4i\bar{\psi}_{B}\gamma_{A}\psi_{C}\\ & +i\tfrac{1}{3}\bar{\rho}'\bar{\gamma}_{ABC}\rho'-2i\bar{\rho}'\bar{\gamma}_{BC}\psi'_{A}-i\tfrac{1}{3}\bar{\psi}'^{p}\bar{\gamma}_{ABC}\psi'_{p}+4i\bar{\psi}'_{B}\bar{\gamma}_{A}\psi'_{C}\,,\end{aligned}$$

which corresponds to the solution for 1.5 formalism.

The master derivatives in the fermionic kinetic terms are twofold:
$\mathcal{D}^{\star}_{A}$ for the unprimed fermions and $\mathcal{D}'^{\star}_{A}$ for the primed fermions, set by

$$\begin{aligned}\Gamma^{\star}_{ABC} &= \Gamma_{ABC}-i\tfrac{11}{96}\bar{\rho}\gamma_{ABC}\rho+i\tfrac{5}{4}\bar{\rho}\gamma_{BC}\psi_{A}+i\tfrac{5}{24}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}}-2i\bar{\psi}_{B}\gamma_{A}\psi_{C}+i\tfrac{5}{2}\bar{\rho}'\bar{\gamma}_{BC}\psi'_{A}\,,\\ \Gamma'^{\star}_{ABC} &= \Gamma_{ABC}-i\tfrac{11}{96}\bar{\rho}'\bar{\gamma}_{ABC}\rho'+i\tfrac{5}{4}\bar{\rho}'\bar{\gamma}_{BC}\psi'_{A}+i\tfrac{5}{24}\bar{\psi}'^{p}\bar{\gamma}_{ABC}\psi'_{p}-2i\bar{\psi}'_{B}\bar{\gamma}_{A}\psi'_{C}+i\tfrac{5}{2}\bar{\rho}\gamma_{BC}\psi_{A}\,.\end{aligned}$$

- **The $\mathcal{N}=2$ supersymmetry transformation rules are**

$$
\begin{aligned}
\delta_\varepsilon d &= -i\tfrac{1}{2}(\bar{\varepsilon}\rho + \bar{\varepsilon}'\rho')\,,\\
\delta_\varepsilon V_{Ap} &= i\bar{V}_A{}^{\bar{q}}(\bar{\varepsilon}'\bar{\gamma}_{\bar{q}}\psi'_p - \bar{\varepsilon}\gamma_p\psi_{\bar{q}})\,,\\
\delta_\varepsilon \bar{V}_{A\bar{p}} &= iV_A{}^q(\bar{\varepsilon}\gamma_q\psi_{\bar{p}} - \bar{\varepsilon}'\bar{\gamma}_{\bar{p}}\psi'_q)\,,\\
\delta_\varepsilon \mathcal{C} &= i\tfrac{1}{2}(\gamma^p\varepsilon\bar{\psi}'_p - \varepsilon\bar{\rho}' - \psi_{\bar{p}}\bar{\varepsilon}'\bar{\gamma}^{\bar{p}} + \rho\bar{\varepsilon}') + \mathcal{C}\delta_\varepsilon d - \tfrac{1}{2}(\bar{V}^A{}_{\bar{q}}\,\delta_\varepsilon V_{Ap})\gamma^{(d+1)}\gamma^p\mathcal{C}\bar{\gamma}^{\bar{q}}\,,\\
\delta_\varepsilon \rho &= -\gamma^p\hat{\mathcal{D}}_p\varepsilon + i\tfrac{1}{2}\gamma^p\varepsilon\,\bar{\psi}'_p\rho' - i\gamma^p\psi^{\bar{q}}\bar{\varepsilon}'\bar{\gamma}_{\bar{q}}\psi'_p\,,\\
\delta_\varepsilon \rho' &= -\bar{\gamma}^{\bar{p}}\hat{\mathcal{D}}'_{\bar{p}}\varepsilon' + i\tfrac{1}{2}\bar{\gamma}^{\bar{p}}\varepsilon'\,\bar{\psi}_{\bar{p}}\rho - i\bar{\gamma}^{\bar{q}}\psi'_p\bar{\varepsilon}\gamma^p\psi_{\bar{q}}\,,\\
\delta_\varepsilon \psi_{\bar{p}} &= \hat{\mathcal{D}}_{\bar{p}}\varepsilon + (\mathcal{F} - i\tfrac{1}{2}\gamma^q\rho\,\bar{\psi}'_q + i\tfrac{1}{2}\psi^{\bar{q}}\,\bar{\rho}'\bar{\gamma}_{\bar{q}})\bar{\gamma}_{\bar{p}}\varepsilon' + i\tfrac{1}{4}\varepsilon\bar{\psi}_{\bar{p}}\rho + i\tfrac{1}{2}\psi_{\bar{p}}\bar{\varepsilon}\rho\,,\\
\delta_\varepsilon \psi'_p &= \hat{\mathcal{D}}'_p\varepsilon' + (\bar{\mathcal{F}} - i\tfrac{1}{2}\bar{\gamma}^{\bar{q}}\rho'\bar{\psi}_{\bar{q}} + i\tfrac{1}{2}\psi'^q\bar{\rho}\gamma_q)\gamma_p\varepsilon + i\tfrac{1}{4}\varepsilon'\bar{\psi}'_p\rho' + i\tfrac{1}{2}\psi'_p\bar{\varepsilon}'\rho'\,,
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{\Gamma}_{ABC} &= \Gamma_{ABC} - i\tfrac{17}{48}\bar{\rho}\gamma_{ABC}\rho + i\tfrac{5}{2}\bar{\rho}\gamma_{BC}\psi_A + i\tfrac{1}{4}\bar{\psi}^{\bar{p}}\gamma_{ABC}\psi_{\bar{p}} - 3i\bar{\psi}'_B\bar{\gamma}_A\psi'_C\,,\\
\hat{\Gamma}'_{ABC} &= \Gamma_{ABC} - i\tfrac{17}{48}\bar{\rho}'\bar{\gamma}_{ABC}\rho' + i\tfrac{5}{2}\bar{\rho}'\bar{\gamma}_{BC}\psi'_A + i\tfrac{1}{4}\bar{\psi}'^p\bar{\gamma}_{ABC}\psi'_p - 3i\bar{\psi}_B\gamma_A\psi_C\,.
\end{aligned}
$$

