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Chapter Two:

Mathematical Preliminaries

This dissertation borrows a lot of vocabulary, notation, concepts, and techniques from the surface of two major areas of mathematics: homological algebra and symplectic geometry. In an effort to make this work as self-contained as possible and not before some debate, I managed to convince myself that rather than succumbing to encyclopædistic tendencies and fill this dissertation with appendices which, on the one hand, will probably not be read; and, on the other hand, would upset the linear order of the discussion; I would devote a medium sized chapter to getting these prerequisites out of the way. Moreover, since if this chapter is to be read at all it should be done so at the beginning, I decided to make it the first real chapter in the dissertation. Needless to say, the reader is strongly urged to at least skim this chapter for notation.

This chapter is organized as follows. In Section 1 we review the basic facts of homological algebra. Although none of the concepts are too difficult (except perhaps spectral sequences), as usual with algebra, there are a lot of names. In this section we set our notation and vocabulary concerning differential complexes. In particular we discuss resolutions which will be very important conceptually throughout this dissertation. We then introduce the reader to spectral sequences. This is possibly the toughest concept in this chapter but it proves to be an invaluable tool when computing cohomology. As a special illustration we then take a look at the two canonical spectral sequences associated to a double complex and as an application of this we prove the algebraic Künneth formula. If the reader comes out with the compulsion that the first thing to try when faced with a double complex is to go ahead and compute the first two terms of the two spectral sequences, this chapter will have served its purpose. Finally, and because we will find ample use for these concepts, we briefly review the highlights of Lie algebra cohomology.

Section 2 is another introductory section which sets the language for the other important subject in this work: symplectic geometry. Everything is this section is familiar in one way or another to every working theoretical physicist; although the names I have used may not be so readily distinguishable. As a particularly nice application of the concepts and methods of symplectic geometry, we give a derivation from first principles of the Dirac bracket. We also cover symplectic reduction with respect to a coisotropic submanifold. This is not the most general case of symplectic reduction, but it is the one we shall be interested in. We then make contact with the theory of constraints. We prove that the constrained submanifold associated to a set of first (resp. second) class constraints is a coisotropic (resp. symplectic) submanifold. We then discuss a very special case of symplectic reduction: the one arising from the action of a Lie group. The first class constraints are nothing but the components of the moment map. Finally we discuss a special case of the moment map. This is the symplectic reduction of a phase space. In this case we show how any action of the configuration space automatically gives rise to an equivariant moment map in the configuration space which is linear in the momenta.

1. Basic Facts of Homological Algebra

In this section we assemble the basic definitions, notation, and facts of homological algebra that will be used in the sequel; as well as some less elementary material on spectral sequences which is nevertheless instrumental for this dissertation. We also give a brief introduction to the basic ideas of Lie algebra cohomology. These will come in handy when we discuss the semi-infinite cohomology of Feigin in Chapters VI-VIII. Homological algebra is a topic which lends itself easily to generalizations which would, however, only obscure the concepts of relevance to our discussion. Therefore we have attempted to suppress almost all evidence of "abstract nonsense" and keep the discussion as elementary as possible while still covering in detail the necessary background. Fuller treatments to which no justice could possibly be done in a few pages are to be found in the books by Lang **[62]**, Hilton & Stammbach **[63]**, and MacLane **[64]**. Lie algebra cohomology is treated in the books of Jacobson **[65]**, Hilton & Stammbach (*op. cit.*), and in the seminal paper of Chevalley & Eilenberg **[66]**. The cohomology of infinite dimensional Lie algebras is discussed with a wealth of examples in the book of Fuks **[67]**.

Basic Definitions

Homological algebra centers itself on the study of complexes and their cohomologies. Let C be a vector space and let $d : C \to C$ be a linear map which obeys $d^2 = 0$. Such a pair (C, d) is called a **differential complex**, and d is called the **differential**. Associated to the differential there are two subspaces of C:

$$Z \equiv \{v \in C \mid dv = 0\} = \ker d \tag{II.1.1}$$

$$B \equiv \{dv \mid v \in C\} = \operatorname{im} d \ , \tag{II.1.2}$$

the **kernel** and the **image** of d respectively. Because $d^2 = 0$, $B \subset Z$. The obstruction to the reverse inclusion is measured by the **cohomology** of d, written $H_d(C)$, and defined by

$$H_d(C) \equiv Z/B \ . \tag{II.1.3}$$

Whenever there is no risk of confusion we will omit all explicit mention of the differential and simply write $H(C)$ for the cohomology. The elements of C, Z, and B are called **cochains**, **cocycles**, and **coboundaries** respectively.

Therefore, $H(C)$ consists of equivalence classes of cocycles, where two cocycles v, w are said to be **cohomologous**—*i.e.*, in the same cohomology class— if their difference is a coboundary. In symbols,

$$[v] = [w] \iff v - w = du \quad (\exists u) \ . \tag{II.1.4}$$

In particular, a coboundary is cohomologous to zero. Although $H(C)$ is a vector space it is worth remarking that it is not a subspace of C. Rather it is a **subquotient**: the quotient of a subspace. Of course, we can always choose a set of cocycles $\{v_i\}$ whose cohomology classes $\{[v_i]\}$ form a basis for $H(C)$ and then complete this set to a basis $\{v_i, w_j\}$ for C. The subspace of C spanned by $\{v_i\}$ is isomorphic to $H(C)$ but this is not canonical. That is, there is no privileged representative cocycle for a given cohomology class. We will see later on, when we discuss BRST cohomology, that this is precisely the algebraic analog of picking a gauge. The situation may, of course, differ if C has some more structure, *e.g.*, an inner product. This will, in fact, be the main theme in Chapter V.

The life of a chain complex with so little structure is rather dull. To relieve this boredom let us add a grading. That is, suppose that C is a $\mathbb{Z}$–graded vector space

$$C = \bigoplus_{n\in\mathbb{Z}} C^n \tag{II.1.5}$$

and that d has degree one with respect to this grading

$$d : C^n \longrightarrow C^{n+1} \ . \tag{II.1.6}$$

We call (C, d) in this case a **graded complex**. A useful graphical depiction of a graded complex is a sequence of vector spaces with linear maps (arrows) between them:

$$\cdots \longrightarrow C^{-1} \xrightarrow{d} C^0 \xrightarrow{d} C^1 \longrightarrow \cdots \ . \tag{II.1.7}$$

We can refine our notions of cocycle and coboundary as follows. Define the subspace Z^n of

n-cocycles and the subspace B^n of n-coboundaries as follows

$$Z^n \equiv Z \cap C^n = \{v \in C^n \mid dv = 0\} \tag{II.1.8}$$

$$B^n \equiv B \cap C^n = \left\{dv \mid v \in C^{n-1}\right\} \ . \tag{II.1.9}$$

Then the n^{th} cohomology group $H^n(C)$ is defined as the quotient

$$H^n(C) \equiv Z^n / B^n \ . \tag{II.1.10}$$

Clearly

$$H(C) = \bigoplus_{n\in\mathbb{Z}} H^n(C) \tag{II.1.11}$$

making the cohomology into a graded vector space. We will often call the degree n the **dimension**; and we refer to $H^n(C)$ as the cohomology of the complex (C, d) in n^{th} dimension.

Perhaps the prime example of a cohomology theory is that of de Rham. Let M be a differentiable manifold and let $\Omega(M)$ denote the graded ring of differential forms. The exterior derivative d is a differential of degree one. The cocycles are called **closed forms**, whereas the coboundaries are called **exact**. The de Rham cohomology is denoted $H_{dR}(M)$ and is one of the simplest topological invariants of M that one can compute.

Now let $\operatorname{End} C$ denote the vector space of **endomorphisms** of C; that is, the linear transformations of C. The $\mathbb{Z}$–grading of C induces a $\mathbb{Z}$–grading of $\operatorname{End} C$ in the obvious way. We say that a linear transformation $f \in \operatorname{End} C$ has degree n if

$$f : C^p \longrightarrow C^{p+n} \qquad \forall\, p\ ; \tag{II.1.12}$$

and we write $f \in \operatorname{End}_n C$. Clearly

$$\operatorname{End} C = \bigoplus_{n\in\mathbb{Z}} \operatorname{End}_n C\ . \tag{II.1.13}$$

We can turn $\operatorname{End} C$ into a Lie superalgebra by defining the bracket of homogeneous elements $f \in \operatorname{End}_i C$ and $g \in \operatorname{End}_j C$ as the graded commutator

$$\left[f\,,\,g\right] \equiv f \circ g - (-1)^{ij} g \circ f\ , \tag{II.1.14}$$

where $\circ$ stands for composition of linear transformations. In particular $d \in \operatorname{End}_1 C$ and hence the fact that $d^2 = 0$ is equivalent to the Lie algebraic statement that the subalgebra of $\operatorname{End} C$ it generates is abelian— a non-trivial statement since d is odd.

We can make a graded complex out of $\operatorname{End} C$ as follows. Define the linear map

$$\text{ad } d : \operatorname{End}_n C \to \operatorname{End}_{n+1} C \tag{II.1.15}$$

by

$$f \mapsto \left[d\,,\,f\right]\ . \tag{II.1.16}$$

Since $d^2 = 0$ and $(\text{ad } d)^2 = \text{ad } d^2$ the above map is a differential of degree one making $(\operatorname{End} C, \text{ad } d)$ into a graded complex. The cocycles are linear transformations of C which (anti)commute with d and are called **chain maps**; whereas the coboundaries are linear transformations which can be written as some (anti)commutator of d and are called **chain homotopic to zero**. If $f = \left[d\,,\,g\right]$ is chain homotopic to zero, g is called the **chain homotopy**. More generally, any two linear transformations (not necessarily chain maps) are said to be **chain homotopic** if their difference is a d (anti)commutator.

It turns out that we can understand the cohomology $H(\operatorname{End} C)$ in terms of $H(C)$ as follows. If $f \in \operatorname{End} C$ is a chain map, it induces a linear transformation f_* in $H(C)$ by

$$f_*[v] \equiv [fv]\ . \tag{II.1.17}$$

This linear transformation is clearly well-defined, *i.e.*, it does not depend on the choice of representative cocycle for the class $[v]$: for if $w = v + du$ then $fw = fv + fdu = fv \pm dfu$. Similarly if f and g are chain homotopic chain maps they induce the same map in $H(C)$. In fact, for any cocycle v, $fv - gv = \left[d\,,\,h\right]v = dhv$ and thus $[fv] = [gv]$, whence $f_* = g_*$. Therefore we have a natural linear map

$$H(\operatorname{End} C) \to \operatorname{End} H(C) \tag{II.1.18}$$

defined by

$$[f] \mapsto f_* \qquad \text{(II.1.19)}\ .$$

Two very natural questions pose themselves:

(i) Are all linear transformations of $H(C)$ induced by chain maps?

(ii) If a chain map induces the zero map in $H(C)$, is it necessarily chain homotopic to zero?

An affirmative answer to the first (resp. second) question is equivalent to the surjectivity (resp. injectivity) of the map $f \mapsto f_*$. Both answers are positive in the special case of C a finite dimensional vector space. We will give a proof in Chapter V in the context of the operator BRST cohomology.

Notice that $H(\mathrm{End}\, C)$ has a further algebraic structure. Namely it is a graded algebra with a multiplication

$$H^p(\mathrm{End}\, C) \otimes H^q(\mathrm{End}\, C) \longrightarrow H^{p+q}(\mathrm{End}\, C) \tag{II.1.20}$$

induced from composition of endomorphisms. To see this notice that

$$\mathrm{ad}\ d(\varphi \circ \psi) = (\mathrm{ad}\ d\,\varphi) \circ \psi + (-1)^g \varphi \circ (\mathrm{ad}\ d\psi) \qquad \text{for } \varphi \in \mathrm{End}_g C\ . \tag{II.1.21}$$

Therefore composition of endomorphisms maps

$$\begin{aligned} \ker\ \mathrm{ad}\ d \otimes \ker\ \mathrm{ad}\ d &\longrightarrow \ker\ \mathrm{ad}\ d \\ \ker\ \mathrm{ad}\ d \otimes \mathrm{im}\ \mathrm{ad}\ d &\longrightarrow \mathrm{im}\ \mathrm{ad}\ d\ , \end{aligned}$$

which makes the following operation well defined

$$[\varphi] \cdot [\psi] \longrightarrow [\varphi \circ \psi]\ . \tag{II.1.22}$$

Now we come to a very important concept which will underlie most of the work described in this dissertation: resolutions. In essence, a resolution of a given object consists of giving it a cohomological description in terms of simpler ones. The fundamental example of a resolution surfaces in Chapter III in our discussion of classical BRST cohomology; although its practical utility will become apparent in Chapters V-VIII. The main idea is very simple. Suppose for definiteness that we have a graded complex (C, d) with the property that all its cohomology resides in zeroth dimension. In other words,

$$H^n(C) = \begin{cases} 0 & \text{for } n \neq 0 \\ H & \text{for } n = 0 \end{cases}\ . \tag{II.1.23}$$

Then we say that the complex (C, d) provides a **resolution** of H. Of course, the utility of a resolution depends on the simplicity of the spaces C^n.

Let us see how one can use a resolution in order to simplify calculations. For this let us assume that C is a finite dimensional vector space so that $C^n = 0$ except for a finite number of n. Suppose further that f is a linear transformation of C which is also a chain map for d. We let f_* denote the linear transformation it induces on H. Then the following formula holds

$$\mathrm{Tr}_H f_* = \sum_{n\in\mathbb{Z}} (-1)^n \mathrm{Tr}_{C^n} f \ . \tag{II.1.24}$$

In particular if f is the identity we have

$$\dim H = \sum_{n\in\mathbb{Z}} (-1)^n \dim C^n \ , \tag{II.1.25}$$

which perhaps is more familiar if we realize that because of (II.1.23) $\dim H$ is really the **Euler characteristic** $\chi(C)$ of the complex (C,d):

$$\chi(C) \equiv \sum_{n\in\mathbb{Z}} (-1)^n \dim H^n(C) \ . \tag{II.1.26}$$

Formula (II.1.24) will be especially useful when we discuss no ghost theorems in Chapters VI-VIII.

A very special kind of resolution is one in which $C^n = 0$ for all $n > 0$. Then the complex can be pictured as follows

$$\cdots \longrightarrow C^{-2} \xrightarrow{d} C^{-1} \xrightarrow{d} C^0 \longrightarrow 0 \ . \tag{II.1.27}$$

The cohomology is given by

$$H^n(C) = \begin{cases} C^0/dC^{-1} \equiv H & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases} \ . \tag{II.1.28}$$

We call such resolutions **projective**. We can augment the complex as follows. We define d acting on C^0 to be the canonical surjection $C^0 \twoheadrightarrow C^0/dC^{-1}$ and we append this space as C^1 to the complex. This yields the following sequence

$$\cdots \longrightarrow C^{-2} \xrightarrow{d} C^{-1} \xrightarrow{d} C^0 \xrightarrow{d} H \longrightarrow 0 \ , \tag{II.1.29}$$

which has the property that the kernel of any arrow is precisely the image of the preceding one. Hence this is an **exact sequence**. Therefore we see that a projective resolution of H consists in constructing an exact sequence with H sitting at the right.

Spectral Sequences

After this brief introduction to the most basic concepts of homological algebra it is upon us to introduce the reader to one of the most powerful gadgets at our disposal when trying to compute cohomologies: the spectral sequence. For the proofs of the theorems we quote in this section, the reader is referred to the books by Lang **[62]**, and Griffiths & Harris **[68]**. A more unified treatment of spectral sequences using Massey's concept of an exact couple can be found in the books by Bott & Tu **[69]**, and Hilton & Stammbach **[63]**. A complete treatment with applications can be found in the book by MacLane **[64]**.

Spectral sequences can be thought of as perturbation theory for cohomology, since it essentially allows us to approximate the cohomology of a complex by computing the cohomology of bigger and bigger chunks. By definition a **spectral sequence** is a sequence $\{(E_r, d_r)\}_{r=0,1,\dots}$ of differential complexes where E_{r+1} is the cohomology of the preceding complex (E_r, d_r). In many cases of interest one has that for $r > R$, $E_r = E_{r+1} = \cdots = E_\infty$. In this case one says that the spectral sequence **converges** to E_∞ and one writes $(E_r) \Rightarrow E_\infty$.

The following is the typical use to which spectral sequences are put to in practice. Suppose we are interested in investigating the cohomology H of a certain complex. If we are lucky we may be able to show (if at all, usually by very general arguments) that there exists a spectral sequence converging to H, whose early (first and/or second) terms are easily computable. Thus one begins to approximate H. It may be that after the first or second term the differentials $\{d_r\}$ are identically zero. Then that term is already isomorphic to the limit term E_∞, in which case the spectral sequence is said to **degenerate** at the E_1 or E_2 terms. In that case we have reduced the computation of H to the computation of the cohomology of a much simpler complex. We will see plenty of examples of this phenomenon in the following chapters.

Sometimes however we are not so lucky and the spectral sequence does not degenerate early, yet it still provides us with a lot of useful information. In particular it can be used to obtain vanishing theorems. Let us elaborate on this. Throughout this work we will consider spectral sequences associated to graded complexes which will converge to the desired cohomology H in a way that will respect the grading. In other words, we will have convergence in each dimension: $(E_r^n) \Rightarrow H^n$ for all n. From the definition of the spectral sequence we notice that E_{r+1}^n is a subquotient of E_r^n and hence if for any r we have a vanishing of cohomology, say, $E_r^n = 0$ for some n, then the vanishing will persist and $H^n = 0$. This propagation of vanishing of cohomology is, in a nutshell, the essence of the vanishing theorems we will be concerned with in this work.

We now describe in some detail the spectral sequences with which we shall be concerned. Since they are all special cases of the spectral sequence which arises from a filtered complex, we start by considering these.

Let (C, d) be a differential complex. By a **filtration** of C we mean a sequence (not necessarily finite) of subspaces $FC = \{F^pC\}$ indexed by an integer p—called the **filtration degree**—such that, for all p, $F^pC \supseteq F^{p+1}C$ and such that $\cup_p F^pC = C$. We will deal exclusively with filtrations which are **bounded**: that is, there exist p_0 and p_1 such that

$$F^pC = \begin{cases} C & \text{for } p \leq p_0 \\ 0 & \text{for } p \geq p_1 \end{cases} . \tag{II.1.30}$$

If the differential respects the filtration, that is, $d\,F^pC \subseteq F^pC$, then (FC, d) is called a **filtered differential complex**.

