Lecture 2: Symplectic reduction

In this lecture we discuss group actions on symplectic manifolds and symplectic reduction. We start with some generalities about group actions on manifolds.

2.1 Differentiable group actions

Let $G$ be a connected Lie group and $\mathfrak{g}$ its Lie algebra. Suppose $G$ acts smoothly on a differentiable manifold $M$. Letting $\mathcal{X}(M)$ denote the vector fields on $M$, we have a map

$$\mathfrak{g} \rightarrow \mathcal{X}(M)$$

$$X \rightarrow \xi_X$$

associating to each $X \in \mathfrak{g}$ a vector field $\xi_X$ on $M$. This map is a Lie algebra homomorphism: $\xi_{[X,Y]} = [\xi_X, \xi_Y]$, where in the RHS we have the Lie bracket of vector fields. On a function $f \in C^\infty(M)$,

$$\xi_X f (m) = \frac{d}{dt} f(e^{-tX} \cdot m)|_{t=0}.$$

This is an example of the Lie derivative. If $\eta \in \mathcal{X}(M)$, then $\mathfrak{g}$ acts on it via

$$X \cdot \eta = [\xi_X, \eta].$$

Similarly, if $\theta \in \Omega^1(M)$ in a one-form, then for all $\eta \in \mathcal{X}(M)$,

$$(X \cdot \theta)(\eta) := X \cdot \theta(\eta) - \theta(X \cdot \eta)$$

$$= \xi_X\theta(\eta) - \theta([\xi_X, \eta]).$$

In general if $\omega \in \Omega^p(M)$ is a $p$-form,

$$X \cdot \omega := (d i(\xi_X) + i(\xi_X) d) \omega,$$

where $d$ is the exterior derivative and $i$ is the contraction operator defined by

$$(i(\xi) \omega) (\eta_1, \ldots, \eta_{p-1}) = \omega(\xi, \eta_1, \ldots, \eta_{p-1}).$$

As a check of this formula, notice it agrees on functions and on one-forms.

Let $\xi$ be a vector field and let $\mathcal{L}_\xi$ denote the Lie derivative on differential forms: $\mathcal{L}_\xi = d i(\xi) + i(\xi) d$. Then the following identities are easy to prove:

- $i(\xi) i(\eta) = -i(\eta) i(\xi)$,
- $\mathcal{L}_\xi i(\eta) - i(\eta) \mathcal{L}_\xi = i([\xi, \eta])$,
- $\mathcal{L}_\xi \mathcal{L}_\eta - \mathcal{L}_\eta \mathcal{L}_\xi = \mathcal{L}_{[\xi, \eta]},$

for all vector fields $\eta, \xi$. 
2.2 Symplectic group actions

Now let \((M, \omega)\) be a symplectic manifold. That is, \(\omega \in \Omega^2(M)\) is a closed non-degenerate 2-form. In other words, \(d\omega = 0\) and the natural map

\[
b : \mathscr{X}(M) \to \Omega^1(M)
\]

\[
\xi \mapsto \xi^\flat = i(\xi)\omega,
\]
is an isomorphism with inverse \(\xi : \Omega^1(M) \to \mathscr{X}(M)\). In local coordinates,

\[
\omega = \frac{1}{2} \omega_{ij} dx^i \wedge dx^j,
\]

nondegeneracy means that \(\det(\omega_{ij}) \neq 0\).

We now take a connected Lie group \(G\) acting on \(M\) via symplectomorphisms, i.e., diffeomorphisms which preserve \(\omega\). Infinitesimally, this means that if \(X \in \mathfrak{g}\) then

\[
0 = X \cdot \omega = dt(\xi_X)\omega + t(\xi_X)d\omega = dt(\xi_X)\omega,
\]

whence the one-form \(t(\xi_X)\omega\) is closed. A vector field \(\xi\) such that \(t(\xi)\omega\) is closed is said to be symplectic. Let \(\mathfrak{sym}(M)\) denote the space of symplectic vector fields. It is clear that the symplectic vector fields are the image of the closed forms under \(\flat\):

\[
\mathfrak{sym}(M) = \sharp \left( \Omega^1_{\text{closed}}(M) \right).
\]