Type II $\mathcal{N}=2$ $D=10$ SDFT [1210.5078]

- **Lagrangian** :

$$\mathcal{L}_{\rm Type\,II} = e^{-2d}\Big[\tfrac{1}{8}(P^{AB}P^{CD}-\bar{P}^{AB}\bar{P}^{CD})S_{ACBD} + \tfrac{1}{2}{\rm Tr}(\mathcal{F}\bar{\mathcal{F}}) - i\bar{\rho}\mathcal{F}\rho' + i\bar{\psi}_{\bar{p}}\gamma_{q}\mathcal{F}\bar{\gamma}^{\bar{p}}\psi'^{q}$$

$$+i\tfrac{1}{2}\bar{\rho}\gamma^{p}\mathcal{D}^{\star}_{p}\rho - i\bar{\psi}^{\bar{p}}\mathcal{D}^{\star}_{\bar{p}}\rho - i\tfrac{1}{2}\bar{\psi}^{\bar{p}}\gamma^{q}\mathcal{D}^{\star}_{q}\psi_{\bar{p}} - i\tfrac{1}{2}\bar{\rho}'\bar{\gamma}^{\bar{p}}\mathcal{D}'^{\star}_{\bar{p}}\rho' + i\bar{\psi}'^{p}\mathcal{D}'^{\star}_{p}\rho' + i\tfrac{1}{2}\bar{\psi}'^{p}\bar{\gamma}^{\bar{q}}\mathcal{D}'^{\star}_{\bar{q}}\psi'_{p}\Big]\,.$$

- The Lagrangian is **pseudo** : It is necessary to impose a **self-duality** of the R-R field strength by hand,

$$\tilde{\mathcal{F}}_{-} := \left(1-\gamma^{(D+1)}\right)\left(\mathcal{F} - i\tfrac{1}{2}\rho\bar{\rho}' + i\tfrac{1}{2}\gamma^{p}\psi_{\bar{q}}\bar{\psi}'_{p}\bar{\gamma}^{\bar{q}}\right) \equiv 0\,.$$

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Type II $\mathcal{N}=2$ $D=10$ SDFT [1210.5078]