Let FC be a bounded filtered complex. Then each F^pC is, in its own right, a complex under d and, therefore, its cohomology can be defined. The inclusion $F^pC \subseteq C$ induces a map in cohomology $H(F^pC) \to H(C)$ which, however, is generally not injective. To understand this notice that a cocycle in F^pC may be the differential of a cochain which does not belong to F^pC but to $F^{p-1}C$. Therefore the cohomology class it defines may not be trivial in $H(F^pC)$ but it may be in $H(C)$. Let us denote by $F^pH(C) \subseteq H(C)$ the image of $H(F^pC)$ under the aforementioned map. It is easy to verify that $FH(C)$ defines a filtration of $H(C)$ which is bounded if FC is.

To every filtered vector space FC we can associate a graded vector space $\mathrm{Gr}\, C = \bigoplus_p \mathrm{Gr}^p C$ where

$$\mathrm{Gr}^p C \equiv F^pC/F^{p+1}C \ . \tag{II.1.31}$$

It is easy to see that as vector spaces C and $\mathrm{Gr}\, C$ are isomorphic; although, since C is not necessarily graded, this isomorphism does not extend to an isomorphism of graded spaces.

If (FC, d) is a filtered differential complex then the associated graded space $\mathrm{Gr}\, C$ is also a complex whose differential is induced by d. Notice that if FC is bounded then $\mathrm{Gr}\, C$ is actually finite. Since d respects the filtration, upon passage to the quotient we obtain a map, also called d, which maps $d\colon \mathrm{Gr}^p C \to \mathrm{Gr}^p C$, whose cohomology is denoted by $H(\mathrm{Gr}\, C)$. Notice that although $\mathrm{Gr}\, C$ is graded, the differential has degree zero. This cohomology is usually easier to calculate than $H(C)$ or $H(FC)$; the reason being that the differential in the associated graded complex is usually a simpler operator: parts of d have positive filtration degree, mapping $F^pC \to F^{p+1}\,C$, in which case this is already zero in $\mathrm{Gr}^p C$.

The spectral sequence of a filtered complex relates the two spaces $\mathrm{Gr}\, H(C)$ and $H(\mathrm{Gr}\, C)$. In fact we have the following theorem:

Theorem II.1.32. *Let FC be a bounded filtered complex and $\operatorname{Gr} C$ its associated graded complex. Then there exists a spectral sequence $\{(E_r, d_r)\}$ of graded spaces*

$$E_r = \bigoplus_p E_r^p$$

with

$$d_r \colon E_r^p \to E_r^{p+r}$$

and such that

$$E_0^p \cong \operatorname{Gr}^p C \ ,$$
$$E_1^p \cong H(\operatorname{Gr}^p C) \ ,$$

and

$$E_\infty^p \cong \operatorname{Gr}^p H(C) \ .$$

Moreover the spectral sequence converges finitely to the limit term.

Now suppose that C is a graded complex and let FC be a filtration of C. In this case we can grade the filtration as follows: $F^pC = \bigoplus_n F^pC^n$ where $F^pC^n = F^pC \cap C^n$. The associated graded complex is now bigraded as follows $\operatorname{Gr} C = \bigoplus_{p,n} \operatorname{Gr}^p C^n$ with the obvious definition for $\operatorname{Gr}^p C^n$. Supposing that the filtration is bounded in each dimension we get a slightly modified version of the previous theorem:

Theorem II.1.33. *Let C be a graded complex, FC be a filtration which is bounded in each dimension and $\operatorname{Gr} C$ its associated graded complex. Then there exists a spectral sequence $\{(E_r, d_r)\}$ of bigraded spaces*

$$E_r = \bigoplus_{p,q} E_r^{p,q}$$

with

$$d_r \colon E_r^{p,q} \to E_r^{p+r, q-r+1}$$

and such that

$$E_0^{p,q} \cong \operatorname{Gr}^p C^{p+q} \ ,$$
$$E_1^{p,q} \cong H^{p+q}(\operatorname{Gr}^p C) \ ,$$

and

$$E_\infty^{p,q} \cong \operatorname{Gr}^p H^{p+q}(C) \ .$$

Moreover the spectral sequence converges finitely to the limit term.

There is a small *caveat* we must emphasize. The limit term of the spectral sequence is not the total cohomology but the graded object associated to the induced filtration. Of course, as vector spaces they are isomorphic but that is the end of the isomorphism. If the total cohomology has an extra algebraic structure (say it is an algebra, for instance) the theorem does not guarantee that the limit term E_∞ and the total cohomology as isomorphic as algebras.

The Spectral Sequences of a Double Complex

Two very important special cases of a filtered complex arise from a double complex. A **double complex** is a bigraded vector space $K = \bigoplus_{p,q} K^{p,q}$ (where, for definiteness, we take p,q integral; although this is not essential) and two differentials

$$D'\colon K^{p,q} \to K^{p+1,q} \tag{II.1.34}$$

$$D''\colon K^{p,q} \to K^{p,q+1} \tag{II.1.35}$$

which anticommute. It is often convenient to represent the double complex pictorially as follows

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & K^{p,q+1} & \xrightarrow{D'} & K^{p+1,q+1} & \longrightarrow & \cdots \\ & & \uparrow{\scriptstyle D''} & & \uparrow{\scriptstyle D''} & & \\ \cdots & \longrightarrow & K^{p,q} & \xrightarrow{D'} & K^{p+1,q} & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \\ & & \vdots & & \vdots & & \end{array} \tag{II.1.36}$$

Hence we shall refer to D' and D'' as the horizontal and vertical differentials, respectively.

As far as the operator D' is concerned, the above double complex decomposes into a direct sum of graded complexes (the rows)

$$\cdots \longrightarrow K^{p,q} \xrightarrow{D'} K^{p+1,q} \longrightarrow \cdots \ ; \tag{II.1.37}$$

whose cohomology shall be denoted ${}'H^p(K^{\bullet,q})$ where the $^\bullet$ just reminds us of which is the index running along with the cohomology we are taking. In other words,

$${}'H^p(K^{\bullet,q}) \equiv \frac{\ker\, D' : K^{p,q} \to K^{p+1,q}}{\mathrm{im}\, D' : K^{p-1,q} \to K^{p,q}} \ . \tag{II.1.38}$$

Since D'' anticommutes with D' (*i.e.*, it is a D'–chain map) it induces a map in ${}'H(K)$ which is also a differential since D'' is and which turns the columns of the double complex (after having taking D' cohomology) into graded complexes

$$\cdots \longrightarrow {}'H^p(K^{\bullet,q}) \xrightarrow{D''} {}'H^p(K^{\bullet,q+1}) \longrightarrow \cdots \ , \tag{II.1.39}$$

where, abusing a little the notation, we have called the differential also D''. We can therefore take D'' cohomology to obtain the spaces $''H^q('H^p(K))$ defined by

$$''H^q('H^p(K)) \equiv \frac{\text{ker } D'' : 'H^p(K^{\bullet,q}) \to 'H^p(K^{\bullet,q+1})}{\text{im } D'' : 'H^p(K^{\bullet,q-1}) \to 'H^p(K^{\bullet,q})} \ . \tag{II.1.40}$$

Reversing the roles of D' and D'' we obtain the cohomologies $'H^p(''H^q(K))$ by taking D' cohomology on the D'' cohomologies $''H^q(K^{p,\bullet})$.

What good are these cohomology groups? They will turn out to be first and second order approximations to the same "total" cohomology. Defining the total degree of vectors in $K^{p,q}$ as $p+q$ we may form a graded complex called the **total complex** and denoted by $\text{Tot}\, K = \bigoplus_n \text{Tot}^n K$ where

$$\text{Tot}^n K = \bigoplus_{p+q=n} K^{p,q} \ . \tag{II.1.41}$$

The differential in the total complex is $D = D' + D''$ and is called the **total differential**. Since the total differential has total degree 1

$$D\colon \text{Tot}^n K \to \text{Tot}^{n+1} K \ , \tag{II.1.42}$$

$(\text{Tot}\, K, D)$ becomes a graded complex. We shall deal exclusively with double complexes which satisfy the following finiteness condition: for each n there are only a finite number of non-zero $K^{p,q}$ with $p+q=n$.

There are two canonical filtrations associated to the graded complex $\text{Tot}\, K$. Define

$$'F^p \text{Tot}\, K = \bigoplus_q \bigoplus_{i \geq p} K^{i,q} \tag{II.1.43}$$

and

$$''F^q \text{Tot}\, K = \bigoplus_p \bigoplus_{j \geq q} K^{p,j} \ . \tag{II.1.44}$$

Fix n and define

$$'F^p \text{Tot}^n K = \bigoplus_{i \geq p} K^{i,n-i} \tag{II.1.45}$$

and

$$''F^q \text{Tot}^n K = \bigoplus_{j \geq q} K^{n-j,j} \ . \tag{II.1.46}$$

The finiteness condition for the double complex imply that the above filtrations are bounded for each n. Therefore, for each n, there exist p_0, p_1, q_0, and q_1—which depend on n—such that

$$'F^p\mathrm{Tot}^n K = \begin{cases} \mathrm{Tot}^n K & \text{for } p \le p_0 \\ 0 & \text{for } p \ge p_1 \end{cases} , \tag{II.1.47}$$

and

$$''F^q\mathrm{Tot}^n K = \begin{cases} \mathrm{Tot}^n K & \text{for } q \le q_0 \\ 0 & \text{for } q \ge q_1 \end{cases} . \tag{II.1.48}$$

By the previous theorem there is a spectral sequence associated to each of the filtrations defined above which converges finitely to the **total cohomology**, *i.e.*, the cohomology of the total complex $(\mathrm{Tot}\, K, D)$. What makes this example so important is that the earliest terms in the spectral sequence are easily described in terms of the original data (K, D', D''). In fact, one finds for the horizontal filtration:

Theorem II.1.49. *Associated to the filtration* $'F\mathrm{Tot}\, K$ *there exists a spectral sequence* $\{('E_r, d_r)\}_{r=0,1,\ldots}$ *of bigraded vector spaces*

$$'E_r = \bigoplus_{p,q} {'E_r^{p,q}}$$

with

$$d_r\colon {'E_r^{p,q}} \to {'E_r^{p+r,q-r+1}}$$

such that

$$'E_0^{p,q} \cong K^{p,q} \ ,$$
$$'E_1^{p,q} \cong {''H^q}(K^{p,\bullet}) \ ,$$
$$'E_2^{p,q} \cong {'H^p}({''H^q}(K)) \ ,$$

and

$$'E_\infty^{p,q} \cong \mathrm{Gr}^p\, H^{p+q}(\mathrm{Tot}\, K) \ .$$

Similarly for the vertical filtration we have the following

Theorem II.1.50. *Associated to the filtration* $''F\mathrm{Tot}\, K$ *there exists a spectral sequence* $\{(''E_r, d_r)\}_{r=0,1,\ldots}$ *of bigraded vector spaces*

$$''E_r = \bigoplus_{p,q} {''E_r^{q,p}}$$

with

$$d_r\colon {''E_r^{q,p}} \to {''E_r^{q+r,p-r+1}}$$

such that

$$''E_0^{q,p} \cong K^{p,q} \ ,$$
$$''E_1^{q,p} \cong {'H^p}(K^{\bullet,q}) \ ,$$
$$''E_2^{q,p} \cong {''H^q}({'H^p}(K)) \ ,$$

and

$$''E_\infty^{q,p} \cong \mathrm{Gr}^q\, H^{p+q}(\mathrm{Tot}\, K) \ .$$

As an application of the spectral theorems associated to a double complex let us prove a simple version of the algebraic Künneth formula. This formula relates the cohomology of a tensor product with the tensor product of the cohomologies. In general the relation between these two objects is governed by a universal coefficient theorem, but in the simple case we deal with, they will turn out to be isomorphic.

Suppose that (E, d) and (F, δ) are real **differential graded algebras**. That is, E (resp. F) is a real $\mathbb{Z}$-graded graded-commutative associative algebra $E = \bigoplus_{n\geq 0} E^n$ (resp. $F = \bigoplus_{n\geq 0} F^n$) such that each graded level is finite-dimensional and such that d (resp. δ) is a linear derivation on the algebra of degree 1 obeying $d^2 = 0$ (resp. $\delta^2 = 0$). Define a derivation D on $C \equiv E \otimes F$ as follows:

$$D(e \otimes f) = de \otimes f + (-1)^{\deg\, e} e \otimes \delta f \ . \tag{II.1.51}$$

It is easy to compute that $D^2 = 0$. C admits a bigrading $C^{p,q} \equiv E^p \otimes F^q$; although D does not have any definite properties with respect to it. Define $K^n \equiv \bigoplus_{p+q=n} C^{p,q}$. Then D has degree 1 with respect to this grading. In fact, C becomes a double complex under d and δ whose total complex is (K, D). Notice that for a fixed n, K^n consists of a finite number of $C^{p,q}$'s. Therefore the canonical filtrations associated to this double complex are bounded and we can use Theorem II.1.49 and Theorem II.1.50. One of the spectral sequences is enough to prove the Künneth formula so, for definiteness, we choose to use the horizontal filtration $'FK$. The $'E_1$ term in the spectral sequence is just the δ cohomology of the vertical complexes (indexed by p)

$$\cdots \longrightarrow C^{p,q-1} \xrightarrow{\delta} C^{p,q} \xrightarrow{\delta} C^{p,q+1} \longrightarrow \cdots \ . \tag{II.1.52}$$

But since $C^{p,q} = E^p \otimes F^q$, both E and F are vector spaces, and δ only acts on F^q, the cohomology of (II.1.52) is simply

$$'E_1^{p,q} = E^p \otimes H^q(F) \ . \tag{II.1.53}$$

The $'E_2$ term is the cohomology of the complexes (indexed by q)

$$\cdots \longrightarrow E^{p-1} \otimes H^q(F) \xrightarrow{d} E^p \otimes H^q(F) \xrightarrow{d} E^{p+1} \otimes H^q(F) \longrightarrow \cdots \ ; \tag{II.1.54}$$

which after similar reasoning allows us to conclude that its cohomology is simply

$$'E_2^{p,q} = H^p(E) \otimes H^q(F) \ . \tag{II.1.55}$$

Since the higher differentials d_r are essentially induced by the original differentials and these are already zero at the $'E_2$ level (since they are acting on their respective cohomologies) we see that the spectral sequence degenerates yielding the result

$$\boxed{H^n_D(E\otimes F)\cong \bigoplus_{p+q=n} H^p(E)\otimes H^q(F)} \tag{II.1.56}$$

which is the celebrated **Künneth formula**.

Lie Algebra Cohomology

A very interesting cohomology theory which is intimately linked to BRST cohomology is the cohomology theory of Chevalley & Eilenberg[66] for Lie algebras. For definiteness we shall only treat finite dimensional Lie algebras in this section.

Let $\mathfrak{g}$ be a finite dimensional real Lie algebra and $\mathfrak{M}$ a $\mathfrak{g}$–module affording the representation

$$\begin{aligned} \mathfrak{g}\times\mathfrak{M} &\longrightarrow \mathfrak{M} \\ (X,m) &\longrightarrow X\cdot m \ . \end{aligned} \tag{II.1.57}$$

Let $C^p(\mathfrak{g},\mathfrak{M})$ denote the vector space of linear maps $\bigwedge^p\mathfrak{g}\to\mathfrak{M}$. That is, $C^p(\mathfrak{g},\mathfrak{M})\equiv \mathrm{Hom}(\bigwedge^p\mathfrak{g},\mathfrak{M})\cong\bigwedge^p\mathfrak{g}^*\otimes\mathfrak{M}$. The $C^p(\mathfrak{g},\mathfrak{M})$ are called the p–**Lie algebra cochains** of $\mathfrak{g}$ with coefficients in $\mathfrak{M}$. Next we define a map $d:\mathfrak{M}\to C^1(\mathfrak{g},\mathfrak{M})$ by $(dm)(X)=X\cdot m$ for all $X\in\mathfrak{g}$ and $m\in\mathfrak{M}$. Clearly, $\ker d=\mathfrak{M}^{\mathfrak{g}}$, *i.e.*, the $\mathfrak{g}$–invariant elements of $\mathfrak{M}$.

We now extend d to a map $d:C^1(\mathfrak{g},\mathfrak{M})\to C^2(\mathfrak{g},\mathfrak{M})$ by defining it on monomials $\alpha\otimes m\in\mathfrak{g}^*\otimes\mathfrak{M}\cong C^1(\mathfrak{g},\mathfrak{M})$ as

$$d(\alpha\otimes m)=d\alpha\otimes m-\alpha\wedge dm \ , \tag{II.1.58}$$

where $d\alpha\in\bigwedge^2\mathfrak{g}^*$ is given by

$$(d\alpha)(X,Y)=-\alpha(\left[X\,,\,Y\right]) \ . \tag{II.1.59}$$

In other words, the map $d:\mathfrak{g}^*\to\bigwedge^2\mathfrak{g}^*$ is the negative transpose to the Lie bracket $\left[\,,\,\right]:\bigwedge^2\mathfrak{g}\to\mathfrak{g}$. Next we extend d inductively to an odd derivation

$$\begin{aligned} &d:C^p(\mathfrak{g},\mathfrak{M})\to C^{p+1}(\mathfrak{g},\mathfrak{M}) \\ &d(\omega\otimes m)=d\omega\otimes m+(-1)^p\omega\wedge dm \ . \end{aligned} \tag{II.1.60}$$

We claim that d so defined is actually a differential. Since d is an odd derivation, d^2 is an even derivation and one need only check it on generators: $\alpha\in\mathfrak{g}^*$ and $m\in\mathfrak{M}$. It is trivial to check that $d^2 m = 0$ due to the fact that $X \cdot (Y \cdot m) - Y \cdot (X \cdot m) = \big[X\,,\,Y\big] \cdot m$. Similarly, $d^2\alpha = 0$ due to the Jacobi identity. Therefore, $d^2 = 0$ and

$$C^0(\mathfrak{g}, \mathfrak{M}) \xrightarrow{d} C^1(\mathfrak{g}, \mathfrak{M}) \xrightarrow{d} C^2(\mathfrak{g}, \mathfrak{M}) \xrightarrow{d} \cdots \tag{II.1.61}$$

is a graded complex whose cohomology $H(\mathfrak{g}, \mathfrak{M})$ is called the **Lie algebra cohomology** of $\mathfrak{g}$ with coefficients in $\mathfrak{M}$. In particular, $H^0(\mathfrak{g}, \mathfrak{M}) = \mathfrak{M}^{\mathfrak{g}}$.