If \(\xi^\flat\) is actually exact, we say that \(\xi\) is a hamiltonian vector field. This means that there exists \(\phi_\xi \in C^\infty(M)\) such that

\[
\xi^\flat + d\phi_\xi = 0.
\]

This function is not unique because we can add to it a locally-constant function and still satisfy the above equation. We let \(\mathfrak{ham}(M)\) denote the space of hamiltonian vector fields. Then we have that

\[
\mathfrak{ham}(M) = \sharp \left( \Omega^1_{\text{exact}}(M) \right).
\]

We can summarise the preceding discussion with the following sequence of maps

\[
0 \longrightarrow H^0_{\text{dr}}(M) \overset{i}{\longrightarrow} C^\infty(M) \overset{\text{deg}}{\longrightarrow} \mathfrak{sym}(M) \overset{\flat}{\longrightarrow} H^1_{\text{dr}}(M) \longrightarrow 0,
\]

where the kernel of each map is precisely the image of the preceding. Such sequences are called exact.

A \(G\)-action on \(M\) is said to be hamiltonian if to every \(X \in \mathfrak{g}\) we can assign a function \(\phi_X\) on \(M\) such that \(\xi^\flat_X + d\phi_X = 0\). In this case we have a map \(\mathfrak{g} \to C^\infty(M)\).
In a symplectic manifold, the functions define a **Poisson algebra**: if \( f, g \in C^\infty(M) \) we define their **Poisson bracket** by
\[
\{f, g\} = \omega(\xi_f, \xi_g),
\]
where \( \xi_f \) is the hamiltonian vector field such that \( \xi_f^\flat + df = 0 \). The Poisson bracket is clearly skew-symmetric and obeys the Jacobi identity (since \( d\omega = 0 \)) and moreover obeys
\[
\{f, gh\} = \{f, g\}h + g\{f, h\}.
\]
In particular it gives \( C^\infty(M) \) the structure of a Lie algebra. A hamiltonian action is said to be **Poisson** if there is a Lie algebra homomorphism \( g \to C^\infty(M) \) sending \( X \) to \( \phi_X \) in such a way that \( \xi_X^\flat + d\phi_X = 0 \) and that
\[
\phi_{[X, Y]} = \{\phi_X, \phi_Y\}.
\]
The obstruction for a symplectic group action to be Poisson can be measured cohomologically. Indeed, it is a mixture of the de Rham cohomology of \( M \) and the Chevalley–Eilenberg cohomology of \( g \). For example, it is not hard to see that if \( g \) is semisimple then the is no obstruction. In fact, the obstruction can be more succinctly expressed in terms of the **equivariant** cohomology of \( M \).

### 2.3 Symplectic reduction

If the \( G \)-action on \( M \) is Poisson we can define the **moment(um) map(ping)**
\[
\Phi : M \to g^* 
\]
by \( \Phi(m)(X) = \phi_X(m) \) for every \( X \in g \) and \( m \in M \). In a sense, this map is dual to the map \( g \to C^\infty(M) \) coming from the Poisson action. The group \( G \) acts both on \( M \) and on \( g^* \) via the coadjoint representation and the momentum mapping \( \Phi \) is \( G \)-equivariant, intertwining between the two actions. Indeed, since the group is connected, it suffices to prove equivariance under the action of the Lie algebra, but this is simply the fact that
\[
\xi_X\Phi_Y = \{\phi_X, \phi_Y\} = \phi_{[X, Y]}.
\]
The equivariance of the moment map means that the \( G \)-action preserves the level set
\[
M_0 := \{m \in M | \Phi(m) = 0\},
\]
which is a closed embedded submanifold of \( M \) provided that \( 0 \in g^* \) is a regular value of \( \Phi \). In this case, we can take the quotient \( M_0/G \), which, if the \( G \)-action is free and proper, will be a smooth manifold. In general, it may only be an orbifold. The following theorem is a centerpiece of this whole subject.
**Theorem 2.1** (Marsden–Weinstein). Let $(M, \omega)$ be a symplectic manifold and let $G$ be a connected Lie group acting on $M$ with an equivariant momentum mapping $\Phi : M \to g^*$. Let $M_0 = \Phi^{-1}(0)$ and let $\tilde{M} := M_0 / G$. If $\tilde{M}$ is a manifold, then it is symplectic and the symplectic form is uniquely defined as follows. Let $i : M_0 \to M$ and $\pi : M_0 \to \tilde{M}$ the natural maps: $i$ is the inclusion and $\pi$ sends every point in $M_0$ to the orbit it lies in. Then there exists a unique symplectic form $\tilde{\omega} \in \Omega^2(\tilde{M})$ such that $i^* \omega = \pi^* \tilde{\omega}$.