- Under the $\mathcal{N}=2$ SUSY transformation rule, the Lagrangian transforms, disregarding total derivatives, as

$$\delta_{\varepsilon}\mathcal{L}_{\rm Type\,II} \simeq -\tfrac{1}{8}e^{-2d}\bar{V}^{A}{}_{\bar{q}}\delta_{\varepsilon}V_{Ap}\,{\rm Tr}\left(\gamma^{p}\tilde{\mathcal{F}}_{-}\bar{\gamma}^{\bar{q}}\overline{\tilde{\mathcal{F}}_{-}}\right)\,,$$

  where

$$\tilde{\mathcal{F}}_{-} := \left(1-\gamma^{(D+1)}\right)\left(\mathcal{F}-i\tfrac{1}{2}\rho\bar{\rho}^{\prime}+i\tfrac{1}{2}\gamma^{p}\psi_{\bar{q}}\bar{\psi}^{\prime}_{p}\bar{\gamma}^{\bar{q}}\right)\,.$$

  This verifies, to the full order in fermions, **the supersymmetric invariance of the action, modulo the self-duality.**

- For a **nontrivial consistency check,** the supersymmetric variation of the self-duality relation is precisely closed by the equations of motion for the gravitinos,

$$\delta_{\varepsilon}\tilde{\mathcal{F}}_{-} = -i\left(\tilde{\mathcal{D}}_{\bar{p}}\rho+\gamma^{p}\tilde{\mathcal{D}}_{p}\psi_{\bar{p}}-\gamma^{p}\mathcal{F}\bar{\gamma}_{\bar{p}}\psi^{\prime}{}_{p}\right)\bar{\varepsilon}^{\prime}\bar{\gamma}^{\bar{p}} - i\gamma^{p}\varepsilon\left(\tilde{\mathcal{D}}^{\prime}_{p}\bar{\rho}^{\prime}+\tilde{\mathcal{D}}^{\prime}_{\bar{p}}\bar{\psi}^{\prime}_{p}\bar{\gamma}^{\bar{p}}-\bar{\psi}_{\bar{p}}\gamma_{p}\mathcal{F}\bar{\gamma}^{\bar{p}}\right)\,.$$

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Equations of Motion for Bosons

- DFT-vielbein:

$$S_{p\bar{q}}+\mathrm{Tr}(\gamma_p\mathcal{F}\bar{\gamma}_{\bar{q}}\bar{\mathcal{F}})+i\bar{\rho}\gamma_p\tilde{\mathcal{D}}_{\bar{q}}\rho+2i\bar{\psi}_{\bar{q}}\tilde{\mathcal{D}}_p\rho-i\bar{\psi}^{\bar{p}}\gamma_p\tilde{\mathcal{D}}_{\bar{q}}\psi_{\bar{p}}+i\bar{\rho}'\bar{\gamma}_{\bar{q}}\tilde{\mathcal{D}}_p\rho'+2i\bar{\psi}'{}_p\tilde{\mathcal{D}}_{\bar{q}}\rho'-i\bar{\psi}'^q\bar{\gamma}_{\bar{q}}\tilde{\mathcal{D}}_p\psi'_q=0.$$

This is DFT-generalization of Einstein equation.

- DFT-dilaton:

$$\mathcal{L}_{\mathrm{Type\,II}}=0\,.$$

Namely, the on-shell Lagrangian vanishes!

- R-R potential:

$$\mathcal{D}^0_-\left(\mathcal{F}-i\rho\bar{\rho}'+i\gamma^r\psi_{\bar{s}}\bar{\psi}'_r\bar{\gamma}^{\bar{s}}\right)=0\,,$$

which is automatically met by the self-duality, together with the nilpotency of $\mathcal{D}^0_+$,

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- The 1.5 formalism works: The variation of the Lagrangian induced by that of the connection is trivial, $\delta\mathcal{L}_{\mathrm{Type\,II}}=\delta\Gamma_{ABC}\times 0\,.$

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Equations of Motion for Fermions

- DFT-dilationos,

$$\gamma^{p}\tilde{\mathcal{D}}_{p}\rho-\tilde{\mathcal{D}}_{\bar{p}}\psi^{\bar{p}}-\mathcal{F}\rho'=0\,,\qquad\qquad \bar{\gamma}^{\bar{p}}\tilde{\mathcal{D}}_{\bar{p}}\rho'-\tilde{\mathcal{D}}_{p}\psi'^{p}-\bar{\mathcal{F}}\rho=0\,.$$