In particular, if $\mathbb{R}$ denotes the trivial $\mathfrak{g}$ module, we have that $H(\mathfrak{g}, \mathbb{R}) \cong \mathbb{R}$. The first and second cohomology $H^1(\mathfrak{g}, \mathbb{R})$ and $H^2(\mathfrak{g}, \mathbb{R})$ have useful algebraic interpretations. Let $\alpha \in \mathfrak{g}^*$. Then $d\alpha = 0$ if and only if, for every $X, Y \in \mathfrak{g}$, $\alpha(\big[X\,,\,Y\big]) = 0$, *i.e.*, if the linear functional α is identically zero in the first derived ideal $\big[\mathfrak{g}\,,\,\mathfrak{g}\big]$. In other words, we have an isomorphism

$$H^1(\mathfrak{g}, \mathbb{R}) \cong \mathfrak{g}/\big[\mathfrak{g}\,,\,\mathfrak{g}\big] \ , \tag{II.1.62}$$

from which we deduce that $H^1(\mathfrak{g}, \mathbb{R}) = 0 \iff \big[\mathfrak{g}\,,\,\mathfrak{g}\big] = \mathfrak{g}$. Similarly, let $c \in \bigwedge^2 \mathfrak{g}^*$ obey $dc = 0$. This is equivalent to the cocycle condition

$$c(\big[X\,,\,Y\big], Z) + c(\big[Y\,,\,Z\big], X) + c(\big[Z\,,\,X\big], Y) = 0 \ , \tag{II.1.63}$$

for all $X, Y, Z \in \mathfrak{g}$. To interpret this algebraically, toss in an extra abstract generator k and consider the augmented space $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus k\mathbb{R}$ and define a new bracket by

$$\big[X\,,\,Y\big]_c = \big[X\,,\,Y\big] + c(X, Y)\,k \ , \tag{II.1.64}$$

and by the requirement that k be central. Then the cocycle condition (II.1.63) is equivalent to the Jacobi identities for the new bracket. Hence $\widehat{\mathfrak{g}}$ becomes a Lie algebra. In fact, it is a one-dimensional central extension of $\mathfrak{g}$. If $c = d\alpha$ for some linear functional $\alpha \in \mathfrak{g}^*$ then we can define $\widetilde{X} = X - \alpha(X)\,k \in \widehat{\mathfrak{g}}$ so that

$$\big[\widetilde{X}\,,\,\widetilde{Y}\big]_c = \big[\widetilde{X\,,\,Y}\big] \ ; \tag{II.1.65}$$

hence the central element drops out. Therefore $H^2(\mathfrak{g}, \mathbb{R})$ is in bijective correspondence with the equivalence classes of non-trivial central extensions of $\mathfrak{g}$.

There is a classic theorem in Lie algebra cohomology known as the **Whitehead lemma**:

Theorem II.1.66. *If $\mathfrak{g}$ is a finite dimensional real semisimple Lie algebra then $H^1(\mathfrak{g}, \mathbb{R}) = H^2(\mathfrak{g}, \mathbb{R}) = 0$.*

Cohomologywise semisimple Lie algebras are not very exciting. In fact, an equivalent characterization of semisimple finite dimensional Lie algebras is that their cohomology groups $H^p(\mathfrak{g}, \mathfrak{M})$ vanish for any non-trivial irreducible module $\mathfrak{M}$.

We shall have more to say about Lie algebra cohomology in Chapter VI when we relate BRST to the semi-infinite cohomology of Feigin.

2. Symplectic Reduction and Dirac's Theory of Constraints

In this section we establish the vocabulary and notation concerning symplectic geometry and phrase Dirac's theory of constraints in a slightly more geometric language. We also discuss symplectic reduction, as this will be a dominant theme in our treatment of classical BRST cohomology. This section is not meant to be expository but rather a brief reacquaintance with the classical mechanics of constrained systems from a slightly more geometric approach in the coordinate-free language of modern differential geometry. Any and all proofs missing from our treatment can be found in varying degrees of mathematical sophistication in the books by Arnold **[70]**, Abraham & Marsden **[71]**, Guillemin & Sternberg **[72]**, and in the excellent notes of Weinstein **[73]**. The classical treatment of constraints is to be found in Dirac's wonderful notes **[16]**.

We start by setting up the notation we will adhere to throughout the rest of our discussion. We then discuss symplectic reduction with respect to a coisotropic submanifold, which will be the geometric framework in which Dirac's theory of first class constraints will be treated. We end the section with a look at a very important special case of first class constraints: those arising from a moment map. Since we are eventually interested in classical BRST cohomology we are mostly concerned with first class constraints. However, second class constraints have an equally solid geometric underpinning, known as symplectic restriction, which, in an attempt to offer the reader unfamiliar with this language another reference point, we have decided to cover as well.

Elementary Symplectic Geometry

A **symplectic manifold** is a pair (M, Ω) consisting of a differentiable manifold M and a closed smooth non-degenerate 2–form Ω. The condition of non-degeneracy refers to the property that the induced map $\Omega^\flat$ taking vector fields to 1–forms and defined by $X \mapsto \Omega(X, \cdot)$ is an isomorphism. In other words, that if $\Omega(X, Y) = 0$ for all vector fields Y, then this implies that $X = 0$. Notice that this requires M to be even dimensional.

The prime example of a symplectic manifold is the cotangent bundle T^*N of a differentiable manifold. This corresponds to the phase space of the configuration space N.

Choose local coordinates q^i for N and let p^i denote coordinates for the covectors. Then the symplectic form for T^*N is given by $\Omega = -d\theta$, where θ is the canonical 1-form on T^*N given locally by $\sum_i p^i \, dq^i$.

The **symplectic form** Ω allows us to define a bracket in the ring $C^\infty(M)$ of smooth functions on M as follows. Given a function $f \in C^\infty(M)$ we define its associated **hamiltonian vector field** X_f as the unique vector field on M satisfying

$$\Omega^\flat(X_f) + df = 0 \; . \tag{II.2.1}$$

We then define the **Poisson bracket** of two functions $f, g \in C^\infty(M)$ as

$$\{f \, , \, g\} = \Omega(X_f, X_g) \; . \tag{II.2.2}$$

The Poisson bracket is clearly antisymmetric and, moreover, because Ω is closed, obeys the Jacobi indentity. Therefore it makes $C^\infty(M)$ into a Lie algebra. Since functions can be added and multiplied, $C^\infty(M)$ is also a commutative, associative algebra; and both of these structures are further linked by the following relation

$$\{f \, , \, gh\} = \{f \, , \, g\}h + g\{f \, , \, h\} \; , \tag{II.2.3}$$

valid for any $f, g, h \in C^\infty(M)$. A commutative, associative algebra possessing, in addition, a Lie bracket obeying (II.2.3) is called a **Poisson algebra**.

A classic theorem of Darboux says that locally on any symplectic manifold we can always find coordinates (p^i, q^i) such that the symplectic form takes the classic form

$$\Omega = \sum_i dq^i \wedge dp^i \; . \tag{II.2.4}$$

Therefore if f is a smooth function, its hamiltonian vector field is given by

$$X_f = \sum_i \left(\frac{\partial f}{\partial q^i} \frac{\partial}{\partial p^i} - \frac{\partial f}{\partial p^i} \frac{\partial}{\partial q^i} \right) \; , \tag{II.2.5}$$

and if f, g are smooth functions their Poisson bracket takes the familiar form

$$\{f \, , \, g\} = \sum_i \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p^i} - \frac{\partial f}{\partial p^i} \frac{\partial g}{\partial q^i} \right) \; , \tag{II.2.6}$$

which is nothing but $X_f(g)$. Therefore Darboux's theorem just says that locally any symplectic manifold looks just like a phase space of a linear configuration space.

Now fix a point $p \in M$ and look at the vector space T_pM of tangent vectors to M at p; *i.e.*, the space of velocities at p. The symplectic form—being tensorial—restricts nicely to a non-degenerate antisymmetric form on T_pM, making it into a **symplectic vector space**. In a symplectic vector space V, there are four kinds of subspaces which merit our attention. If W is a subspace of V, we let $W^\perp$ denote its **symplectic complement** relative to the symplectic form Ω:

$$W^\perp = \{X \in V \mid \Omega(X,Y) = 0 \; \forall Y \in V\} \ . \tag{II.2.7}$$

Notice that if W is one dimensional, $W \subseteq W^\perp$ due to the antisymmetry of Ω. Subspaces W obeying $W \subseteq W^\perp$ are called **isotropic** and they necessarily obey $\dim W \leq \frac{1}{2}\dim V$. On the other hand, if $W \supseteq W^\perp$, W is called **coisotropic** and it must obey $\dim W \geq \frac{1}{2}\dim V$. If W is both isotropic and coisotropic, then it is its own symplectic complement, it obeys $\dim W = \frac{1}{2}\dim V$ and it is called a **lagrangian** subspace. Finally, if $W \cap W^\perp = 0$, W is called **symplectic**.

Notice that if W is isotropic and, in particular, lagrangian, the restriction of Ω to W is identically zero; whereas if W is symplectic, Ω restricts nicely to a symplectic form. In particular, symplectic subspaces are even dimensional. The most interesting case for us is when W is coisotropic. In this case Ω restricts to a non-zero antisymmetric bilinear form on W but which, nevertheless, is degenerate since any vector in $W^\perp \subseteq W$ is symplectically orthogonal to all of W. But it then follows that the quotient $W/W^\perp$ inherits a well defined symplectic form and hence becomes a symplectic vector space. The passage from V to $W/W^\perp$ (which is a subquotient) is known as the **symplectic reduction** of V relative to the coisotropic subspace W. The next subsection is devoted to the globalization of this procedure.

Symplectic Reduction

A submanifold M_o of a symplectic manifold M is called **isotropic**, **coisotropic**, **lagrangian**, or **symplectic** according to whether at all points $p \in M_o$, T_pM_o is an isotropic, coisotropic, lagrangian, or symplectic subspace of T_pM, respectively.

Suppose that M_o is a coisotropic submanifold of M and let $i : M_o \hookrightarrow M$ denote the inclusion. We let $\Omega_o \equiv i^*\Omega$ denote the pull back of the symplectic form of M onto M_o. It defines a distribution (in the sense of Frobenius), which we call $TM_o^\perp$, as follows. For $p \in M_o$ we let $(TM_o^\perp)_p = (T_pM_o)^\perp$. We will first show that this distribution is involutive. To this effect, let $X, Y \in TM_o^\perp$. Since Ω_o is closed, for all vector fields Z tangent to M_o, we have that

$$
\begin{aligned}
0 =& d\Omega_o(X,Y,Z) \\
=& X\Omega_o(Y,Z) - Y\Omega_o(X,Z) + Z\Omega_o(X,Y) \\
& - \Omega_o(\big[X\,,\,Y\big],Z) + \Omega_o(\big[X\,,\,Z\big],Y) - \Omega_o(\big[Y\,,\,Z\big],X)\ .
\end{aligned}
\tag{II.2.8}
$$

But all terms except the fourth are automatically zero since they involve Ω_o contractions between TM_o and $TM_o^\perp$. Therefore the fourth term is also zero, whence $\big[X\,,\,Y\big] \in TM_o^\perp$. Therefore, by Frobenius' theorem, $TM_o^\perp$ are the tangent spaces to a foliation of M_o which we denote $\mathcal{M}_o^\perp$. We define $\widetilde{M} \equiv M_o/\mathcal{M}_o^\perp$ to be the space of leaves of the foliation and we let $\pi : M_o \twoheadrightarrow \widetilde{M}$ be the natural surjection mapping a point in M_o to the unique leaf it belongs to. Then locally (and also globally, if the foliation is sufficiently well behaved) $\widetilde{M}$ is a smooth manifold, whose tangent space at a leaf is isomorphic to $T_pM_o/T_pM_o^\perp$ for any point p lying in that leaf. We can therefore give $\widetilde{M}$ a symplectic structure $\widetilde{\Omega}$ by demanding that $\pi^*\widetilde{\Omega} = \Omega_o$. In other words, let $\widetilde{X}$, $\widetilde{Y}$ be vectors tangent to $\widetilde{M}$ at a leaf. To compute $\widetilde{\Omega}(\widetilde{X},\widetilde{Y})$ we merely lift $\widetilde{X}$ and $\widetilde{Y}$ to vectors X_o and Y_o tangent to M_o at a point p in the leaf and then compute $\Omega_o(X_o,Y_o)$. The result is clearly independent of the particular lift, since the difference of any two lifts is in $TM_o^\perp$; and, moreover, it is also independent of the particular point p of the leaf since, if Z is a tangent vector to the leaf, the Lie derivative of Ω_o by Z:

$$
\mathcal{L}_Z\Omega_o = d\,\imath(Z)\Omega_o + \imath(Z)d\Omega_o \tag{II.2.9}
$$

vanishes since $d\Omega_o = 0$ and $\imath(Z)\Omega_o = 0$. Therefore $(\widetilde{M},\widetilde{\Omega})$ becomes a symplectic manifold (at least locally) and it is called the **symplectic reduction** of (M,Ω) relative to the coisotropic submanifold (M_o,Ω_o).

Suppose now that M_o is a symplectic submanifold of M and let $i : M_o \hookrightarrow M$ denote the inclusion. We can give M_o a symplectic structure merely by pulling back Ω to M_o. Hence, if $\Omega_o \equiv i^*\Omega$, (M_o,Ω_o) becomes a symplectic manifold, called the **symplectic restriction** of M onto M_o. In this case we can work out fairly explicitly the Poisson bracket of M_o in terms of the Poisson bracket of M: obtaining, as a special case, the celebrated Dirac bracket. We will impose, for convenience, the additional technical assumption that M_o is a closed imbedded submanifold of M. This is necessary and sufficient[**74**] to be able to extend any smooth function on M_o to a smooth function on M and to guarantee that all smooth functions on M_o can be obtained by restriction of smooth functions on M. Most cases that arise in practice satisfy this condition; although this could be precisely why these are the cases that arise in practice.

Let f and g be smooth functions on M_o and let us extend them to smooth functions on M which, allowing ourselves some notational abuse, will also be denoted by f and g, respectively. Let X_f and X_g be their respective hamiltonian vector fields on M, *i.e.*, computed with Ω. Since M_o is symplectic, the tangent space of M at every point $p \in M_o$ can written as the following direct sum

$$T_pM = T_pM_o \oplus (T_pM_o)^\perp \ ,$$

according to which a vector field X can be decomposed as the sum of two vectors: X_T, tangent to M_o; and $X^\perp$ symplectically perpendicular to M_o. Then the Poisson bracket of the two functions f and g on M_o is simply given by

$$\left\{f \, , \, g\right\}_o = \Omega(X_f - X_f^\perp, X_g - X_g^\perp) \ . \tag{II.2.10}$$

Now suppose that $\{Z_\alpha\}$ is a local basis for $TM_o^\perp$.[7] Then, given any vector X we can expand its normal part $X^\perp$ as linear combinations of the Z_α whose coefficients are easily determined as follows. Write

$$X^\perp = \sum_\alpha \lambda^\alpha X_\alpha \ . \tag{II.2.11}$$

Then notice that

$$\Omega(X, Z_\alpha) = \Omega(X^\perp, Z_\alpha) = \sum_\beta \lambda^\beta \Omega(Z_\beta, Z_\alpha) \ . \tag{II.2.12}$$

Because M_o is a symplectic submanifold, the square matrix $\mathbb{M}$ whose entries are given by $\mathbb{M}_{\alpha\beta} = \Omega(Z_\alpha, Z_\beta)$ is invertible. Let $\mathbb{M}^{\alpha\beta}$ be defined by

$$\sum_\beta \mathbb{M}_{\alpha\beta}\mathbb{M}^{\beta\gamma} = \delta_\alpha^\gamma \ . \tag{II.2.13}$$

Then the coefficients λ^α are given by

$$\lambda^\beta = \sum_\alpha \Omega(X, X_\alpha)\mathbb{M}^{\alpha\beta} \ . \tag{II.2.14}$$

---

[7] A sufficient and necessary condition[**75**] for the existence of a global basis is for M_o to be expressible as the zero locus of $(\dim M - \dim M_o)$ smooth functions $\{\chi_\alpha\}$. In that case, the global basis is just given by the hamiltonian vector fields associated to the $\{\chi_\alpha\}$. In general one can easily show that there exist functions $\{\chi_\alpha\}$ which locally describe M_o as their zero locus and whose hamiltonian vector fields provide a local basis for the normal vectors.

Plugging (II.2.14) into (II.2.11) and this into (II.2.10) we find that

$$\{f\,,\,g\}_o = \{f\,,\,g\} - \sum_{\alpha\beta} \Omega(X_f, Z_\alpha)\mathbb{M}^{\alpha\beta}\Omega(Z_\beta, X_g)\ . \tag{II.2.15}$$

If we further suppose that the $\{Z_\alpha\}$ are the hamiltonian vector fields associated (via Ω) to functions $\{\chi_\alpha\}$, then

$$\boxed{\{f\,,\,g\}_o = \{f\,,\,g\} - \sum_{\alpha\beta} \{f\,,\,\chi_\alpha\}\mathbb{M}^{\alpha\beta}\{\chi_\beta\,,\,g\}}\ , \tag{II.2.16}$$

where $\mathbb{M}^{\alpha\beta}$ is now the matrix inverse to the $\{\chi_\alpha\,,\,\chi_\beta\}$. Therefore, $\{\,,\,\}_o$ in nothing but the **Dirac bracket** associated to the second class constraints $\{\chi_\alpha\}$.

