Lecture 1: Lie algebra cohomology

In this lecture we will introduce the Chevalley–Eilenberg cohomology of a Lie algebra, which will be morally one half of the BRST cohomology.

1.1 Cohomology

Let C be a vector space and $d : C \to C$ a linear transformation. If $d^2 = 0$ we say that (C, d) is a **(differential) complex**. We call C the **cochains** and d the **differential**. Vectors in the kernel $Z = \ker d$ are called **cocycles** and those in the image $B = \operatorname{im} d$ are called **coboundaries**. Because $d^2 = 0$, $B \subset Z$ and we can define the **cohomology**

$$H(C, d) := Z/B\,.$$

It is an important observation that H is *not* a subspace of Z, but a quotient. It is a subquotient of C. Elements of H are equivalence classes of cocycles—two cocycles being equivalent if their difference is a coboundary.

Having said this, with additional structure it is often the case that we can choose a privileged representative cocycle for each cohomology class and in this way view H as a subspace of C. For example, if C has a (positive-definite) inner product and if d^* is the adjoint with respect to this inner product, then one can show that every cohomology class contains a unique cocycle which is annihilated also by d^*.

Most complexes we will meet will be **graded**. This means that $C = \bigoplus_n C^n$ and d has degree 1, so it breaks up into a sequence of maps $d_n : C^n \to C^{n+1}$, which satisfy $d_{n+1} \circ d_n = 0$. Such complexes are usually denoted $(C^\bullet, d)$ and depicted as a sequence of linear maps

$$\cdots \longrightarrow C^{n-1} \xrightarrow{d_{n-1}} C^n \xrightarrow{d_n} C^{n+1} \longrightarrow \cdots$$

the composition of any two being zero. The cohomology is now also a graded vector space $H(C^\bullet, d) = \bigoplus_n H^n$, where

$$H^n = Z_n/B_n\,,$$

with $Z_n = \ker d_n : C^n \to C^{n+1}$ and $B_n = \operatorname{im} d_{n-1} : C^{n-1} \to C^n$.

The example most people meet for the first time is the de Rham complex of differential forms on a smooth m-dimensional manifold M, where $C^n = \Omega^n(M)$ and $d : \Omega^n(M) \to \Omega^{n+1}(M)$ is the exterior derivative. This example is special in that it has an additional structure, namely a graded commutative multiplication given by the wedge product of forms. Moreover the exterior derivative is a derivation over the wedge product, turning $(\Omega^\bullet(M), d)$ into a **differential graded algebra**. In particular the de Rham cohomology $H^\bullet(M)$ has a well-defined multiplication induced from the wedge product. If M is riemannian, compact and orientable one has the celebrated Hodge decomposition theorem stating that in every de Rham cohomology class there is a unique smooth harmonic form.

The second example most people meet is that of a Lie group G. The de Rham complex $\Omega^\bullet(G)$ has a subcomplex consisting of the left-invariant differential forms. (They form a subcomplex because the exterior derivative commutes with pullbacks.) A left-invariant p-form is uniquely determined by its value at the identity, where it defines a linear map $\Lambda^p\mathfrak{g} \to \mathbb{R}$, where we have identified the tangent space at the identity with the Lie algebra $\mathfrak{g}$—in other words, an element of $\Lambda^p\mathfrak{g}^*$. The exterior derivative then induces a linear map also called $d : \Lambda^p\mathfrak{g}^* \to \Lambda^{p+1}\mathfrak{g}^*$. When G is compact one can show that the cohomology of the left-invariant subcomplex is isomorphic to the de Rham cohomology of G, thus reducing in effect a topological calculation (the de Rham cohomology) to a linear algebra problem (the so-called Lie algebra cohomology). Indeed, one can show that every de Rham class has a unique bi-invariant representative and these are precisely the harmonic forms relative to a bi-invariant metric.