- Gravitinos,

$$\tilde{\mathcal{D}}_{\bar{p}}\rho+\gamma^{p}\tilde{\mathcal{D}}_{p}\psi_{\bar{p}}-\gamma^{p}\mathcal{F}\bar{\gamma}_{\bar{p}}\psi'_{p}=0\,,\qquad\qquad \tilde{\mathcal{D}}_{p}\rho'+\bar{\gamma}^{\bar{p}}\tilde{\mathcal{D}}_{\bar{p}}\psi'_{p}-\bar{\gamma}^{\bar{p}}\bar{\mathcal{F}}\gamma_{p}\psi_{\bar{p}}=0\,.$$

Equations of Motion for Fermions

- DFT-dilationos,

$$\gamma^{p}\tilde{\mathcal{D}}_{p}\rho-\tilde{\mathcal{D}}_{\bar{p}}\psi^{\bar{p}}-\mathcal{F}\rho^{\prime}=0\,,\qquad\qquad\bar{\gamma}^{\bar{p}}\tilde{\mathcal{D}}_{\bar{p}}\rho^{\prime}-\tilde{\mathcal{D}}_{p}\psi^{\prime p}-\bar{\mathcal{F}}\rho=0\,.$$

- Gravitinos,

$$\tilde{\mathcal{D}}_{\bar{p}}\rho+\gamma^{p}\tilde{\mathcal{D}}_{p}\psi_{\bar{p}}-\gamma^{p}\mathcal{F}\bar{\gamma}_{\bar{p}}\psi^{\prime}{}_{p}=0\,,\qquad\qquad\tilde{\mathcal{D}}_{p}\rho^{\prime}+\bar{\gamma}^{\bar{p}}\tilde{\mathcal{D}}_{\bar{p}}\psi^{\prime}{}_{p}-\bar{\gamma}^{\bar{p}}\bar{\mathcal{F}}\gamma_{p}\psi_{\bar{p}}=0\,.$$

Truncation to $\mathcal{N}=1$ $D=10$ SDFT [1112.0069]

- Turning off the primed fermions and the R-R sector truncates the $\mathcal{N}=2$ $D=10$ SDFT to $\mathcal{N}=1$ $D=10$ SDFT,

$$\mathcal{L}_{\mathcal{N}=1} = e^{-2d}\Big[\tfrac{1}{8}\left(P^{AB}P^{CD}-\bar{P}^{AB}\bar{P}^{CD}\right)S_{ACBD} + i\tfrac{1}{2}\bar{\rho}\gamma^{A}\mathcal{D}^{\star}_{A}\rho - i\bar{\psi}^{A}\mathcal{D}^{\star}_{A}\rho - i\tfrac{1}{2}\bar{\psi}^{B}\gamma^{A}\mathcal{D}^{\star}_{A}\psi_{B}\Big]\,.$$

- $\mathcal{N}=1$ **Local SUSY:**

$$\begin{aligned}
\delta_{\varepsilon}d &= -i\tfrac{1}{2}\bar{\varepsilon}\rho\,,\\
\delta_{\varepsilon}V_{Ap} &= -i\bar{\varepsilon}\gamma_{p}\psi_{A}\,,\\
\delta_{\varepsilon}\bar{V}_{A\bar{p}} &= i\bar{\varepsilon}\gamma_{A}\psi_{\bar{p}}\,,\\
\delta_{\varepsilon}\rho &= -\gamma^{A}\hat{\mathcal{D}}_{A}\varepsilon\,,\\
\delta_{\varepsilon}\psi_{\bar{p}} &= \bar{V}^{A}{}_{\bar{p}}\hat{\mathcal{D}}_{A}\varepsilon - i\tfrac{1}{4}(\bar{\rho}\psi_{\bar{p}})\varepsilon + i\tfrac{1}{2}(\bar{\varepsilon}\rho)\psi_{\bar{p}}\,.
\end{aligned}$$

$\mathcal{N}=1$ SUSY Algebra [1112.0069]

- Commutator of supersymmetry reads

$$[\delta_{\varepsilon_1},\delta_{\varepsilon_2}] \equiv \hat{\mathcal{L}}_{X_3} + \delta_{\varepsilon_3} + \delta_{\mathbf{so}(1,9)_L} + \delta_{\mathbf{so}(9,1)_R} + \delta_{\mathrm{trivial}}\,.$$

where

$$X_3^A = i\bar{\varepsilon}_1\gamma^A\varepsilon_2\,, \qquad \varepsilon_3 = i\tfrac{1}{2}\left[(\bar{\varepsilon}_1\gamma^p\varepsilon_2)\gamma_p\rho + (\bar{\rho}\varepsilon_2)\varepsilon_1 - (\bar{\rho}\varepsilon_1)\varepsilon_2\right]\,, \qquad \text{etc.}$$

and $\delta_{\mathrm{trivial}}$ corresponds to the fermionic equations of motion.