<u>First and Second Class Constraints</u>

The purpose of this subsection is to show that the submanifold defined by a set of first class (resp. second class) constraints is coisotropic (resp. symplectic). But first we review Dirac's treatment of constraints. Throughout this subsection (M, Ω) shall be a fixed symplectic manifold on which we have singled out a privileged set of smooth functions $\{\psi_a\}$ which are called **constraints**. That is, the allowed "phase space" of the relevant dynamical system is the zero locus of the constraints

$$\{p \in M \mid \psi_a(p) = 0\ \forall\, a\}\ . \tag{II.2.17}$$

Of course the truly physically relevant information that the constraints convey is their zero locus. Any other set of functions with the same zero locus gives an equivalent description of the physics and this is why, in the modern literature (*cf.* **[71]** and references therein) on constrained dynamics, it is often the subvariety defined by (II.2.17) which is called the constraint. However in practice one needs an algebraic description of the constraints and there the $\{\psi_a\}$ play a crucial rôle; although we should (and will) at the end of the day make sure that none of our constructions depend on the particular choice of functions $\{\psi_a\}$.

Following Dirac let us denote by Ψ the linear subspace of $C^\infty(M)$ generated by the $\{\psi_a\}$; in other words, Ψ consists of linear combinations of the $\{\psi_a\}$ with constant coefficients. Let us also denote by J the ideal of $C^\infty(M)$ they generate. That is, linear combinations of the $\{\psi_a\}$ whose coefficients are arbitrary smooth functions. Then let F be a maximal subspace of Ψ with the property that

$$\left\{F\,,\,\Psi\right\} \subset J\ . \tag{II.2.18}$$

Let $\{\phi_i\}_{i=1}^{l}$ be a basis for F. The $\{\phi_i\}$ are linear combinations with constant coefficients of the $\{\psi_a\}$. Dirac calls the aforementioned basis for F **first class constraints**. Let the subspace S of Ψ complementary to F be spanned by $\{\chi_\alpha\}_{\alpha=1}^{k}$. Dirac calls these functions **second class constraints**. In terms of these functions, (II.2.18) just says that

$$\left\{\phi_i\,,\,\phi_j\right\} = f_{ij}{}^{k}\phi_k + f_{ij}{}^{\alpha}\chi_\alpha \tag{II.2.19}$$

$$\left\{\phi_i\,,\,\chi_\alpha\right\} = f_{i\alpha}{}^{j}\phi_j + f_{i\alpha}{}^{\beta}\chi_\beta \tag{II.2.20}$$

for arbitrary smooth functions $f_{ij}{}^{k}$, $f_{ij}{}^{\alpha}$, $f_{i\alpha}{}^{j}$, and $f_{i\alpha}{}^{\beta}$.

Dirac goes on to prove[16] that the matrix of functions $\left\{\chi_\alpha\,,\,\chi_\beta\right\}$ is nowhere degenerate. This, we will now show, is nothing but the statement that the submanifold defined by the second class constraints is symplectic. We will work under the additional technical assumption that zero is a regular value for the function $\Xi : M \to \mathbb{R}^k$ whose components are the second class constraints, *i.e.*, $\Xi(m) = (\chi_1(m), \ldots, \chi_k(m))$. This will guarantee[74] that the submanifold $N \equiv \Xi^{-1}(0)$ defined by the second class constraints is a closed imbedded submanifold of M. Then the vectors tangent to N are precisely those vectors which are perpendicular to the gradients of the constraints. That is, X is a tangent vector to N if, and only if, $d\chi_\alpha(X) = 0$ for all α. By the definition of the hamiltonian vector fields associated to the constraints, and denoting these by Z_α, the above condition translates into

$$X \in TN \iff \Omega(X, Z_\alpha) = 0\ \forall\,\alpha\ . \tag{II.2.21}$$

Let us denote by $\langle Z_\alpha\rangle$ the span of the vector fields Z_α. Then $TN = \langle Z_\alpha\rangle^\perp$. Since $\Omega(Z_\alpha, Z_\beta) = \left\{\chi_\alpha\,,\,\chi_\beta\right\}$ is non-degenerate, $\langle Z_\alpha\rangle \cap TN = 0$. Taking symplectic complements, $TN \cap TN^\perp = 0$, whence N is a symplectic submanifold of M. Therefore we can restrict ourselves to the symplectic manifold N with the Poisson bracket given by (II.2.16).

We now restrict the first class constraints $\{\phi_i\}$ to N. Allowing a little abuse of notation we still denote them $\{\phi_i\}$. Due to (II.2.19) and (II.2.16) they are still first class constraints. We again put them together in a function $\Phi : N \to \mathbb{R}^l$ and assume that 0 is a regular value of Φ, so that the submanifold $N_o \equiv \Phi^{-1}(0)$ defined by them is a closed imbedded submanifold. We now claim that N_o is a coisotropic submanifold of N. Again the tangent vectors to N_o are those vectors tangent to N such that they are annihilated by the gradients of the constraints

$$X \in TN_o \iff d\phi_i(X) = 0 \; \forall\, i \tag{II.2.22}$$

which, using the definition of the hamiltonian vector fields $\{X_i\}$ associated to the constraints $\{\phi_i\}$, translates into

$$TN_o = \langle X_i \rangle^{\perp} \; . \tag{II.2.23}$$

But—since the constraints are first class—

$$d\phi_i(X_j) = \big\{\phi_i \, , \, \phi_j\big\} = c_{ij}^k \phi_k \; , \tag{II.2.24}$$

which is zero on N_o. Therefore the X_i are tangent to N_o. This is equivalent, taking the symplectic complement of (II.2.23), to

$$TN_o^{\perp} \subset TN_o \; ; \tag{II.2.25}$$

and, hence, to the coisotropy of N_o in N.

The Moment Map

A very special example of first class constraints arises in some cases when (M, Ω) admits a group action which preserves the symplectic structure. A diffeomorphism φ of M is called a **symplectomorphism** if $\varphi^*\Omega = \Omega$, *i.e.*, if it preserves the symplectic structure. Let $\mathrm{Symp}(M)$ denote the Lie subgroup of $\mathrm{Diff}(M)$ consisting of symplectomorphisms. Its Lie algebra $\mathfrak{symp}(M)$ is the Lie subalgebra of the Lie algebra of smooth vector fields on M consisting of those vector fields X obeying $\mathcal{L}_X \Omega = 0$. Such vector fields are called **symplectic**. Since Ω is closed this is equivalent to $\imath(X)\Omega$ being closed. Hence $\mathfrak{symp}(M)$ is the image of the closed 1–forms via the map $\Omega^{\sharp}$ inverse to $\Omega^{\flat}$. The image of the exact 1–forms is an ideal $\mathfrak{ham}(M) \subseteq \mathfrak{symp}(M)$ known as the **hamiltonian vector fields**. In fact, more is true:

$$\big[\mathfrak{symp}(M) \, , \, \mathfrak{symp}(M)\big] \subseteq \mathfrak{ham}(M) \; . \tag{II.2.26}$$

Now suppose that G is a Lie group acting on M via symplectomorphisms. Then this action defines a Lie algebra morphism $\mathfrak{g} \to \mathfrak{symp}(M)$ sending a vector $X \in \mathfrak{g}$ to a symplectic vector field $\widetilde{X}$. If for all $X \in \mathfrak{g}$, $\widetilde{X}$ is a hamiltonian vector field, then the G action is called **hamiltonian**. Notice that because of (II.2.26), if $\big[\mathfrak{g} \, , \, \mathfrak{g}\big] = \mathfrak{g}$—*i.e.*, if $H^1(\mathfrak{g}, \mathbb{R}) = 0$—then this is automatically satisfied. Also if all closed forms are exact, *i.e.*, $H^1_{dR}(M) = 0$, the action is also hamiltonian. Hence we see that the obstructions to a symplectic action being hamiltonian are cohomological in nature.

Suppose then that the G action is hamiltonian. That is, there exist functions ϕ_X for each $X \in \mathfrak{g}$ obeying

$$\imath(\widetilde{X})\Omega + d\phi_X = 0 \ . \tag{II.2.27}$$

The existence of these functions provides a linear map $\mathfrak{g} \to C^\infty(M)$, sending $X \to \phi_X$ which, nevertheless, may fail to be a Lie algebra morphism. To identify the obstruction in this case let us compute.

$$\begin{aligned} d\{\phi_X \, , \, \phi_Y\} &= d\Omega(\widetilde{X},\widetilde{Y}) \\ &= d\imath(\widetilde{Y})\imath(\widetilde{X})\Omega \\ &= \mathcal{L}_{\widetilde{Y}}\imath(\widetilde{X})\Omega && \text{since } \widetilde{X} \in \mathfrak{symp}(M) \\ &= \left[\mathcal{L}_{\widetilde{Y}} \, , \, \imath(\widetilde{X})\right]\Omega && \text{since } \widetilde{Y} \in \mathfrak{symp}(M) \\ &= \imath(\left[\widetilde{Y} \, , \, \widetilde{X}\right])\Omega \\ &= d\phi_{\left[X \, , Y\right]} \ . \end{aligned}$$

Therefore,

$$c(X,Y) \equiv \{\phi_X \, , \, \phi_Y\} - \phi_{\left[X \, , Y\right]} \tag{II.2.28}$$

is locally constant. We shall assume for simplicity that M is connected and hence it is an honest constant. It is evident that c is antisymmetric and also that it obeys the cocycle conditions

$$c(\left[X \, , \, Y\right], Z) + c(\left[Y \, , \, Z\right], X) + c(\left[Z \, , \, X\right], Y) = 0 \ . \tag{II.2.29}$$

Therefore it defines a projective representation of $\mathfrak{g}$. Notice that ϕ_X are defined up to a constant (*cf.* (II.2.27)) and hence $c(X,Y)$ is defined up to the addition of a term $b(\left[X \, , \, Y\right])$ where b is an arbitrary linear functional on $\mathfrak{g}$. If by redefining the ϕ_X in this way we can shift c to zero, we have an honest representation and we say that the action is **Poisson**. If this is the case, the $\{\phi_i\}$, associated to a basis $\{X_i\}$ for $\mathfrak{g}$, are first class constraints. In particular, if $H^2(\mathfrak{g},\mathbb{R}) = 0$, $\mathfrak{g}$ admits non non-trivial central extension and the action is, again, Poisson. So we see again that the obstruction is cohomological in nature. A very nice derivation of these obstructions in terms of equivariant cohomology is given in the notes of Weinstein[**73**].

Let us suppose that we have a Poisson action of G on (M,Ω). We define the **moment map** $\Phi : M \to \mathfrak{g}^*$ dual to $\mathfrak{g} \to C^\infty(M)$ by

$$\langle \Phi(m), X \rangle = \phi_X(m) \ , \tag{II.2.30}$$

where $\langle , \rangle$ is the dual pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. The Poisson property of the action guarantees that this map is **equivariant**: intertwining between the action of $\mathfrak{g}$ on M and the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$. Let $M_o \equiv \Phi^{-1}(0)$. If 0 is a regular value then M_o is a G–invariant coisotropic closed imbedded submanifold of M. In particular, the symplectic Killing vectors $\widetilde{X}$ are tangent to M_o and they define a foliation $\mathcal{G}$ of M_o whose leaves are the orbits of the G action, *i.e.*, the **gauge orbits**. The space of orbits $\widetilde{M} \equiv M_o/\mathcal{G}$ is (at least locally) a symplectic manifold and is a special case of the symplectic reduction of Marsden & Weinstein[**76**].

Symplectic Reduction of a Phase Space

In physics most symplectic manifolds are phase spaces, *i.e.*, cotangent bundles T^*N of a suitable configuration space N. Moreover many of the symmetries that arise in the study of dynamical systems are already symmetries of the configuration space. For example, in Yang-Mills the configuration space is the (convex) space $\mathfrak{A}$ of gauge fields (=connection 1-forms in a principal bundle over spacetime) and the gauge transformations $\mathfrak{G}$ have a well defined action on the connections. The physical configuration space is the space of gauge orbits $\mathfrak{A}/\mathfrak{G}$. Another example is given by bosonic string theory. The configuration space is the space of smooth maps $\mathrm{Map}(S^1, M)$ from the string to spacetime; whereas the physical configurations cannot distinguish between two smooth maps which are related by a reparametrization of the string. Hence the physical configurations are the space of orbits under $\mathrm{Diff}\, S^1$. Finally another example is general relativity in the hamiltonian description. Fixing a spacelike hypersurface Σ in spacetime, the configuration space is the "superspace" consisting of riemannian metrics on Σ. Just like in the string, to obtain the physical configurations we must identify configurations which are related by a diffeomorphism of Σ.

It turns out that whenever the configuration space N admits a smooth group action, the action automatically lifts to the phase space T^*N in such a way that it does not just preserves the symplectic form, but it also gives rise to an equivariant moment map which is linear in the momenta. That the action on N lifts to a symplectic action on T^*N follows from the fact that the canonical 1-form θ on T^*N is a diffeomorphism invariant of N. In other words, let $\varphi : N \to N$ be a diffeomorphism and let $T^*\varphi$ denote the induced diffeomorphism on T^*N. Then $(T^*\varphi)^*\theta = \theta$. Hence it also preserves the symplectic form $\Omega = -d\theta$.

So let G act on N via diffeomorphisms. Then if $X \in \mathfrak{g}$ is a vector in the Lie algebra, it gives rise to a Killing vector $\widetilde{X}$ on N and a Killing vector $\widehat{X}$ in T^*N. Since the canonical 1–form θ is G invariant, we have that

$$\begin{aligned} 0 =& \mathcal{L}_{\widehat{X}}\theta \\ =& d\imath(\widehat{X})\theta + \imath(\widehat{X})d\theta \end{aligned}$$

$$=d\imath(\widehat{X})\theta - \imath(\widehat{X})\Omega \ ,$$

Hence $\imath(\widehat{X})\Omega = d\imath(\widehat{X})\theta$, whence the hamiltonian function associated to X is $\phi_X = -\theta(\widehat{X})$. Therefore the G action is hamiltonian. But for $X, Y \in \mathfrak{g}$,

$$\begin{aligned}
\phi_{[X\,,\,Y]} &= -\,\imath([\widehat{X}\,,\,\widehat{Y}])\theta \\
&= -\,[\mathcal{L}_{\widehat{X}}\,,\,\imath(\widehat{Y})]\theta \\
&= -\,\mathcal{L}_{\widehat{X}}\imath(\widehat{Y})\theta && \text{since } \mathcal{L}_{\widehat{X}}\theta = 0 \\
&= -\,\imath(\widehat{X})d\imath(\widehat{Y})\theta \\
&= \imath(\widehat{X})\imath(\widehat{Y})d\theta && \text{since } \mathcal{L}_{\widehat{Y}}\theta = 0 \\
&= \Omega(\widehat{X}, \widehat{Y}) \\
&= \{\phi_X\,,\,\phi_Y\} \ . && \text{(II.2.31)}
\end{aligned}$$

Therefore the action is also Poisson.

The induced equivariant moment map is easy to write down explicitly. Let $\alpha \in T^*N$ be thought of as a 1-form on N at the point $\widetilde{\pi}(\alpha) \in N$, where $\widetilde{\pi} : T^*N \twoheadrightarrow N$ is the canonical projection sending a covector on N to the point on which it is defined. Then the moment map $\Phi : T^*N \to \mathfrak{g}^*$ is given by

$$\langle \Phi(\alpha), X \rangle = \langle \alpha, \widetilde{X} \rangle_{\widetilde{\pi}(\alpha)} \ , \tag{II.2.32}$$

where the right hand side of this equation refers to the dual pairing between tangent vectors and covectors on N at the point $\widetilde{\pi}(\alpha)$. Given local coordinates (p, q) on T^*N associated to local coordinates q for N, we have that the components of the moment map are

$$\phi_X(p, q) = p_i \widetilde{X}^i(q) \ , \tag{II.2.33}$$

whence linear in the momenta. Conversely, if a transformation on phase space induces a transformation on the configuration space, its associated hamiltonian function (which always exists locally) must be linear in the momenta, since its Poisson brackets with a function on configuration space $f(q)$ cannot depend on the momenta.

The symplectic reduction in this case, $\Phi^{-1}(0)/\mathcal{G}$, is nothing but the phase space of the reduced configuration space:

$$\Phi^{-1}(0)/\mathcal{G} \cong T^*(N/G) \ ; \tag{II.2.34}$$

hence the name **reduced phase space**.

Chapter Three:

Classical BRST Cohomology

In this chapter we discuss the BRST construction in a classical mechanics setting. Classical BRST is a cohomology theory which, in a sense to be made precise below, is dual to symplectic reduction. As explained in Section II.2, in symplectic reduction one starts with a symplectic manifold (M, Ω) and a given coisotropic submanifold $i : M_o \hookrightarrow M$ and constructs another symplectic manifold $\widetilde{M}$ defined as the space of leaves of the characteristic (null) foliation associated to the 2–form $i^*\Omega$ on M_o. What the BRST construction achieves is a cohomological description of this procedure. That such a cohomological description exists should not come as a complete surprise since after all both symplectic reduction and cohomology are subquotients. The goal of the BRST construction is to make this heuristic observation precise; and in order to do so we must learn how to describe geometric objects algebraically.

Dual to a manifold M we have the commutative algebra $C^\infty(M)$ of its smooth functions which characterize it completely. The correspondence goes roughly as follows. To every point $p \in M$ there corresponds an ideal $I(p)$ of $C^\infty(M)$ consisting of those functions vanishing at p. Since it is the kernel (via the evaluation map) of a homomorphism onto a field this ideal is maximal. Moreover with respect to any topology on $C^\infty(M)$ relative to which the evaluation map is continuous, $I(p)$ is closed. Hence we have an assignment of a maximal closed ideal of $C^\infty(M)$ to every point in M. It turns out that these are all the maximal closed ideals there are. So that as a set M is just the set $\mathcal{M}$ of maximal closed ideals of $C^\infty(M)$. In fact, one can topologize and give a differentiable structure to $\mathcal{M}$ in such a way that the set isomorphism $\mathcal{M} \cong M$ is really a diffeomorphism.

Similarly if $i : M_o \hookrightarrow M$ is a submanifold, it can be described by an ideal $I(M_o)$ consisting of the smooth functions vanishing on M_o. Clearly $I(M_o) = \cap_{p \in M_o} I(p)$. For a special type of submanifolds M_o, $I(M_o)$ is finitely generated. This corresponds to submanifolds which are described as the regular zero locus of a set of smooth functions. Then these functions generate $I(M_o)$ over $C^\infty(M)$. This will be the case of interest in this chapter. The rôle of the submanifold M_o will be played by the zero locus of a set of first class constraints on a symplectic manifold.