1.2 Lie algebra cohomology

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $\mathfrak{M}$ a representation, with $\varrho : \mathfrak{g} \to \operatorname{End}\mathfrak{M}$ the structure map:

$$\varrho(X)\varrho(Y) - \varrho(Y)\varrho(X) = \varrho([X,Y]) \tag{1}$$

for all $X, Y \in \mathfrak{g}$. We will refer to $\mathfrak{M}$ together with the map ϱ as a **$\mathfrak{g}$-module**. (The nomenclature stems from the fact that $\mathfrak{M}$ is an honest module over an honest ring: the universal enveloping algebra of $\mathfrak{g}$.)

Define the space of linear maps

$$C^p(\mathfrak{g};\mathfrak{M}) := \operatorname{Hom}(\Lambda^p\mathfrak{g}, \mathfrak{M}) \cong \Lambda^p\mathfrak{g}^* \otimes \mathfrak{M}$$

which we call the space of **p-forms on $\mathfrak{g}$ with values in $\mathfrak{M}$**.

We now define a differential $d : C^p(\mathfrak{g};\mathfrak{M}) \to C^{p+1}(\mathfrak{g};\mathfrak{M})$ as follows:

- for $m \in \mathfrak{M}$, let $dm(X) = \varrho(X)m$ for all $X \in \mathfrak{g}$;
- for $\alpha \in \mathfrak{g}^*$, let $d\alpha(X,Y) = -\alpha([X,Y])$ for all $X, Y \in \mathfrak{g}$;
- extend it to $\Lambda^\bullet\mathfrak{g}^*$ by

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d\beta\,, \tag{2}$$

- and extend it to $\Lambda^\bullet\mathfrak{g}^* \otimes \mathfrak{M}$ by

$$d(\omega \otimes m) = d\omega \otimes m + (-1)^{|\omega|}\omega \wedge dm\,. \tag{3}$$

We check that $d^2 m = 0$ for all $m \in \mathfrak{M}$ using (1) and that $d^2\alpha = 0$ for all $\alpha \in \mathfrak{g}^*$ because of the Jacobi identity. It then follows by induction using (2) and (3) that $d^2 = 0$ everywhere.

We have thus defined a graded differential complex

$$\cdots \longrightarrow C^{p-1}(\mathfrak{g};\mathfrak{M}) \xrightarrow{d} C^{p}(\mathfrak{g};\mathfrak{M}) \xrightarrow{d} C^{p+1}(\mathfrak{g};\mathfrak{M}) \longrightarrow \cdots$$

called the **Chevalley–Eilenberg** complex of $\mathfrak{g}$ with values in $\mathfrak{M}$. Its cohomology

$$H^p(\mathfrak{g};\mathfrak{M}) = \frac{\ker d : C^p(\mathfrak{g};\mathfrak{M}) \to C^{p+1}(\mathfrak{g};\mathfrak{M})}{\operatorname{im} d : C^{p-1}(\mathfrak{g};\mathfrak{M}) \to C^{p}(\mathfrak{g};\mathfrak{M})}$$

is called the **Lie algebra cohomology of $\mathfrak{g}$ with values in $\mathfrak{M}$.**

It is easy to see that

$$H^0(\mathfrak{g};\mathfrak{M}) = \mathfrak{M}^{\mathfrak{g}} := \left\{ m \in \mathfrak{M} \middle| \varrho(X) m = 0 \quad \forall X \in \mathfrak{g} \right\} ;$$

that is, the invariants of $\mathfrak{M}$. This simple observation will be crucial to the aim of these lectures.

It is not hard to show that $H^p(\mathfrak{g};\mathfrak{M} \oplus \mathfrak{N}) \cong H^p(\mathfrak{g};\mathfrak{M}) \oplus H^p(\mathfrak{g};\mathfrak{N})$.

We can take $\mathfrak{M}$ to be the trivial one-dimensional module, in which case we write simply $H^\bullet(\mathfrak{g})$ for the cohomology. A simplified version of the **Whitehead lemmas** say that if $\mathfrak{g}$ is semisimple then $H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = 0$. Indeed, it is not hard to show that

$$H^1(\mathfrak{g}) \cong \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] ,$$

where $[\mathfrak{g},\mathfrak{g}]$ is the first derived ideal.