Reduction to SUGRA

Now we are going to

- **parametrize the DFT-field-variables in terms of Riemannian variables,**
- **discuss the 'unification',**
- **choose a diagonal gauge of Spin**$(1,D-1)_L \times$ **Spin**$(D-1,1)_R$**,**
- **and reduce SDFT to SUGRAs.**

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- As stressed before, one of the characteristic features in our construction of $\mathcal{N}=2$ $D=10$ SDFT is the usage of the $\mathbf{O}(D,D)$ covariant, genuine DFT-field-variables.
- However, the relation to an ordinary supergravity can be established only after we solve the defining algebraic relations of the DFT-vielbeins and parametrize the solution in terms of Riemannian variables, *i.e.* zehnbeins and B-field.
- Assuming that the upper half blocks are non-degenerate, the DFT-vielbein takes the most general form,

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  Hitchin; Grana, Minasian, Petrini, Waldram

- For example, the $\mathbf{O}(D,D)$ covariant Dirac operators become

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Unification of type IIA and IIB SUGRAs

- Since the two zehnbeins correspond to the same spacetime metric, they are related by a Lorentz rotation,

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Diagonal gauge fixing and Reduction to SUGRA

- Setting the diagonal gauge,

$$e_\mu{}^p \equiv \bar{e}_\mu{}^{\bar{p}}$$

with $\eta_{pq} = -\bar{\eta}_{\bar{p}\bar{q}}$, $\bar{\gamma}^{\bar{p}} = \gamma^{(D+1)}\gamma^p$, $\bar{\gamma}^{(D+1)} = -\gamma^{(D+1)}$, breaks the local Lorentz symmetry,

$$\mathbf{Spin}(1,D{-}1)_L \times \mathbf{Spin}(D{-}1,1)_R \quad\Longrightarrow\quad \mathbf{Spin}(1,D{-}1)_D\,.$$

- And it reduces SDFT to SUGRA:

$\mathcal{N}=2$ $D=10$ **SDFT** $\Longrightarrow$ $10D$ **Type II democratic SUGRA**

Bergshoeff, *et al.*; Coimbra, Strickland-Constable, Waldram

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$\mathcal{N} = 1$ $D = 10$ **SDFT** $\Longrightarrow$ $10D$ **minimal SUGRA** Chamseddine; Bergshoeff *et al.*

- **To the full order in fermions, $\mathcal{N}=1$ SDFT reduces to $10D$ minimal SUGRA:**

$$\begin{aligned}\mathcal{L}_{10D} = \det e \times e^{-2\phi}\Big[& R + 4\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{12}H_{\lambda\mu\nu}H^{\lambda\mu\nu} \\ & + i2\sqrt{2}\bar{\rho}\gamma^m[\partial_m\rho + \tfrac{1}{4}(\omega + \tfrac{1}{6}H)_{mnp}\gamma^{np}\rho] - i4\sqrt{2}\bar{\psi}^p[\partial_p\rho + \tfrac{1}{4}(\omega + \tfrac{1}{2}H)_{pqr}\gamma^{qr}\rho] \\ & - i2\sqrt{2}\bar{\psi}^p\gamma^m[\partial_m\psi_p + \tfrac{1}{4}(\omega + \tfrac{1}{6}H)\gamma^{np}\psi_p + \omega_{mpq}\psi^q - \tfrac{1}{2}H_{mpq}\psi^q] \\ & + \tfrac{1}{24}(\bar{\psi}^q\gamma_{mnp}\psi_q)(\bar{\psi}^r\gamma^{mnp}\psi_r) - \tfrac{1}{48}(\bar{\psi}^q\gamma_{mnp}\psi_q)(\bar{\rho}\gamma^{mnp}\rho)\ \Big] .\end{aligned}$$