The BRST construction will follow three steps. The first step is to construct a cohomological description (a resolution) of the smooth functions on M_o from the smooth functions on M. The second step, which is independent from the first, is to describe cohomologically the functions on $\widetilde{M}$ starting from the functions on M_o. Finally the third step combines these two into a cohomology theory (BRST) which describes the smooth functions on $\widetilde{M}$ from the smooth functions (plus some extra ingredients) on M.

This chapter is organized as follows. In Section 1 we study the first step of the subquotient: the restriction to the subspace. Suppose $i : M_o \hookrightarrow M$ is a closed embedded submanifold of codimension k corresponding to the zero set (assumed regular) of a smooth function $\Phi : M \to \mathbb{R}^k$. We then define a Koszul-like complex associated to this embedding, which will play a central rôle in the constructions of the BRST cohomology theory. This complex yields a free acyclic resolution for $C^\infty(M_o)$ thought of as a $C^\infty(M)$-module. We give a novel proof of the acyclicity of this complex in which we introduce a double complex completely analogous to the Čech-de Rham complex introduced by Weil in order to prove the de Rham theorem. We call it the Čech-Koszul complex.

In Section 2 we tackle the second step of the subquotient: the quotient of the subspace. We define a cohomology theory associated to the foliation determined by the null distribution of $i^*\Omega$ on M_o. This is a de Rham-like cohomology theory of differential forms (co)tangent to the leaves of the foliation (vertical forms) relative to the exterior derivative along the leaves of the foliation (vertical derivative). If the foliation fibers onto a smooth manifold $\widetilde{M}$—the symplectic quotient of M by M_o—the zeroth cohomology is naturally isomorphic to $C^\infty(\widetilde{M})$. We then lift this cohomology theory via the Koszul resolution obtained in Section 1 to a cohomology theory (BRST) in a certain bigraded complex. The existence of this cohomology theory must be proven since the vertical derivative does not lift to a differential operator, *i.e.*, its square is not zero. However its square is chain homotopic to zero (relative to the Koszul differential) and the acyclicity of the Koszul resolution allows us to construct the desired differential.

In Section 3 we place the BRST construction in a truly symplectic setting. It should be emphasized that the BRST procedure *per se* is not really tied down to symplectic geometry. It should be amply evident from Sections 1 and 2, that we never make essential use of the symplectic structure of M. However when we take advantage of the symplectic structure, the BRST construction becomes so much more natural and manageable from a computational point of view. In this section we first review the basics of Poisson superalgebras and we then show that the BRST cohomology constructed in Section 2 is naturally expressed in this context. This allows us to prove that not only the ring and module structures are preserved under BRST cohomology but, more importantly, the Poisson structures also correspond. In fact, the BRST cohomology can be interpreted as the cohomology of an inner derivation on the ring of "smooth" functions of a De Witt supermanifold; although we will not follow this point of view here.

Finally in Section 4 we compute the classical BRST cohomology in terms of initial data. In particular we show that the BRST cohomology only depends on the constrained submanifold $i : M_o \hookrightarrow M$ eliminating in this way the fictitious dependence on the actual form of the constraints used to define it. The cleanest results arise from the case of a group action. We show that the classical BRST cohomology is given by the smooth functions on the reduced symplectic manifold taking values in the de Rham cohomology of the Lie group. We also prove a duality theorem for the BRST cohomology.

1. The Čech-Koszul Complex

We saw in our discussion on symplectic reduction that the reduction process was essentially a subquotient, consisting of two steps:

(i) restriction to the constrained submanifold; and

(ii) identifying points lying in the same leaf of the foliation; *i.e.*, taking a topological quotient.

In this section we describe algebraically the "restriction" part of the process. It is of a more general nature than the symplectic reduction, as should be amply evident to the reader. In particular, we never make use of the symplectic structure. So throughout this section M is an arbitrary smooth manifold and the "constraints" are arbitrary smooth functions. The key idea of this section is to construct a projective resolution for the smooth functions of the constrained submanifold M_o in terms of the smooth functions of M. This will allow us to, in effect, work with the functions on M_o without actually having to restrict ourselves to M_o.

For M_o a closed imbedded submanifold, any smooth function on M_o extends to a smooth function on M and the difference of any two such extensions vanishes on M_o. Hence if we let $I(M_o)$ denote the (multiplicative) ideal of $C^\infty(M)$ consisting of functions which vanish at M_o, we have the following isomorphism

$$C^\infty(M_o) \cong C^\infty(M)/I(M_o) \ . \tag{III.1.1}$$

This is still not satisfactory since $I(M_o)$ is not a very manageable object. It will turn out that

$I(M_o)$ is precisely the ideal J generated by the constraints. Still this is not completely satisfactory because we would rather work with the constraints themselves than with the ideal they generate. The solution of this problem relies on a construction due to Koszul[77],[78]. We will see that there is a differential complex (the Koszul complex)

$$\cdots \longrightarrow K^2 \longrightarrow K^1 \longrightarrow C^\infty(M) \longrightarrow 0 \; , \tag{III.1.2}$$

whose cohomology in positive dimensions is zero and in zero dimension is precisely $C^\infty(M_o)$. We shall refer to this fact as the **quasi-acyclicity** of the Koszul complex. It will play a fundamental rôle in all our constructions.

The Local Koszul Complex

We will first discuss the construction on $\mathbb{R}^m$ and later we will globalize to M. We start with an elementary observation.

Lemma III.1.3. *Let $\mathbb{R}^m$ be given coordinates*

$$(y, x) = (y^1, \ldots, y^k, x^1, \ldots, x^{m-k}) \; .$$

Let $f : \mathbb{R}^m \to \mathbb{R}$ be a smooth function such that $f(\mathbf{0}, x) = 0$. Then there exist k smooth functions $h_i : \mathbb{R}^m \to \mathbb{R}$ such that $f = \sum_{i=1}^k \phi_i\, h_i$, where the ϕ_i are the functions defined by $\phi_i(y, x) = y^i$.

Proof: Notice that

$$\begin{aligned} f(y,x) &= \int_0^1 dt\, \frac{d}{dt} f(ty,x) \\ &= \int_0^1 dt\, \sum_{i=1}^k y^i\, (D_i\, f)(ty,x) \\ &= \sum_{i=1}^k y^i \int_0^1 dt\, (D_i\, f)(ty,x) \\ &= \sum_{i=1}^k \phi_i(y,x) \int_0^1 dt\, (D_i\, f)(ty,x) \; , \end{aligned}$$

where D_i is the i^{th} partial derivative. Defining

$$h_i(y,x) \stackrel{\text{def}}{=} \int_0^1 dt\, (D_i\, f)(ty,x) \tag{III.1.4}$$

the proof is complete. ■

Therefore, if we let $P \subset \mathbb{R}^m$ denote the subspace defined by $y^i = 0$ for all i, the ideal of $C^\infty(\mathbb{R}^m)$ consisting of functions which vanish on P is precisely the ideal generated by the functions ϕ_i.

Definition III.1.5. Let R be a commutative ring with unit. A sequence $(\phi_i)_{i=1}^k$ of elements of R is called **regular** if for all $j = 1, \ldots, k$, ϕ_j is not a zero divisor in R/I_{j-1}, where I_j is the ideal generated by $\phi_1, \ldots, \phi_j$ and $I_0 = 0$. In other words, if $f \in R$ and for any $j = 1, \ldots, k$, $\phi_j\, f \in I_{j-1}$ then $f \in I_{j-1}$ to start out with. In particular, ϕ_1 is not identically zero.

Proposition III.1.6. *Let $\mathbb{R}^m$ be given coordinates*

$$(y, x) = (y^1, \ldots, y^k, x^1, \ldots, x^{m-k}) \ .$$

Then the sequence (ϕ_i) in $C^\infty(\mathbb{R}^m)$ defined by $\phi_i(y, x) = y^i$ is regular.

Proof: First of all notice that ϕ_1 is not identically zero. Next suppose that $(\phi_1, \ldots, \phi_j)$ is regular. Let P_j denote the hyperplane defined by $\phi_1 = \cdots = \phi_j = 0$. Then by Lemma III.1.3, $C^\infty(P_j) = C^\infty(\mathbb{R}^m)/I_j$. Let $[f]_j$ denote the class of a $f \in C^\infty(\mathbb{R}^m)$ modulo I_j. Then ϕ_{j+1} gives rise to a function $[\phi_{j+1}]_j$ in $C^\infty(P_j)$ which, if we think of P_j as coordinatized by

$$(y^{j+1}, \ldots, y^k, x^1, \ldots, x^{m-k}) \ ,$$

turns out to be defined by

$$[\phi_{j+1}]_j(y^{j+1}, \ldots, y^k, x^1, \ldots, x^{m-k}) = y^{j+1} \ . \tag{III.1.7}$$

This is clearly not identically zero and, therefore, the sequence $(\phi_1\ \ldots, \phi_{j+1})$ is regular. By induction we are done. ■

We now come to the definition of the Koszul complex. Let R be a ring and let $\Phi = (\phi_1, \ldots, \phi_k)$ be a sequence of elements of R. We define a complex $K(\Phi)$ as follows: $K^0(\Phi) = R$ and for $p > 0$, $K^p(\Phi)$ is defined to be the free R module with basis $\{b_{i_1} \wedge \cdots \wedge b_{i_p} \mid 0 < i_1 < \cdots < i_p \leq k\}$.

Define a map $\delta_K : K^p(\Phi) \to K^{p-1}(\Phi)$ by $\delta_K b_i = \phi_i$ and extending to all of $K(\Phi)$ as an R-linear antiderivation. That is, δ_K is identically zero on $K^0(\Phi)$ and

$$\delta_K(b_{i_1} \wedge \cdots \wedge b_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1} \phi_{i_j}\, b_{i_1} \wedge \cdots \wedge \widehat{b_{i_j}} \wedge \cdots \wedge b_{i_p} \ , \tag{III.1.8}$$

where a $\widehat{\ }$ adorning a symbol denotes its omission. It is trivial to verify that $\delta_K^2 = 0$, yielding a complex

$$0 \longrightarrow K^k(\Phi) \xrightarrow{\delta_K} K^{k-1}(\Phi) \longrightarrow \cdots \longrightarrow K^1(\Phi) \longrightarrow R \longrightarrow 0 \ , \tag{III.1.9}$$

called the **Koszul complex**.

The following theorem is a classical result in homological algebra whose proof is completely straight-forward and can be found, for example, in **[62]**.

Theorem III.1.10. *If $(\phi_1, \ldots, \phi_k)$ is a regular sequence in R then the cohomology of the Koszul complex is given by*

$$\boxed{H^p(K(\Phi)) \cong \begin{cases} \mathbf{0} & \text{for } p > 0 \\ R/J & \text{for } p = 0 \end{cases}} \;, \tag{III.1.11}$$

where J is the ideal generated by the ϕ_i.

Therefore the complex $K(\Phi)$ provides an acyclic resolution (known as the **Koszul resolution**) for the R-module R/J. Therefore if $R = C^\infty(\mathbb{R}^m)$ and Φ is the sequence $(\phi_1, \ldots, \phi_k)$ of Proposition III.1.6, the Koszul complex gives an acyclic resolution of $C^\infty(\mathbb{R}^m)/J$ which by Lemma III.1.3 is just $C^\infty(P_k)$, where P_k is the subspace defined by $\phi_1 = \cdots = \phi_k = 0$. The $\{b_i\}$ in the Koszul complex are the classical **antighosts**.

Globalization: The Čech-Koszul Complex

We now globalize this construction. Let M be our original symplectic manifold and $\Phi : M \to \mathbb{R}^k$ be the function whose components are the first class constraints constraints, *i.e.*, $\Phi(m) = (\phi_1(m), \ldots, \phi_k(m))$. We assume that 0 is a regular value of Φ so that $M_o \equiv \Phi^{-1}(0)$ is a closed embedded submanifold of M. Therefore for each point $m \in M_o$ here exists an open set $U \in M$ containing m and a chart $\Psi : U \to \mathbb{R}^m$ such that Φ has components $(\phi_1, \ldots, \phi_k, x^1, \ldots, x^{m-k})$ and such that the image under Φ of $U \cap M_o$ corresponds exactly to the points $(\underbrace{0, \ldots, 0}_{k}, x^1, \ldots, x^{m-k})$. Let $\mathcal{U}$ be an open cover for M consisting of sets like these. Of course, there will be some sets $V \in \mathcal{U}$ for which $V \cap M_o = \varnothing$.

To motivate the following construction let's understand what is involved in proving, for example, that the ideal J generated by the constraints coincides with the ideal $I(M_o)$ of smooth functions which vanish on M_o. It is clear that $J \subset I(M_o)$. We want to show the converse. That is, if f is a smooth function vanishing on M_o then there are smooth functions h^i such that $f = \sum_i h^i \phi_i$. This is always true locally. That is, restricted to any set $U \in \mathcal{U}$ such that $U \cap M_o \neq \varnothing$, Lemma III.1.3 implies that there will exist functions $h^i_U \in C^\infty(U)$ such that on U

$$f_U = \sum_i \phi_i \, h^i_U \;, \tag{III.1.12}$$

where f_U denotes the restriction of f to U. If, on the other hand, $V \in \mathcal{U}$ is such that $V \cap M_o = \varnothing$, then not all of the ϕ_i vanish and the statement is also true. There is a certain ambiguity in the choice of h^U_i. In fact, if δ_K denotes the Koszul differential we notice that (III.1.12) can be written as $f_U = \delta_K h_U$, where $h_U = \sum_i h^i_U b_i$ is a Koszul 1-cochain on U. Therefore, the ambiguity in h_U is precisely a Koszul 1-cocycle on U, but by Theorem III.1.10, the Koszul complex on U is quasi-acyclic and hence every 1-cocycle is a 1-coboundary. What we would like to show is that this ambiguity can be exploited to choose the h_U in such a way that $h_U = h_V$ on all non-empty overlaps $U \cap V$. This condition is precisely the condition for h_U to be a Čech 0-cocycle. In order to analyze this problem it is useful to make use of the machinery of Čech cohomology with coefficients in a sheaf. For a review of the necessary material we refer the reader to **[69]**; and, in particular, to their discussion of the Čech-de Rham complex. Our construction is very close in spirit to that one: in fact, it should properly be called the Čech-Koszul complex.

Let $\mathcal{E}_M$ denote the sheaf of germs of smooth functions on M and let $\mathcal{K} = \bigoplus_p \mathcal{K}^p$ denote the free sheaf of $\mathcal{E}_M$-modules which appears in the Koszul complex: $\mathcal{K}^p = \bigwedge^p \mathbb{V} \otimes \mathcal{E}_M$, where $\mathbb{V}$ is the vector space with basis $\{b_i\}$. Let $C^p(\mathcal{U};\mathcal{K}^q)$ denote the Čech p-cochains with coefficients in the Koszul subsheaf $\mathcal{K}^q$. This becomes a double complex under the two differentials

$$\check{\delta} : C^p(\mathcal{U};\mathcal{K}^q) \to C^{p+1}(\mathcal{U};\mathcal{K}^q) \qquad \text{“Čech”}$$

and

$$\delta_K : C^p(\mathcal{U};\mathcal{K}^q) \to C^p(\mathcal{U};\mathcal{K}^{q-1}) \qquad \text{“Koszul”}$$

which clearly commute, since they are independent. We can therefore define the complex $CK^n = \bigoplus_{p-q=n} C^p(\mathcal{U};\mathcal{K}^q)$ and the differential $D = \check{\delta} + (-1)^p \delta_K$ on $C^p(\mathcal{U};\mathcal{K}^q)$. The total differential has total degree one $D : CK^n \to CK^{n+1}$ and moreover obeys $D^2 = 0$. Since the double complex is bounded, *i.e.*, for each n, CK^n is the direct sum of a finite number of $C^p(\mathcal{U};\mathcal{K}^q)$'s, Theorem II.1.49 and Theorem II.1.50 guarantee the existence of two spectral sequences converging to the total cohomology. We now proceed to compute them. In doing so we will find it convenient to depict our computations graphically. The original double complex is depicted by the following diagram:

$C^0(\mathcal{U};\mathcal{K}^2)$	$C^1(\mathcal{U};\mathcal{K}^2)$	$C^2(\mathcal{U};\mathcal{K}^2)$	
$C^0(\mathcal{U};\mathcal{K}^1)$	$C^1(\mathcal{U};\mathcal{K}^1)$	$C^2(\mathcal{U};\mathcal{K}^1)$	
$C^0(\mathcal{U};\mathcal{K}^0)$	$C^1(\mathcal{U};\mathcal{K}^0)$	$C^2(\mathcal{U};\mathcal{K}^0)$	

Upon taking cohomology with respect to the horizontal differential (*i.e.*, Čech cohomology) and using the fact that the sheaves $\mathcal{K}^q$ are fine, being free modules over the structure sheaf

$\mathcal{E}_M$, we get

$K^2(\Phi)$	0	0	
$K^1(\Phi)$	0	0	
$K^0(\Phi)$	0	0	

where $K^p(\Phi) \cong \bigwedge^p \mathbb{V} \otimes C^\infty(M)$ are the spaces in the Koszul complex on M. Taking vertical cohomology yields the Koszul cohomology

$H^2(K(\Phi))$	0	0	
$H^1(K(\Phi))$	0	0	
$H^0(K(\Phi))$	0	0	

Since the next differential in the spectral sequence necessarily maps across columns it must be identically zero. The same holds for the other differentials and we see that the spectral sequence degenerates at the E_2 term. In particular the total cohomology is isomorphic to the Koszul cohomology:

$$H_D^n \cong H^n(K(\Phi)) \ . \tag{III.1.13}$$

To compute the other spectral sequence we first start by taking vertical cohomology, *i.e.*, Koszul cohomology. Because of the choice of cover $\mathcal{U}$ we can use Theorem III.1.10 and Lemma III.1.3 to deduce that the vertical cohomology is given by

0	0	0	
0	0	0	
$C^0(\mathcal{U}; \mathcal{E}_M/\mathcal{J})$	$C^1(\mathcal{U}; \mathcal{E}_M/\mathcal{J})$	$C^2(\mathcal{U}; \mathcal{E}_M/\mathcal{J})$	

where $\mathcal{E}_M/\mathcal{J}$ is defined by the exact sheaf sequence

$$0 \to \mathcal{J} \to \mathcal{E}_M \to \mathcal{E}_M/\mathcal{J} \to 0 \ , \tag{III.1.14}$$

where $\mathcal{J}$ is the subsheaf of $\mathcal{E}_M$ consisting of germs of smooth functions belonging to the ideal generated by the ϕ_i. Because of our choice of cover, Lemma III.1.3 implies that $\mathcal{J}(U)$

agrees, for all $U \in \mathcal{U}$, with those smooth functions vanishing on $U \cap M_o$, and hence we have an isomorphism of sheaves $\mathcal{E}_M/\mathcal{J} \cong \mathcal{E}_{M_o}$, where $\mathcal{E}_{M_o}$ is the sheaf of germs of smooth functions on M_o. Next we notice that $\mathcal{E}_{M_o}$ is a fine sheaf and hence all its Čech cohomology groups vanish except the zeroth one. Thus the E_2 term in this spectral sequence is just

0	0	0	
0	0	0	
$C^\infty(M_o)$	0	0	

Again we see that the higher differentials are automatically zero and the spectral sequence collapses. Since both spectral sequences compute the same cohomology we have the following corollary.