In general, the second cohomology $H^2(\mathfrak{g})$ is isomorphic to the space of equivalence classes of central extensions of $\mathfrak{g}$.

We can take $\mathfrak{M} = \mathfrak{g}$ with the adjoint representation $\varrho = \mathrm{ad}$. The groups $H^\bullet(\mathfrak{g};\mathfrak{g})$ contain structural information about $\mathfrak{g}$. It can be shown, for example, that $H^1(\mathfrak{g};\mathfrak{g})$ is the space of outer derivations, whereas $H^2(\mathfrak{g};\mathfrak{g})$ is the space of nontrivial infinitesimal deformations. Similarly the obstructions to integrating (formally) an infinitesimal deformation live in $H^3(\mathfrak{g};\mathfrak{g})$.

One can also show that a Lie algebra $\mathfrak{g}$ is semisimple if and only if $H^1(\mathfrak{g};\mathfrak{M}) = 0$ for every *finite-dimensional* module $\mathfrak{M}$.

Using Lie algebra cohomology one can give elementary algebraic proofs of important results such as Weyl's reducibility theorem, which states that every finite-dimensional module of a semisimple Lie algebra is isomorphic to a direct sum of irreducibles, and the Levi-Mal'čev theorem, which states that a finite-dimensional Lie algebra is isomorphic to the semidirect product of a semisimple and a solvable Lie algebra (the radical).

1.3 An operator expression for d

On $\Lambda^\bullet\mathfrak{g}^*$ we have two natural operations. If $\alpha \in \mathfrak{g}^*$ we define $\varepsilon(\alpha) : \Lambda^p\mathfrak{g}^* \to \Lambda^{p+1}\mathfrak{g}^*$ by wedging with α:

$$\varepsilon(\alpha)\omega = \alpha \wedge \omega \ .$$

Similarly, if $X \in \mathfrak{g}$, then we define $\imath(X) : \Lambda^p\mathfrak{g}^* \to \Lambda^{p-1}\mathfrak{g}^*$ by contracting with X:

$$\imath(X)\alpha = \alpha(X) \qquad \text{for } \alpha \in \mathfrak{g}^*$$

and extending it as an odd derivation

$$\imath(X)(\alpha \wedge \beta) = \imath(X)\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge \imath(X)\beta$$

to all of $\Lambda^\bullet\mathfrak{g}^*$. Notice that $\varepsilon(\alpha)\imath(X) + \imath(X)\varepsilon(\alpha) = \alpha(X)\,\mathrm{id}$.

Let (X_i) and (α^i) be canonically dual bases for $\mathfrak{g}$ and $\mathfrak{g}^*$ respectively. In terms of these operations and the structure map of the $\mathfrak{g}$-module $\mathfrak{M}$, we can write the differential as

$$d = \varepsilon(\alpha^i)\varrho(X_i) - \tfrac{1}{2}\varepsilon(\alpha^i)\varepsilon(\alpha^j)\imath([X_i, X_j]) \ ,$$

where we here in the sequel we use the Einstein summation convention.

It is customary to introduce the **ghost** $c^i := \varepsilon(\alpha^i)$ and the **antighost** $b_i := \imath(X_i)$, in terms of which, and abstracting the structure map ϱ, we can rewrite the differential as

$$d = c^i X_i - \tfrac{1}{2} f_{ij}^k c^i c^j b_k \ ,$$

where $[X_i, X_j] = f_{ij}^k X_k$ are the structure functions in this basis. To show that the above operator is indeed the Chevalley–Eilenberg differential, one simply shows that it agrees with it on generators

$$dm = \alpha^i \otimes X_i m \qquad \text{and} \qquad d\alpha^k = -\tfrac{1}{2} f_{ij}^k \alpha^i \wedge \alpha^j \ .$$

Finally, let us remark that using $c^i b_j + b_j c^i = \delta^i_j$ and $X_i X_j - X_j X_i = f_{ij}^k X_k$, it is also possible to show directly that $d^2 = 0$.