$$\delta_\varepsilon\phi = i\tfrac{1}{2}\bar{\varepsilon}(\rho + \gamma^a\psi_a)\,, \qquad \delta_\varepsilon e_\mu^a = i\bar{\varepsilon}\gamma^a\psi_\mu\,, \qquad \delta_\varepsilon B_{\mu\nu} = -2i\bar{\varepsilon}\gamma_{[\mu}\psi_{\nu]}\,,$$

$$\begin{aligned}\delta_\varepsilon\rho = & -\tfrac{1}{\sqrt{2}}\gamma^a[\partial_a\varepsilon + \tfrac{1}{4}(\omega + \tfrac{1}{6}H)_{abc}\gamma^{bc}\varepsilon - \partial_a\phi\varepsilon] \\ & + i\tfrac{1}{48}(\bar{\psi}^d\gamma_{abc}\psi_d)\gamma^{abc}\varepsilon + i\tfrac{1}{192}(\bar{\rho}\gamma_{abc}\rho)\gamma^{abc}\varepsilon + i\tfrac{1}{2}(\bar{\varepsilon}\gamma_{[a}\psi_{b]})\gamma^{ab}\rho\,,\end{aligned}$$

$$\begin{aligned}\delta_\varepsilon\psi_a = & \tfrac{1}{\sqrt{2}}[\partial_a\varepsilon + \tfrac{1}{4}(\omega + \tfrac{1}{2}H)_{abc}\gamma^{bc}\varepsilon] \\ & - i\tfrac{1}{2}(\bar{\rho}\varepsilon)\psi_a - i\tfrac{1}{4}(\bar{\rho}\psi_a)\varepsilon + i\tfrac{1}{8}(\bar{\rho}\gamma_{bc}\psi_a)\gamma^{bc}\varepsilon + i\tfrac{1}{2}(\bar{\varepsilon}\gamma_{[b}\psi_{c]})\gamma^{bc}\psi_a\,.\end{aligned}$$

Diagonal gauge fixing and Reduction to SUGRA

- After the diagonal gauge fixing, we may parameterize the R-R potential as

$$\mathcal{C} \equiv \left(\tfrac{1}{2}\right)^{\frac{D+2}{4}} \textstyle\sum_p' \frac{1}{p!}\, \mathcal{C}_{a_1a_2\cdots a_p}\gamma^{a_1a_2\cdots a_p}$$

  and obtain the field strength,

$$\mathcal{F} := \mathcal{D}^0_+\mathcal{C} \equiv \left(\tfrac{1}{2}\right)^{\frac{D}{4}} \textstyle\sum_p' \frac{1}{(p+1)!}\, \mathcal{F}_{a_1a_2\cdots a_{p+1}}\gamma^{a_1a_2\cdots a_{p+1}}$$

  where $\sum_p'$ denotes the odd p sum for Type IIA and even p sum for Type IIB, and

$$\mathcal{F}_{a_1a_2\cdots a_p} = p\left(D_{[a_1}\mathcal{C}_{a_2\cdots a_p]} - \partial_{[a_1}\phi\,\mathcal{C}_{a_2\cdots a_p]}\right) + \tfrac{p!}{3!(p-3)!}\,H_{[a_1a_2a_3}\mathcal{C}_{a_4\cdots a_p]}$$

- The pair of nilpotent differential operators, $\mathcal{D}^0_+$ and $\mathcal{D}^0_-$, reduce to a 'twisted K-theory' exterior derivative and its dual, after the diagonal gauge fixing,

$$\mathcal{D}^0_+ \quad\Longrightarrow\quad \mathrm{d} + (H - \mathrm{d}\phi)\wedge$$

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- **In this way, ordinary SUGRA $\equiv$ gauge-fixed SDFT,**

$$\textbf{Spin}(1,D{-}1)_L \times \textbf{Spin}(D{-}1,1)_R \quad\Longrightarrow\quad \textbf{Spin}(1,D{-}1)_D\,.$$

Modifying $\mathbf{O}(D,D)$ transformation rule

- The diagonal gauge, $e_\mu{}^p \equiv \bar{e}_\mu{}^{\bar{p}}$, is incompatible with the vectorial $\mathbf{O}(D,D)$ transformation rule of the DFT-vielbein.