Corollary III.1.15. *If 0 is a regular value for $\Phi : M \to \mathbb{R}^k$ the Koszul complex $K(\Phi)$ gives an acyclic resolution for $C^\infty(M_o)$. In other words, the cohomology of the Koszul complex is given by*

$$\boxed{H^p(K(\Phi)) \cong \begin{cases} 0 & \text{for } p > 0 \\ C^\infty(M_o) & \text{for } p = 0 \end{cases}}, \tag{III.1.16}$$

where $M_o \equiv \Phi^{-1}(0)$.

Notice that, in particular, this means that the ideal J generated by the constraints is precisely the ideal consisting of functions vanishing on M_o. This is because $C^\infty(M_o) \cong C^\infty(M)/I(M_o)$ since M_o is a closed embedded submanifold. On the other hand, Corollary III.1.15 implies that $C^\infty(M_o) \cong C^\infty(M)/J$. Hence the equality between the two ideals.

It may appear overkill to use the spectral sequence method to arrive at Corollary III.1.15. In fact it is not necessary and the reader is urged to supply a proof using the "tic-tac-toe" methods in **[69]**. This way one gains some valuable intuition on this complex. In particular, one can show that way that the sequence Φ is regular in $C^\infty(M)$ and that $J = I(M_o)$ without having to first prove Corollary III.1.15. Lack of spacetime prevents us from exhibiting both computations and the spectral sequence computation is decidedly shorter.

We now introduce a generalization of the Koszul complex which will be of much use in the sections to come. Let R be a ring and E an R-module. We can then define a complex

$K(\Phi; E)$ associated to any sequence $(\phi_1, \ldots, \phi_k)$ by just tensoring the Koszul complex $K(\Phi)$ with E, that is, $K^p(\Phi; E) = K^p(\Phi) \otimes_R E$ and extending δ_K to $\delta_K \otimes \mathbf{1}$. Let $H(K(\Phi); E)$ denote the cohomology of this complex. It is naturally an R-module. It is easy to show that if E and F are R-modules, then there is an R-module isomorphism

$$H(K(\Phi); E \oplus F) \cong H(K(\Phi); E) \oplus H(K(\Phi); F)\ . \qquad \text{(III.1.17)}$$

Hence, if $F \cong \bigoplus_\alpha R$ is a free R-module then

$$H(K(\Phi); F) \cong \bigoplus_\alpha H(K(\Phi))\ . \qquad \text{(III.1.18)}$$

In particular if Φ is a regular sequence then the generalized Koszul complex with coefficients in a free R-module is quasi-acyclic. Now let P be a projective module, *i.e.*, P is a summand of a free module. Then let N be an R-module such that $P \oplus N = F$, F a free R-module. Then

$$H(K(\Phi); F) \cong H(K(\Phi); P) \oplus H(K(\Phi); N)\ , \qquad \text{(III.1.19)}$$

which, since $H(K(\Phi); F)$ is quasi-acyclic, implies the quasi-acyclicity of $H(K(\Phi); P)$. How about $H^0(K(\Phi); P)$? By definition

$$H^0(K(\Phi); P) \cong R/J \otimes_R P \cong P/JP\ . \qquad \text{(III.1.20)}$$

Therefore we have the following algebraic result

Theorem III.1.21. *If $\Phi = (\phi_1, \ldots, \phi_k)$ is a regular sequence in R, and P is a projective R-module, then the homology of the Koszul complex with coefficients in P is given by*

$$\boxed{H^p(K(\Phi); P) \cong \begin{cases} 0 & \text{for } p > 0 \\ P/JP & \text{for } p = 0 \end{cases}}\ , \qquad \text{(III.1.22)}$$

where J is the ideal generated by the ϕ_i.

The relevance of considering projective modules will come when we discuss geometric quantization. There we will not just have to work with the smooth fucntions on $\widetilde{M}$ but also with sections of vector bundles over $\widetilde{M}$ and these are precisely[79] the finitely generated projective modules over $C^\infty(\widetilde{M})$.

We conclude this section with two philosophical remarks. First, it should be emphasized that the Koszul resolution is independent on the nature of the constraints as long as their zero locus was a regular set. In particular, we never made use of the fact that the constraints were first class or, for that matter, that M had a symplectic structure. Hence also in the case of second class constraints there is a Koszul resolution giving a cohomological description of the smooth functions of the constrained submanifold. This, to my knowledge, has not been used in the physics literature. It would seem to be the natural starting place to extend the BRST quantization to the case of second class constraints and hence give a unified cohomological description of the full Dirac theory.

Second, it is worth pointing out that the restriction to the constraints being regular is not really necessary. With a bit more work a resolution (called the Tate resolution) can be constructed in order to handle this case as well. The method of Tate[**80**] consists of adding new cochains to kill whatever cohomology might exist in positive dimension. These new cochains are the antighosts for the ghosts for ghosts in the treatment of reducible gauge theories. A complete description of this work can be found in the recent paper by Fisch, Henneaux, Stasheff, & Teitelboim [**21**].

2. Classical BRST Cohomology

In this section we complete the construction of the algebraic equivalent of symplectic reduction by first defining a cohomology theory (vertical cohomology) that describes the passage of M_o to $\widetilde{M}$ and then, in keeping with our philosophy of not having to work on M_o, we lift it via the Koszul resolution to a cohomology theory (classical BRST cohomology) which allows us to work with $\widetilde{M}$ from objects defined on M. We shall assume for convenience that the foliation defining $\widetilde{M}$ is such that $\widetilde{M}$ is a smooth manifold and $\pi : M_o \twoheadrightarrow \widetilde{M}$ is a smooth surjection. In other words, the foliation is actually a fibration $M_o \xrightarrow{\pi} \widetilde{M}$ whose fibers are the leaves.

Vertical Cohomology

Since $\widetilde{M}$ is obtained from M_o by collapsing each leaf of the null foliation $\mathcal{M}_o^\perp$ to a point, a smooth function on $\widetilde{M}$ pulls back to a smooth function on M_o which is constant on each leaf. Conversely, any smooth function on M_o which is constant on each leaf defines a smooth function on $\widetilde{M}$. Since the leaves are connected (Frobenius' theorem) a function is constant on the leaves if and only if it is locally constant. Since the hamiltonian vector fields $\{X_i\}$ associated to the constraints $\{\phi_i\}$ form a global basis of the tangent space to the leaves, a function f on M_o is locally constant on the leaves if and only if $X_i\, f = 0$ for all i. In an effort to build a cohomology theory and in analogy to the de Rham theory, we pick a global basis $\{c^i\}$ for the cotangent space to the leaves such that they are dual to the $\{X_i\}$, *i.e.*, $c^i(X_j) = \delta^i_j$. We then define the **vertical derivative** d_V on functions as

$$d_V f = \sum_i (X_i f)\, c^i \qquad \forall\, f \in C^\infty(M_o)\ . \tag{III.2.1}$$

Let $\Omega_V(M_o)$ denote the exterior algebra generated by the $\{c^i\}$ over $C^\infty(M_o)$. We will refer to them as **vertical forms**. We can extend d_V to a derivation

$$d_V : \Omega^p_V(M_o) \to \Omega^{p+1}_V(M_o) \tag{III.2.2}$$

by defining

$$d_V c^i = -\frac{1}{2} \sum_{j,k} f_{jk}{}^i\, c^j \wedge c^k\ , \tag{III.2.3}$$

where the $\{f_{ij}{}^k\}$ are the functions appearing in the Lie bracket of the hamiltonian vector fields associated to the constraints: $\left[X_i\,,\,X_j\right] = \sum_k f_{ij}{}^k\, X_k$; or, equivalently, in the Poisson bracket of the constraints themselves: $\left\{\phi_i\,,\,\phi_j\right\} = \sum_k f_{ij}{}^k\, \phi_k$.

Notice that the choice of $\{c^i\}$ corresponds to a choice of connection on the fiber bundle $M_o \xrightarrow{\pi} \widetilde{M}$. Let V denote the subbundle of TM_o spanned by the $\{X_i\}$. It can be characterized either as ker π_* or as $TM_o^\perp$. A connection is then a choice of complementary subspace H such that $TM_o = V \oplus H$. It is clear that a choice of $\{c^i\}$ implies a choice of H since we can define $X \in H$ if and only if $c^i(X) = 0$ for all i. If we let pr_V denote the projection $TM_o \to V$ it is then clear that acting on vertical forms, $d_V = \mathrm{pr}_V^* \circ d$, where d is the usual exterior derivative on M_o.

It follows therefore that $d_V^2 = 0$. We call its cohomology the **vertical cohomology** and we denote it as $H_V(M_o)$. As we will see in Section 4, it can be computed in terms of the de Rham cohomology of the typical fiber in the fibration $M_o \xrightarrow{\pi} \widetilde{M}$. In particular, from its definition, we already have that

$$H^0_V(M_o) \cong C^\infty(\widetilde{M})\ . \tag{III.2.4}$$

<u>The BRST Construction</u>

However this is not the end of the story since we don't want to have to work on M_o but on M. The results of the previous section suggest that we use the Koszul construction. Notice that $\Omega_V(M_o)$ is isomorphic to $\bigwedge \mathbb{R}^k \otimes C^\infty(M_o)$ where $\mathbb{R}^k$ has basis $\{c^i\}$. The Koszul complex gives a resolution for $C^\infty(M_o)$. Therefore extending the Koszul differential as the identity on $\bigwedge \mathbb{R}^k$ we get a resolution for $\Omega_V(M_o)$. We find it convenient to think of $\mathbb{R}^k$ as $\mathbb{V}^*$, whence the resolution of $\Omega_V(M_o)$ is given by

$$\cdots \longrightarrow \bigwedge \mathbb{V}^* \otimes \mathbb{V} \otimes C^\infty(M) \stackrel{\mathbf{1}\otimes\delta_K}{\longrightarrow} \bigwedge \mathbb{V}^* \otimes C^\infty(M) \longrightarrow 0\ . \tag{III.2.5}$$

This gives rise to a bigraded complex $K = \bigoplus_{c,b} K^{c,b}$, where

$$\boxed{K^{c,b} \equiv \bigwedge^c \mathbb{V}^* \otimes \bigwedge^b \mathbb{V} \otimes C^\infty(M)}\ , \tag{III.2.6}$$

under the Koszul differential $\delta_K : K^{c,b} \to K^{c,b-1}$. The Koszul cohomology of this bigraded complex is zero for $b > 0$ by (III.1.18), and for $b = 0$ it is isomorphic to the vertical forms, where the vertical derivative is defined. Elements of $\bigwedge \mathbb{V}^*$ are the classical **ghosts**. Therefore we see that although the ghosts and antighosts are dual to each other the rôles they play in the BRST construction are very different.

The purpose of the BRST construction is to lift the vertical derivative to K. That is, to define a differential δ_1 on K which anticommutes with the Koszul differential, which induces the vertical derivative upon taking Koszul cohomology, and which obeys $\delta_1^2 = 0$. This would mean that the total differential $D = \delta_K + \delta_1$ would obey $D^2 = 0$ acting on K and its cohomology would be isomorphic to the vertical cohomology. This is possible only in the case of a group action, *i.e.*, when the linear span of the constraints closes under Poisson bracket. In general this is not possible and we will be forced to add further δ_i's to D to ensure $D^2 = 0$. The need to include these extra terms was first pointed out by Fradkin and Fradkina in **[19]**, as was pointed out to me by Marc Henneaux.

We find it convenient to define $\delta_0 = (-1)^c \delta_K$ on $K^{b,c}$. We define δ_1 on functions and ghosts as the vertical derivative[8]

$$\begin{aligned} \delta_1\, f &= \sum_i (X_i f)\, c^i \\ &= \sum_i \{\phi_i\, ,\, f\}\, c^i \end{aligned} \tag{III.2.7}$$

and

---

[8] Notice that the vertical derivative is defined on M_o and hence has no unique extension to M. The choice we make is the simplest and the one that, in the case of a group action, corresponds to the Lie algebra coboundary operator.

$$\delta_1\, c^i = -\frac{1}{2}\sum_{j,k} {f_{jk}}^i\, c^j \wedge c^k \ . \tag{III.2.8}$$

We can then extend it as a derivation to all of $\bigwedge \mathbb{V}^* \otimes C^\infty(M)$. Notice that it trivially anticommutes with δ_0 since it stabilizes $\bigwedge \mathbb{V}^* \otimes C^\infty(M)$ where δ_0 acts trivially. We now define it on antighosts in such a way that it commutes with δ_0 everywhere. This does not define it uniquely but a convenient choice is

$$\delta_1 e_i = \sum_{j,k} {f_{kj}}^i\, \omega^j \wedge e_k \ . \tag{III.2.9}$$

Notice that $\delta_1^2 \neq 0$ in general, although it does in the case where the ${f_{ij}}^k$ are constant. However since it anticommutes with δ_0 it does induce a map in δ_0 (*i.e.*, Koszul) cohomology which precisely agrees with the vertical derivative d_V, which does obey $d_V^2 = 0$. Hence δ_1^2 induces the zero map in Koszul cohomology. This is enough (see algebraic lemma below) to deduce the existence of a derivation $\delta_2 : K^{c,b} \to K^{c+2,b+1}$ such that $\delta_1^2 + \left\{\delta_0\,,\,\delta_2\right\} = 0$, where $\left\{\ ,\ \right\}$ denotes the anticommutator. This suggests that we define $D_2 = \delta_0 + \delta_1 + \delta_2$. We see that

$$D_2^2 = \delta_0^2 \oplus \left\{\delta_0\,,\,\delta_1\right\} \oplus (\delta_1^2 + \left\{\delta_0\,,\,\delta_2\right\}) \oplus \left\{\delta_1\,,\,\delta_2\right\} \oplus \delta_2^2\ , \tag{III.2.10}$$

where we have separated it in terms of different bidegree and arranged them in increasing c-degree. The first three terms are zero but, in general, the other two will not vanish. The idea behind the BRST construction is to keep defining higher $\delta_i : K^{c,b} \to K^{c+i,b+i-1}$ such that their partial sums $D_i = \delta_0 + \cdots + \delta_i$ are nilpotent up to terms of higher and higher c-degree until eventually $D_k^2 = 0$. The proof of this statement will follow by induction from the quasi-acyclicity of the Koszul complex, but first we need to introduce some notation that will help us organize the information.

Let us define $F^pK = \bigoplus_{c\geq p}\bigoplus_b K^{c,b}$. Then $K = F^0K \supseteq F^1K \supseteq \cdots$ is a filtration of K. Let $\operatorname{Der} K$ denote the derivations (with respect to the $\wedge$ product) of K. We say that a derivation has bidegree (i,j) if it maps $K^{c,b} \to K^{c+i,b+j}$. $\operatorname{Der} K$ is naturally bigraded

$$\operatorname{Der} K = \bigoplus_{i,j} \operatorname{Der}^{i,j} K\ , \tag{III.2.11}$$

where $\operatorname{Der}^{i,j} K$ consists of derivations of bidegree (i,j). This decomposition makes $\operatorname{Der} K$ into a bigraded Lie superalgebra under the graded commutator:

$$[\ ,\ ] : \operatorname{Der}^{i,j} K \times \operatorname{Der}^{k,l} K \to \operatorname{Der}^{i+k,j+l} K\ . \tag{III.2.12}$$

We define $F^p\operatorname{Der} K = \bigoplus_{i\geq p}\bigoplus_j \operatorname{Der}^{i,j} K$. Then $F\operatorname{Der} K$ gives a filtration of $\operatorname{Der} K$ associated to the filtration $F\,K$ of K.

The remarks immediately following (III.2.10) imply that $D_2^2 \in F^3\mathrm{Der}\,K$. Moreover, it is trivial to check that $\left[\delta_0\,,\,D_2^2\right] \in F^4\mathrm{Der}\,K$. In fact,

$$\left[\delta_0\,,\,D_2^2\right] = \left[D_2\,,\,D_2^2\right] - \left[\delta_1\,,\,D_2^2\right] - \left[\delta_2\,,\,D_2^2\right] \tag{III.2.13}$$

where the first term vanishes because of the Jacobi identity and the last two terms are clearly in $F^4\mathrm{Der}\,K$. Therefore the part of D_2^2 in $F^3\mathrm{Der}\,K/F^4\mathrm{Der}\,K$ is a δ_0-chain map: that is, $\left[\delta_0\,,\,\left\{\delta_1\,,\,\delta_2\right\}\right] = 0$. Since it has non-zero b-degree, the quasi-acyclicity of the Koszul complex implies that it induces the zero map in Koszul cohomology. By the following algebraic lemma (see below), there exists a derivation δ_3 of bidegree $(3,2)$ such that $\left\{\delta_0\,,\,\delta_3\right\} + \left\{\delta_1\,,\,\delta_2\right\} = 0$. If we define $D_3 = \sum_{i=0}^{3}\delta_i$, this is equivalent to $D_3^2 \in F^4\mathrm{Der}\,K$. But by arguments identical to the ones above we deduce that $\left[\delta_0\,,\,D_3^2\right] \in F^5\,\mathrm{Der}\,K$, and so on. It is not difficult to formalize these arguments into an induction proof of the following theorem:

Theorem III.2.14. *We can define a derivation $D = \sum_{i=0}^{k}\delta_i$ on K, where δ_i are derivations of bidegree $(i, i-1)$, such that $D^2 = 0$.*

Finally we come to the proof of the algebraic lemma used above.

Lemma III.2.15. *Let*

$$\cdots \longrightarrow K_2 \xrightarrow{\delta_0} K_1 \xrightarrow{\delta_0} K_0 \to 0 \tag{III.2.16}$$

denote the Koszul complex where $K_b = \bigoplus_c K^{c,b}$. Let $d : K_b \to K_{b+i}$, $(i \geq 0)$ be a derivation which commutes with δ_0 and which induces the zero map on cohomology. Then there exists a derivation $K : K_b \to K_{b+i+1}$ such that $d = \left\{\delta_0\,,\,K\right\}$.

Proof: Since $C^\infty(M)$ is an $\mathbb{R}$-algebra it is, in particular, a vector space. Let $\{f_\alpha\}$ be a basis for it. Then, since $\delta_0\,f_\alpha = 0$, $\delta_0\,d\,f_\alpha = 0$. Since d induces the zero map in cohomology, there exists λ_α such that $d\,f_\alpha = \delta_0\,\lambda_\alpha$. Define $K\,f_\alpha = \lambda_\alpha$. Similarly, since $\delta_0\,d\,c^i = 0$, there exists μ^i such that $d\,c^i = \delta_0\,\mu^i$. Define $K\,c^i = \mu^i$. Since $C^\infty(M)$ and the $\{c^i\}$ generate K_0, we can extend K to all of K_0 as a derivation and, by construction, in such a way that on K_0, $d = \left\{\delta_0\,,\,K\right\}$. Now, $\delta_0\,d\,b_i = d\,\delta_0\,b_i$. But since $\delta_0\,b_i \in K_0$, $\delta_0\,d\,b_i = \delta_0\,K\,\delta_0\,b_i$. Therefore $\delta_0\,(d\,b_i - K\,\delta_0\,b_i) = 0$. Since $d\,b_i \in K^{i+1}$ for some $i \geq 0$, the quasi-acyclicity of the Koszul complex implies that there exists ξ_i such that $d\,b_i - K\,\delta_0\,b_i = \delta_0\,\xi_i$. Define $K\,b_i = \xi_i$. Therefore, $d\,b_i = \left\{\delta_0\,,\,K\right\}b_i$. We can now extend K as a derivation to all of K. Since d and $\left\{\delta_0\,,\,K\right\}$ are both derivations and they agree on generators, they are equal. ■

Defining the total complex $K = \bigoplus_n K^n$, where $K^n = \bigoplus_{c-b=n} K^{c,b}$, we see that $D : K^n \to K^{n+1}$. Its cohomology is therefore graded, that is, $H_D = \bigoplus_n H_D^n$. D is the **classical BRST operator** and its cohomology is the **classical BRST cohomology**. The total degree is known as the **ghost number**. We now investigate the classical BRST cohomology; although a full description in terms of initial data will have to wait until Section 4. Notice that since all terms in D have non-negative filtration degree with respect to $F\,K$, there exists (Theorem II.1.32) a spectral sequence associated to this filtration which converges to the cohomology of D. The E_1 term is the cohomology of the associated graded object $\mathrm{Gr}^p K \equiv F^p K/F^{p+1}K$, with respect to the induced differential. The induced differential is the part of D of c-degree 0, that is, δ_0. Therefore the E_1 term is given by

$$E_1^{c,b} \cong \bigwedge{}^c \mathbb{V}^* \otimes H^b(K(\Phi)) \ . \tag{III.2.17}$$

That is, $E_1^{c,0} \cong \Omega_V^c(M_o)$ and $E_1^{c,b>0} = 0$.

The E_2 term is the cohomology of E_1 with respect to the induced differential d_1. Tracking down the definitions we see that d_1 is induced by δ_1 and hence it is just the vertical derivative d_V. Therefore, $E_2^{c,0} \cong H_V^c(M_0)$ and $E_2^{c,b>0} = 0$. Notice, however, that the spectral sequence is degenerate at this term, since the higher differentials $d_2, d_3, \ldots$ all have b-degree different from zero. Therefore we have proven the following theorem.

Theorem III.2.18. *The classical BRST cohomology is given by*

$$\boxed{H_D^n \cong \begin{cases} 0 & \text{for } n < 0 \\ H_V^n(M_o) & \text{for } n \geq 0 \end{cases}} \ . \tag{III.2.19}$$

In particular, $H_D^0 \cong C^\infty(\widetilde{M})$.

We have not yet made sure, as we said we should, that the BRST cohomology is independent of the explicit form of the constraints and, thus, that it depends only on the actual constrained submanifold $i : M_o \hookrightarrow M$. Actually since, by Theorem III.2.18, the classic BRST cohomology merely recovers the vertical cohomology we must make sure that it is the vertical cohomology which is independent of the form of the constraints. From its definition the vertical cohomology explicitly depends on the choice of connection H. In other words, whereas the vertical tangent space V is uniquely defined, its complement H is not. We must show that any other choice of connection yields the same vertical cohomology; although, of course, the complexes used to calculate it are different. Instead of proving this directly we will wait until Section 4. There we compute the vertical cohomology and the answer is manifestly independent of the choice of connection.

3. Poisson Structure of Classical BRST

So far in the construction of the BRST complex no use has been made of the Poisson structure of the smooth functions on M. In this section we remedy the situation. It turns out that the complex K introduced in the last section is a Poisson superalgebra and the BRST operator D can be made into a Poisson derivation. It will then follow that in cohomology all constructions based on the Poisson structures will be preserved. This will be of special importance in the context of geometric quantization since all objects there can be defined purely in terms of the Poisson algebra structure of the smooth functions. In this section we review the concepts associated to Poisson algebras. We define the relevant Poisson structures in K and explore its consequences.

Poisson Superalgebras and Poisson Derivations

Recall that a **Poisson superalgebra** is a $\mathbb{Z}_2$-graded vector space $P = P_0 \oplus P_1$ together with two bilinear operations preserving the grading:

$$\begin{aligned} P \times P &\to P \qquad \text{(multiplication)} \\ (a,b) &\mapsto ab \end{aligned}$$

and

$$\begin{aligned} P \times P &\to P \qquad \text{(Poisson bracket)} \\ (a,b) &\mapsto [a\,,\,b] \end{aligned}$$

obeying the following properties

(P1) P is an associative supercommutative superalgebra under multiplication:

$$\begin{aligned} a(bc) &= (ab)c \\ ab &= (-1)^{|a||b|}\, ba \ ; \end{aligned}$$

(P2) P is a Lie superalgebra under Poisson bracket:

$$\begin{aligned} [a\,,\,b] &= (-1)^{|a||b|}\, [b\,,\,a] \\ [a\,,\,[b\,,\,c]] &= [[a\,,\,b]\,,\,c] + (-1)^{|a||b|}\, [b\,,\,[a\,,\,c]] \ ; \end{aligned}$$

(P3) Poisson bracket is a derivation over multiplication:

$$[a\,,\,bc] = [a\,,\,b]c + (-1)^{|a||b|}\, b[a\,,\,c] \ ;$$

for all $a, b, c \in P$ and where $|a|$ equals 0 or 1 according to whether a is even or odd, respectively.

The algebra $C^\infty(M)$ of smooth functions of a symplectic manifold (M,Ω) is clearly an example of a Poisson superalgebra where $C^\infty(M)_1 = 0$. On the other hand, if $\mathbb{V}$ is a finite dimensional vector space and $\mathbb{V}^*$ its dual, then the exterior algebra $\bigwedge(\mathbb{V}\oplus\mathbb{V}^*)$ posseses a Poisson superalgebra structure. The associative multiplication is given by exterior multiplication ($\wedge$) and the Poisson bracket is defined for $u,v\in\mathbb{V}$ and $\alpha,\beta\in\mathbb{V}^*$ by

$$\big[\alpha\,,\,v\big] = \langle\alpha,v\rangle \qquad \big[v\,,\,w\big] = 0 = \big[\alpha\,,\,\beta\big]\ , \tag{III.3.1}$$

where $\langle,\rangle$ is the dual pairing between $\mathbb{V}$ and $\mathbb{V}^*$. We then extend it to all of $\bigwedge(\mathbb{V}\oplus\mathbb{V}^*)$ as an odd derivation. Therefore the classical ghosts/antighosts in BRST possess a Poisson algebra structure. In **[81]** it is shown that this Poisson bracket is induced from the supercommutator in the Clifford algebra $\mathrm{Cl}(\mathbb{V}\oplus\mathbb{V}^*)$ with respect to the non-degenerate inner product on $\mathbb{V}\oplus\mathbb{V}^*$ induced by the dual pairing.

To show that K is a Poisson superalgebra we need to discuss tensor products. Given two Poisson superalgebras P and Q, their tensor product $P\otimes Q$ can be given the structure of a Poisson superalgebra as follows. For $a,b\in P$ and $u,v\in Q$ we define

$$(a\otimes u)(b\otimes v) = (-1)^{|u||b|}\,ab\otimes uv \tag{III.3.2}$$

$$\big[a\otimes u\,,\,b\otimes v\big] = (-1)^{|u||b|}\,\Big(\big[a\,,\,b\big]\otimes uv + ab\otimes\big[u\,,\,v\big]\Big)\ . \tag{III.3.3}$$

The reader is invited to verify that with these definitions (P1)-(P3) are satisfied. From this it follows that $K = C^\infty(M)\otimes\bigwedge(\mathbb{V}\oplus\mathbb{V}^*)$ becomes a Poisson superalgebra.

Now let P be a Poisson superalgebra which, in addition, is $\mathbb{Z}$-graded, that is, $P = \bigoplus_n P^n$ and $P^n\,P^m \subseteq P^{m+n}$ and $\big[P^n\,,\,P^m\big]\subseteq P^{m+n}$; and such that the $\mathbb{Z}_2$-grading is the reduction modulo 2 of the $\mathbb{Z}$-grading, that is, $P_0 = \bigoplus_n P^{2n}$ and $P_1 = \bigoplus_n P^{2n+1}$. We call such an algebra a **graded Poisson superalgebra**. Notice that P^0 is an even Poisson subalgebra of P.

For example, letting $K = C^\infty(M)\otimes\bigwedge(\mathbb{V}\oplus\mathbb{V}^*)$ we can define $K^n = \bigoplus_{c-b=n} K^{c,b}$. This way K becomes a $\mathbb{Z}$-graded Poisson superalgebra. Although the bigrading is preserved by the exterior product, the Poisson bracket does not preserve it. In fact, the Poisson bracket obeys

$$\big[\ ,\ \big] : K^{i,j}\times K^{k,l} \to K^{i+k,j+l}\oplus K^{i+k-1,j+l-1}\ . \tag{III.3.4}$$

By a **Poisson derivation** of degree k we will mean a linear map $D : P^n \to P^{n+k}$ such that

$$D(ab) = (Da)b + (-1)^{k|a|}\, a(Db) \tag{III.3.5}$$
$$D\big[a\,,\,b\big] = \big[Da\,,\,b\big] + (-1)^{k|a|}\,\big[a\,,\,Db\big]\ . \tag{III.3.6}$$

The map $a \mapsto \big[Q\,,\,a\big]$ for some $Q \in P^k$ automatically obeys (III.3.5) and (III.3.6). Such Poisson derivations are called **inner**. Whenever the degree derivation is inner, any Poisson derivation of non-zero degree is inner[**51**] as we now show. The degree derivation N is defined uniquely by $Na = na$ if and only if $a \in P^n$. In the case $P = K$, N is the ghost number operator which is an inner derivation $\big[G\,,\,\cdot\big]$, where $G = \sum_i c^i \wedge b_i$, where $\{b_i\}$ is a basis for $\mathbb{V}$ and $\{c^i\}$ denotes its canonical dual basis. Now if $a \in P^n$, and the degree of D is $k \neq 0$, it follows from (III.3.6) that

$$Da = \frac{-1}{k}\big[DG\,,\,a\big]\ , \tag{III.3.7}$$

and so D is an inner derivation. If, furthermore, D should obey $D^2 = 0$, and be of degree 1, $Q = -DG$ would obey $\big[Q\,,\,Q\big] = 0$. To see this notice that for all $a \in P^n$

$$D^2 a = \big[Q\,,\,\big[Q\,,\,a\big]\big] = \frac{1}{2}\big[\big[Q\,,\,Q\big]\,,\,a\big] = 0\ .$$

But for $a = G$ we get that $\big[Q\,,\,Q\big] = 0$.

<u>The BRST Operator as a Poisson Derivation</u>

The BRST operator D constructed in the previous section is a derivation over the exterior product. Nothing in the way it was defined guarantees that it is a Poisson derivation and, in fact, it need not be so. However one can show that the δ_i's — which were, by far, not unique — can be defined in such a way that the resulting D is a Poisson derivation, from which it would immediately follow that it is inner. It is easier, however, to show the existence of the element $Q \in K^1$ such that $D = \big[Q\,,\,\cdot\big]$. We will show that there exists $Q = \sum_{i\geq 0} Q_i$, where $Q_i \in K^{i+1,i}$, such that $\big[Q\,,\,Q\big] = 0$ and that the cohomology of the operator $\big[Q\,,\,\cdot\big]$ is isomorphic to that of D. This was first proven by Henneaux in [**20**] and later in a completely algebraic way by Stasheff in [**55**]. Our proof is a simplified version of this latter proof.

From the discussion previous to Theorem III.2.18 we know that the only parts of D which affect its cohomology are δ_0, which is the Koszul differential, and δ_1 acting on the Koszul cohomology. Hence we need only make sure that the Q_i we construct realize these differentials. Notice that if $Q_i \in K^{i+1,i}$, $\big[Q_i\,,\,\cdot\big]$ has terms of two different bidegrees $(i+1, i)$

and $(i, i-1)$. Hence the only term which can contribute to the Koszul differential is Q_0. There is a unique element $Q_0 \in K^{1,0}$ such that $\big[Q_0\,,\, b_i\big] = \delta_0\, b_i = \phi_i$. This is given by

$$Q_0 = \sum_i c^i\, \phi_i\ . \tag{III.3.8}$$

Notice that

$$\begin{aligned} \big[Q_0\,,\, b_i\big] &= \delta_0\, b_i = \phi_i && \text{(III.3.9)}\\ \big[Q_0\,,\, c^i\big] &= \delta_0\, c^i = 0 && \text{(III.3.10)}\\ \big[Q_0\,,\, f\big] &= (\delta_0 + \delta_1)\, f = \sum_i \big[\phi_i\,,\, f\big]\, c^i\ . && \text{(III.3.11)} \end{aligned}$$

There is now a unique $Q_1 \in K^{2,1}$ such that $\big[Q_1\,,\, c^i\big] = \delta_1\, c^i$, namely,

$$Q_1 = -\frac{1}{2} \sum_{i,j,k} f_{ij}{}^k\, c^i \wedge c^j \wedge b_k\ . \tag{III.3.12}$$

If we define $R_1 = Q_0 + Q_1$ we then have that

$$\begin{aligned} \big[R_1\,,\, b_i\big] &= (\delta_0 + \delta_1)\, b_i && \text{(III.3.13)}\\ \big[R_1\,,\, c^i\big] &= (\delta_0 + \delta_1)\, c^i && \text{(III.3.14)}\\ \big[R_1\,,\, f\big] &= (\delta_0 + \delta_1 + \delta_2)\, f\ . && \text{(III.3.15)} \end{aligned}$$

In particular, two things are imposed upon us: $\delta_2\, f$ and $\delta_1\, b_i$; the latter imposition agrees with the choice made in (III.2.9).

Letting FK denote the filtration of K defined in the previous section, and using the notation in which, if $O \in K$ is an odd element, O^2 stands for $\frac{1}{2}\big[O\,,\, O\big]$, the following are satisfied:

$$R_1^2 \in F^3K \qquad \text{and} \qquad \big[Q_0\,,\, R_1^2\big] \in F^4K\ . \tag{III.3.16}$$

That means that the part of R_1^2 which lives in F^3K/F^4K is a δ_0-cocycle, since the $(0,-1)$ part of Q_0 is precisely δ_0. By the quasi-acyclicity of the Koszul complex it is a coboundary, say, $-\delta_0\, Q_2$ for some $Q_2 \in K^{3,2}$. In other words, there exists $Q_2 \in K^{3,2}$ such that if $R_2 = Q_0 + Q_1 + Q_2$, then $R_2^2 \in F^4K$. If this is the case then

$$\big[Q_0\,,\, R_2^2\big] = \big[R_2\,,\, R_2^2\big] - \big[Q_1\,,\, R_2^2\big] - \big[Q_2\,,\, R_2^2\big]\ . \tag{III.3.17}$$

But the first term is zero because of the Jacobi identity and the last two terms are clearly in $F^5 K$ due to the fact that, from (III.3.4),

$$\left[F^p K\,,\, F^q K\right] \subseteq F^{p+q-1} K\ . \tag{III.3.18}$$

Hence, $\left[Q_0\,,\,R_2^2\right] \in F^5 K$, from where we can deduce the existence of $Q_3 \in K^{4,3}$ such that $R_3 = Q_0 + Q_1 + Q_2 + Q_3$ obeys $R_3^2 \in F^5 K$, and so on. It is easy to formalize this into an induction proof of the following theorem.

Theorem III.3.19. *There exists $Q = \sum_i Q_i$, where $Q_i \in K^{i+1,i}$ such that $\left[Q\,,\,Q\right] = 0$.*

Now let $D = \left[Q\,,\,\cdot\right]$. Then $D^2 = 0$ and repeating the proof of Theorem III.2.18 we obtain the following.

Theorem III.3.20. *The cohomology of D is given by*

$$\boxed{H_D^n \cong \begin{cases} 0 & \text{for } n < 0 \\ H_V^n(M_o) & \text{for } n \geq 0 \end{cases}}\ . \tag{III.3.21}$$

In particular, $H_D^0 \cong C^\infty(\widetilde{M})$.

From now on we will take $D = \left[Q\,,\,\cdot\right]$ to be the classical BRST operator; although it is common in the physics literature to call Q the classical BRST operator.