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Modifying $\mathbf{O}(D,D)$ transformation rule

- The $\mathbf{O}(D,D)$ rotation must accompany a compensating $\mathbf{Pin}(D{-}1,1)_R$ local Lorentz rotation, $\bar{L}_{\bar{q}}{}^{\bar{p}}$, $S_{\bar{L}}{}^{\bar{\alpha}}{}_{\bar{\beta}}$:

$$\bar{L} = \bar{e}^{-1}\left[\mathbf{a}^t - (g+B)\mathbf{b}^t\right]\left[\mathbf{a}^t + (g-B)\mathbf{b}^t\right]^{-1}\bar{e}\,, \qquad \bar{\gamma}^{\bar{q}}\bar{L}_{\bar{q}}{}^{\bar{p}} = S_{\bar{L}}^{-1}\bar{\gamma}^{\bar{p}}S_{\bar{L}}\,,$$

where

$$\mathbf{ab}^t + \mathbf{ba}^t = 0\,, \qquad \mathbf{cd}^t + \mathbf{dc}^t = 0\,, \qquad \mathbf{ad}^t + \mathbf{bc}^t = 1\,,$$

such that they parametrize a generic $\mathbf{O}(D,D)$ group element,

$$M_A{}^B = \begin{pmatrix} \mathbf{a}^{\mu}{}_{\nu} & \mathbf{b}^{\mu\sigma} \\ \mathbf{c}_{\rho\nu} & \mathbf{d}_{\rho}{}^{\sigma} \end{pmatrix}.$$

Modified O(D,D) Transformation Rule After The Diagonal Gauge Fixing

$$
\begin{array}{ccc}
d & \longrightarrow & d \\
V_A{}^p & \longrightarrow & M_A{}^B\, V_B{}^p \\
\bar{V}_A{}^{\bar{p}} & \longrightarrow & M_A{}^B\, \bar{V}_B{}^{\bar{q}}\, \bar{L}_{\bar{q}}{}^{\bar{p}} \\
\mathcal{C}^\alpha{}_{\bar{\alpha}}\,, \quad \mathcal{F}^\alpha{}_{\bar{\alpha}} & \longrightarrow & \mathcal{C}^\alpha{}_{\bar{\beta}}(S_{\bar{L}}^{-1})^{\bar{\beta}}{}_{\bar{\alpha}}\,, \quad \mathcal{F}^\alpha{}_{\bar{\beta}}(S_{\bar{L}}^{-1})^{\bar{\beta}}{}_{\bar{\alpha}} \\
\rho^\alpha & \longrightarrow & \rho^\alpha \\
\rho'^{\bar{\alpha}} & \longrightarrow & (S_{\bar{L}})^{\bar{\alpha}}{}_{\bar{\beta}}\,\rho'^{\bar{\beta}} \\
\psi^\alpha_{\bar{p}} & \longrightarrow & (\bar{L}^{-1})_{\bar{p}}{}^{\bar{q}}\,\psi^\alpha_{\bar{q}} \\
\psi'^{\bar{\alpha}}_p & \longrightarrow & (S_{\bar{L}})^{\bar{\alpha}}{}_{\bar{\beta}}\,\psi'^{\bar{\beta}}_p
\end{array}
$$

- **All the barred indices are now to be rotated.** Consistent with Hassan
- **The R-R sector can be also mapped to O(D,D) spinors.**

Fukuma, Oota Tanaka; Hohm, Kwak, Zwiebach

Flipping the chirality: IIA $\Leftrightarrow$ IIB

- **If and only if $\det(\bar{L}) = -1$, the modified $\mathbf{O}(D,D)$ rotation flips the chirality of the theory, since**

$$\bar{\gamma}^{(D+1)} S_{\bar{L}} = \det(\bar{L})\, S_{\bar{L}} \bar{\gamma}^{(D+1)}\,.$$

- Thus, the mechanism above naturally realizes the exchange of Type IIA and IIB supergravities under $\mathbf{O}(D,D)$ T-duality.

- However, since $\bar{L}$ explicitly depends on the parametrization of V_{Ap} and $\bar{V}_{A\bar{p}}$ in terms of $g_{\mu\nu}$ and $B_{\mu\nu}$, it is impossible to impose the modified $\mathbf{O}(D,D)$ transformation rule from the beginning on the parametrization-independent covariant formalism.