We now come to an important consequence of the fact that the classical BRST operator is a (inner) Poisson derivation. It is easy to verify that this implies that $\ker D$ becomes a Poisson subalgebra of K and $\operatorname{im} D$ is a Poisson ideal of $\ker D$. Therefore the cohomology space $H_D = \ker D/\operatorname{im} D$ naturally inherits the structure of a Poisson superalgebra. Moreover since K is a graded Poisson superalgebra and D is homogeneous with respect to this grading, the cohomology naturally becomes a graded Poisson superalgebra. In particular, H_D^0 is a Poisson subalgebra and H_D is naturally a graded Poisson module of H_D^0. In particular, since H_D^0 is isomorphic to $C^\infty(\widetilde{M})$ we see that the Poisson brackets get induced. Therefore if we wished to compute the Poisson brackets of two smooth functions on $\widetilde{M}$ we merely need to find suitable BRST cocycles representing them and compute the Poisson bracket in K. It is noteworthy to remark that it is not always possible to choose BRST cocycles which are ghost independent, *i.e.*, in $K^{0,0}$ so that the ghosts and antighosts are an integral ingredient in the formulation.

The Case of a Group Action

Since the case when the constraints arise from a moment map is of special interest, it is worth looking at its classical BRST operator in some detail. We will be able to relate the BRST cohomology with a Lie algebra cohomology group with coefficients in an infinite dimensional (differential) representation.

So let G be a Lie group and $\mathfrak{g}$ its Lie algebra and let there be a Poisson action of G M giving rise to an equivariant moment map $\Phi : M \to \mathfrak{g}^*$. Let $\{b_i\}$ be a basis for $\mathfrak{g}$ and $\{c^i\}$ be the canonical dual basis for $\mathfrak{g}^*$. Notice that the dual of the moment map gives rise to a map $\mathfrak{g} \to C^\infty(M)$ sending $b_i \mapsto \phi_i$, where ϕ_i are the coefficients of the moment map relative to the $\{c^i\}$:

$$\langle \Phi(m), b_i \rangle = \phi_i(m) \ , \tag{III.3.22}$$

which is precisely the map δ_K in the Koszul complex. In particular, we can identify $\mathbb{V}$ with $\mathfrak{g}$. Since the action is Poisson, the functions $\{\phi_i\}$ represent the algebra under the Poisson bracket: $\{\phi_i \, , \, \phi_j\} = \sum_k {f_{ij}}^k \phi_k$, where the ${f_{ij}}^k$ are the structure constants of $\mathfrak{g}$ in the chosen basis. Let $Q = Q_0 + Q_1$ where Q_0 and Q_1 are given by (III.3.8) and (III.3.12), respectively. Since the ${f_{ij}}^k$ are constant and satisfy the Jacobi identity, $\{Q \, , \, Q\} = 0$, and hence the extra $Q_{i>1}$ are not necessary. Hence the classical BRST "operator" is

$$\boxed{Q = \sum_i c^i \, \phi_i - \frac{1}{2} \sum_{i,j,k} {f_{ij}}^k \, c^i \wedge c^j \wedge b_k} \ . \tag{III.3.23}$$

Notice that this is precisely the operator found by Batalin & Vilkoviskii [**18**].

We can now make contact with Lie algebra cohomology. The cohomology of the classical BRST operator is exactly the cohomology of the vertical derivative which is computed by the complex C defined by

$$C^\infty(M_o) \xrightarrow{D} \mathfrak{g}^* \otimes C^\infty(M_o) \xrightarrow{D} \textstyle\bigwedge^2 \mathfrak{g}^* \otimes C^\infty(M_o) \xrightarrow{D} \cdots \ , \tag{III.3.24}$$

where D is defined on the generators by

$$\begin{aligned} Df &= \sum_i c^i \otimes \{\phi_i \, , \, f\} \\ Dc^i &= -\frac{1}{2} \sum_{j,k} {f_{ij}}^k \, c^j \wedge c^k \ . \end{aligned}$$

Comparing with (II.1.61) we deduce that C is nothing but the space of Lie algebra cochains $C(\mathfrak{g}; C^\infty(M_o))$; and comparing with (II.1.59) we deduce that D is nothing but the Lie algebra coboundary operator. Hence, for the case of a Poisson group action, the classical Lie algebra cohomology is just the Lie algebra cohomology of $\mathfrak{g}$ with coefficients in the module $C^\infty(M_o)$: $H(\mathfrak{g}; C^\infty(M_o))$.

4. Topological Characterization

In Section 2 we saw that that there is a geometric interpretation for the classical BRST cohomology as the vertical cohomology acting on differential forms along the leaves of the foliation $\mathcal{M}_o^\perp$ defined by the first class constraints on the coisotropic submanifold M_o traced by their zero locus. In this section we use this geometric interpretation to compute the classical BRST cohomology.

The tangent bundle of M_o breaks up as $TM_o = T\mathcal{M}_o^\perp \oplus N\mathcal{M}_o^\perp$, where $T\mathcal{M}_o^\perp = TM_o^\perp$ is the tangent space to the foliation and $N\mathcal{M}_o^\perp$ is the normal bundle to the foliation. Let $T^*\mathcal{M}_o^\perp$ and $N^*\mathcal{M}_o^\perp$ denote the cotangent and conormal bundles to the foliation, respectively. Under this split, the differential forms, $\Omega(M_o)$, on M_o decompose as

$$\Omega(M_o) = \bigoplus_{p,q} \Omega^{p,q}(M_o) \ , \tag{III.4.1}$$

where $\Omega^{p,q}(M_o)$ is the space of smooth sections through the bundle

$$\bigwedge^p T^*\mathcal{M}_o^\perp \otimes \bigwedge^q N^*\mathcal{M}_o^\perp \ . \tag{III.4.2}$$

The exterior derivative on M_o has a piece

$$d_V : \Omega^{p,q}(M_o) \to \Omega^{p+1,q}(M_o) \ , \tag{III.4.3}$$

which is just the vertical derivative and whose cohomology, acting on the vertical forms $\Omega_V^p(M_o) \equiv \Omega^{p,0}(M_o)$, is precisely the classical BRST cohomology.

In **[82]** the Poincaré lemma for this complex is proven. That is, if ω is a d_V-closed vertical p-form (for $p \geq 1$), then around each point in M_o there exists a neighborhood U and a vertical $(p-1)$-form θ_U defined on U such that $\omega = d_V\theta_U$ on U. A vertical 0–form is just a function on M_o and it is d_V–closed if and only if it is constant on each leaf. Therefore a d_V–closed vertical 0–form is the pull back via π of a function on $\widetilde{M}$. Let $\mathcal{E}_{\widetilde{M}}$ be the sheaf of germs of smooth functions on $\widetilde{M}$ and let Ω_V denote the sheaf of germs of vertical forms on M_o. By the above remarks there is an acyclic resolution

$$0 \longrightarrow \pi^* \mathcal{E}_{\widetilde{M}} \longrightarrow \Omega^0_V \xrightarrow{d_V} \Omega^1_V \longrightarrow \cdots \tag{III.4.4}$$

where the first map is the inclusion. This identifies the vertical cohomology with the sheaf cohomology $H(M_o; \pi^* \mathcal{E}_{\widetilde{M}})$ and thus makes contact with the work of Buchdahl[**83**] on the relative de Rham sequence, of which the vertical cohomology is an important special case.

Buchdahl treats the case of an arbitrary smooth surjective map $f : Y \to X$ between two arbitrary (smooth, paracompact) manifolds. He then obtains a resolution for the pull-back sheaf $f^* \mathcal{E}_X$ in terms of **relative forms** Ω_f. Relative forms are differential forms along the fibers of f and the derivative is the exterior derivative along the fibers; where by a fiber we mean the preimage via f of a point in X. Hence vertical cohomology is a particular case of this construction for a very special f, Y and X. Buchdahl does not characterize the relative cohomology completely, but he proves two results that relate it to the cohomology of the fibers. In the case of vertical cohomology, his results (Propositions 1 and 2 in [**83**]) imply the following two theorems, where F is the typical fiber in the fibration $M_o \stackrel{\pi}{\longrightarrow} \widetilde{M}$ and $H(F)$ stands for the real de Rham cohomology of the typical fiber.

Theorem III.4.5. *$H^1(F) = \mathbf{0}$ implies $H^1_V(M_o) = \mathbf{0}$. If $H^{p-1}(F) = H^p(F) = \mathbf{0}$ for some $p > 1$, then $H^p_V(M_o) = \mathbf{0}$.*

Theorem III.4.6. *If for some $p \geq 1$, $H^p_V(M_o) = H^{p+1}_V(M_o) = \mathbf{0}$, then $H^p(F) = \mathbf{0}$.*

An easy corollary of these two theorems gives a characterization of the vanishing of the BRST cohomology for positive ghost number.

Corollary III.4.7. *A necessary and sufficient condition for the classical BRST cohomology to vanish for positive ghost number is that the gauge orbits have vanishing positive de Rham cohomology.*

In particular in the case of a compact orientable gauge orbit, Poincaré duality already forbids the vanishing of the BRST cohomology of top ghost number.

These results, although already providing a lot of information, are far from fully characterizing the BRST cohomology in terms of the topology of the gauge orbits and the gauge invariant observables. Since the case of interest to us is so special we can obtain stronger results. In fact, we can characterize the vertical cohomology from initial data.

The Main Theorem

To fix the notation, let $F \longrightarrow M_o \overset{\pi}{\longrightarrow} \widetilde{M}$ be a smooth fiber bundle where the typical fiber, F, is connected. Let d_V denote the vertical derivative, $\Omega_V(M_o)$ the vertical forms, and $H_V(M_o)$ the vertical cohomology. By definition, the zeroth vertical cohomology, $H^0_V(M_o)$, consists of those smooth functions on M_o which are locally constant on the fibers; and since the fibers are connected, these functions are constant. The projection π induces an isomorphism, $\pi^* : C^\infty(\widetilde{M}) \to C^\infty(M_o)$, defined by $\pi^* f = f \circ \pi$, onto the smooth functions on M_o which are constant on the fibers. Therefore, there is an isomorphism

$$H^0_V(M_o) \cong C^\infty(\widetilde{M}) \ . \tag{III.4.8}$$

By its definition the vertical derivative d_V obeys

$$d_V(\omega \wedge \theta) = (d_V\omega) \wedge \theta + (-1)^p \omega \wedge (d_V\theta) \ , \tag{III.4.9}$$

for $\omega \in \Omega^p_V(M_o)$ and $\theta \in \Omega_V(M_o)$. Therefore $\wedge$ induces an operation in cohomology

$$\cup : H^p_V(M_o) \times H^q_V(M_o) \longrightarrow H^{p+q}_V(M_o) \ , \tag{III.4.10}$$

defined by $[\omega] \cup [\theta] = [\omega \wedge \theta]$. This operation is well defined because of (III.4.9) and makes the vertical cohomology into a graded ring. In particular,

$$\cup : H^0_V(M_o) \times H^q_V(M_o) \longrightarrow H^q_V(M_o) \tag{III.4.11}$$

makes $H_V(M_o)$ into a graded $H^0_V(M_o) \cong C^\infty(\widetilde{M})$ module.

Let $\mathcal{H}_V$ denote the sheaf of $C^\infty(\widetilde{M})$-modules on $\widetilde{M}$ defined by $\mathcal{H}_V(U) = H_V(\pi^{-1}U)$ for all open $U \subset \widetilde{M}$. By local triviality there exists an open cover $\mathcal{U}$ for $\widetilde{M}$ such that for all $U \in \mathcal{U}$, $\pi^{-1}U \cong U \times F$. Therefore $\mathcal{H}_V(U) \cong H_V(U \times F)$. By a theorem of Kacimi-Alaoui (III (1) in **[84]**) the vertical cohomology of a product is given simply by

$$\boxed{H_V(U \times F) \cong C^\infty(U) \otimes H(F)} \ , \tag{III.4.12}$$

where $H(F)$ is the real de Rham cohomology of F. This implies that $\mathcal{H}_V$ is a locally free sheaf and thus[85] the sheaf of germs of smooth sections of a vector bundle over $\widetilde{M}$ with fiber $H(F)$.

The task ahead is to determine the transition functions of this bundle. Let $\{\psi_U\}$ be the family of diffeomorphisms

$$\psi_U : \pi^{-1}U \longrightarrow U \times F \tag{III.4.13}$$

given by the local triviality of the original bundle $M_o \stackrel{\pi}{\longrightarrow} \widetilde{M}$. The transition functions of this bundle are then given, for all $U \cap V \neq \varnothing$, by $g_{UV} = \psi_U \circ \psi_V^{-1}$, thought of as a map $g_{UV} : U \cap V \to \mathrm{Diff}\, F$.

Recall that there is a natural representation of Diff F as automorphisms of degree zero of the (graded) de Rham cohomology ring $H(F)$. If $\varphi \in \mathrm{Diff}\, F$ then the automorphism is defined by $[\omega] \mapsto [(\varphi^{-1})^*\omega]$. By the homotopy invariance of de Rham cohomology, two diffeomorphisms which are homotopic are represented by the same automorphism in $H(F)$. So any diffeomorphism which is homotopic to the identity will automatically induce the identity automorphism on cohomology.

Composing the transition functions $\{g_{UV}\}$ with this representation provides maps

$$(g_{UV}^{-1})^* : U \cap V \to \mathrm{Aut}\ \ H(F)\ , \tag{III.4.14}$$

which, as we will now see, are the transition functions of the bundle whose sheaf of sections is given by $\mathcal{H}_V$.

To see this notice that for all open sets $U \in \mathcal{U}$

$$(\psi_U^{-1})^* : H_V(\pi^{-1}U) \to H_V(U \times F) \cong C^\infty(\widetilde{M}) \otimes H(F)\ , \tag{III.4.15}$$

allows us to identify vertical cohomology classes on $\pi^{-1}U$ with $H(F)$-valued functions on U. Let ω be a d_V-closed vertical form and $[\omega]$ its class in vertical cohomology. Restricted to $U \cap V$ there are two ways in which one can identify $[\omega]$ with an $H(F)$-valued function on $U \cap V$: either by using the trivialization on U or the one on V. Let $f_U = [(\psi_U^{-1})^*\omega]$ and $f_V = [(\psi_V^{-1})^*\omega]$. The transition functions h_{UV} are precisely the automorphisms of the fiber $H(F)$ relating these two descriptions of the same object. That is, the transition functions obey $f_U = h_{UV} f_V$. But because

$$\begin{aligned} f_U &= [(\psi_U^{-1})^*\omega] \\ &= [(\psi_U^{-1})^* \circ \psi_V^* \circ (\psi_V^{-1})^*\omega] \\ &= [(\psi_U^{-1})^* \circ \psi_V^*\, f_V] \\ &= [(\psi_V \circ \psi_U^{-1})^*\, f_V] \\ &= [(g_{UV}^{-1})^*\, f_V]\ , \end{aligned} \tag{III.4.16}$$

the transition functions are in fact the ones in (III.4.14). Therefore we have proven the following theorem.

Theorem III.4.17. *As a module over $C^\infty(\widetilde{M})$ the BRST cohomology is isomorphic to the smooth sections of the associated bundle $M_o \times_\rho H(F) \longrightarrow \widetilde{M}$ associated to the representation $\rho : \text{Diff}\, F \to \text{Aut}\;\, H(F)$.*

Notice that this associated bundle decomposes naturally as a Whitney sum of vector bundles

$$M_o \times_\rho H(F) = \bigoplus_p M_o \times_\rho H^p(F) \tag{III.4.18}$$

since diffeomorphisms do not alter the degree of a form.

As a corollary of this theorem we have that the vertical cohomology (and hence the classical BRST cohomology) does not depend on the explicit form of the constraints used to describe M_o. In fact, the inclusion $i : M_o \hookrightarrow M$ is all that the cohomology depends on. With this information alone we can determine the pullback 2-form $i^*\Omega$ and hence its null foliation $\mathcal{M}_o^\perp$ and this defines a fibration $F \longrightarrow M_o \xrightarrow{\pi} \widetilde{M}$. By Theorem III.4.17, this is all the classical BRST cohomology depends on.

<u>The Case of a Group Action</u>

When the constraints arise from the hamiltonian action of a connected Lie group G—*i.e.*the constraints are the coefficients of the moment map relative to a fixed basis for the Lie algebra of G—the bundle

$$\begin{array}{ccc} G & \longrightarrow & M_o \\ & & \big\downarrow \pi \\ & & \widetilde{M} \end{array} \tag{III.4.19}$$

is in fact a principal G-bundle and the diffeomorphisms of G defined by the transition functions correspond to right multiplication by an element of the group. Since G is connected, right multiplication by any element $g \in G$ is homotopic to the identity. (*Proof:* Let $t \mapsto g(t)$ be a curve in G such that $g(0) = \mathbf{1}$ and $g(1) = g$. Right multiplication by $g(t)$ gives the desired homotopy.) By the homotopy invariance of de Rham cohomology, the transition functions of the associated bundle $M_o \times_\rho H(G) \longrightarrow \widetilde{M}$ are the identity maps and thus the bundle is trivial. This proves the following corollary.

Corollary III.4.20. *When the constraints arise from the hamiltonian action of a connected Lie group G, the BRST cohomology is isomorphic to the $H(G)$-valued functions on $\widetilde{M}$.*

The Case of Compact Fibers

Finally suppose that the fibers are compact. Since they are also orientable[9], Poincaré duality induces an isomorphism

$$\star : H^p(F) \to H^{n-p}(F) \ , \tag{III.4.21}$$

where n is the dimension of the fiber. This induces a duality in the BRST cohomology as follows. Let σ be a section through $M_o \times_\rho H^p(F)$. Define a section $\widetilde{\star}\sigma$ through $M_o \times_\rho H^{n-p}(F)$ by

$$(\widetilde{\star}\sigma)(m) = \star\,\sigma(m) \qquad \forall\ m \in \widetilde{M} \ . \tag{III.4.22}$$

This is an isomorphism and hence we have the following result.

Corollary III.4.23. *Let the typical fiber F be n-dimensional and compact. Then there is an isomorphism*

$$\boxed{H^p_V(M_o) \cong H^{n-p}_V(M_o)} \ . \tag{III.4.24}$$

It is worth remarking that for the case of reducible constraints the BRST operator also has the same geometric interpretation[21] and hence almost all the results of this section go through unchanged. The only exception is the last subsection where we needed orientability of the fibers. In the reducible case the fibers are no longer parallelizable. I ignore if they are generally orientable and hence, for reducible constraints, the hypothesis in Corollary III.4.23 must be amended to assume that the fibers are orientable.

[9] In fact, they are parallelizable since the $\{X_i\}$ provide a global basis for the tangent bundle.

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