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- With the semi-covariant derivative, we may construct YM-DFT :

$$\mathcal{F}_{AB} := \nabla_A \mathcal{V}_B - \nabla_B \mathcal{V}_A - i\,[\mathcal{V}_A, \mathcal{V}_B]\,, \qquad \mathcal{V}_A = \left(\begin{array}{c} \phi^{\lambda} \\ A_\mu + B_{\mu\nu}\phi^\nu \end{array}\right),$$

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- Similar to topologically twisted Yang-Mills, but differs in detail.
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Comment 1: Double field Yang-Mills theory 1102.0419

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Comment 2: Rank-four tensor 1105.6294

- With DFT-vielbein, it is possible to construct a rank-four tensor which is covariant with respect to $\mathbf{O}(D,D)$ and 'diagonal' local Lorentz symmetry.
- Gauge fixing the two vielbeins equal to each other, $e_{\mu m} = \bar{e}_{\mu \bar{m}}$, gives

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- The section condition is equivalent to the coordinate gauge symmetry,

$$x^A \sim x^A + \varphi\partial^A\varphi' .$$

A 'physical point' is one-to-one identified with a 'gauge orbit' in coordinate space.

- **String propagates in doubled yet gauged spacetime, 1307.8377**

$$\mathcal{S} = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\ \mathcal{L}\,, \qquad \mathcal{L} = -\tfrac{1}{2}\sqrt{-h}h^{ij}D_iX^MD_jX^N\mathcal{H}_{MN}(X) - \epsilon^{ij}D_iX^M\mathcal{A}_{jM}\,,$$

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Conclusion

Summary

- The fundamental field-variables of $\mathcal{N}=2$ $D=10$ SDFT are, besides the fermions, the DFT-dilaton, d, DFT-vielbeins, V_{Ap}, $\bar{V}_{A\bar{p}}$, and the R-R potential, $\mathcal{C}^{\alpha}{}_{\bar{\alpha}}$.

- Novel differential geometic ingredients:
  - ▷ projectors, $P_{AB}=V_{Ap}V_B{}^p$, $\bar{P}_{AB}=\bar{V}_{A\bar{p}}\bar{V}_B{}^{\bar{p}}$, and semi-covariant derivative.

- $\mathcal{N}=2$ $D=10$ SDFT manifests simultaneously the symmetric structures:
  - ▷ $\mathbf{O}(10,10)$ *T-duality*
  - ▷ *DFT-diffeomorphism (generalized Lie derivative)*
  - ▷ *A pair of local Lorentz symmetries,* $\mathbf{Spin}(1,9)_L\times\mathbf{Spin}(9,1)_R$

- While the theory is unique, the solutions are twofold $\Longrightarrow$ Unification of IIA and IIB.

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- After parametrizing the DFT field-variables in terms of Riemannian ones and taking the diagonal gauge, $\mathbf{Spin}(1,9)_L \times \mathbf{Spin}(9,1)_R \longrightarrow \mathbf{Spin}(1,9)_D$, SDFT reduces to SUGRA.

- *A priori*, in the covariant formalism, the R-R sector and all the fermions are $\mathbf{O}(D,D)$ T-duality singlet.

- Yet, the diagonal gauge fixing, $e_\mu{}^p \equiv \bar{e}_\mu{}^{\bar{p}}$, modifies the $\mathbf{O}(D,D)$ transformation rule to call for a compensating $\mathbf{Pin}(D{-}1,1)_R$ rotation, which may flip the chirality of the theory, resulting in the exchange of type IIA and IIB SUGRAs.

- The parametrization of the DFT variables is not unique:
  - $\mathbf{O}(D,D)$ rotations and field redefinitions can be freely applied.
  - In fact, non-Riemannian 'metric-less' backgrounds are also allowed:

$$P_{MN} - \bar{P}_{MN} = \mathcal{H}_{MN} = \begin{pmatrix} 0 & N \\ N^t & 0 \end{pmatrix}, \qquad N^2 = 1 .$$

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- $\mathbf{O}(10,10)$ covariant Killing spinor equation: SUSY and T-duality are compatible!
- Quantization of the string action on doubled yet gauged spacetime.
- Further study on the non-Riemannian, metric-less backgrounds.

- The relaxation of the section condition Geissbuhler; Grana, Marques, Aldazabal; Berman, Musaev, Blair, Malek, Perry; Berman, Kanghoon Lee

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