A common notation for $\tilde{M}$ is $M//G$.

We will actually sketch the proof of a more general result, but before doing so we need to introduce some notation.

### 2.4 Coisotropic reduction

A symplectic vector space $(V, \omega)$ is a vector space $V$ together with a nondegenerate skew-symmetric bilinear form $\omega$. Nondegeneracy means that the linear map $\flat : V \to V^*$ defined by $\nu \mapsto \omega(\nu, -)$ is an isomorphism. The tangent space $T_pM$ at any point $p$ in a symplectic manifold is a symplectic vector space relative to the restriction to $p$ of the symplectic form.

If $W \subset V$ is a linear subspace of a symplectic vector space, we let

$$W^\perp := \{v \in V | \omega(v, w) = 0 \ \forall \ w \in W\}$$

denote the **symplectic perpendicular**. Unlike the case of a positive-definite inner product, $W$ and $W^\perp$ need not be disjoint. Nevertheless, one can show that $\dim W^\perp = \dim V - \dim W$. A subspace $W \subset V$ is said to be

- **isotropic**, if $W \subset W^\perp$;
- **coisotropic**, if $W^\perp \subset W$;
- **lagrangian**, if $W^\perp = W$; and
- **symplectic**, if $W^\perp \cap W = \{0\}$.

It is easy to see that if $W \subset V$ is isotropic, then $\dim W \leq \frac{1}{2} \dim V$, whereas if it is coisotropic, then $\dim W \geq \frac{1}{2} \dim V$. Lagrangian subspaces are both isotropic and coisotropic, whence they are middle-dimensional. Notice that the restriction of the symplectic structure to an isotropic subspace is identically zero, whereas if $W$ is coisotropic, the quotient $W/W^\perp$ inherits a symplectic structure from that of $V$.

Now let $(M, \omega)$ be a symplectic manifold and let $N \subset M$ be a (closed, embedded) submanifold. We say that $N$ is **isotropic** (resp. **coisotropic**, **lagrangian**, **symplectic**) if for every $p \in N$, $T_p \subset T_pM$ is isotropic (resp. coisotropic, lagrangian, symplectic).
If $G$ acts on $(M, \omega)$ giving rise to an equivariant moment mapping $\Phi : M \rightarrow \mathfrak{g}^*$, then the zero locus $M_0$ of the moment mapping turns out to be a coisotropic submanifold. To prove this we need to show that $(T_p M_0)^\perp \subset T_p M_0$ for all $p \in M_0$. This will follow from the following observation. A vector $v \in T_p M$, $p \in M_0$, is tangent to $M_0$ if and only if $d\Phi(v) = 0$. However, for all $X \in \mathfrak{g}$,

$$d\Phi(v)(X) = d\phi_X(v) = \omega(v, \xi_X),$$

which shows that $(T_p M_0)^\perp$ is the subspace of $T_p M$ spanned by the $\xi_X(p)$; in other words, the tangent space of the $G$-orbit $\mathcal{O}$ through $p$. Now $G$ preserves $M_0$, whence $\mathcal{O} \subset M_0$ and hence $(T_p M_0)^\perp = T_p \mathcal{O} \subset T_p M_0$.

We will now leave the case of a $G$-action and consider a general coisotropic submanifold $M_0 \subset M$ and let $i : M_0 \rightarrow M$ denote the inclusion. Let $\omega_0 = i^* \omega$ denote the pull-back of the symplectic form to $M_0$. It is not a symplectic form, because it is degenerate. Indeed, its kernel at $p$ is $(T_p M_0)^\perp \subset T_p M_0$. We will assume that $\dim(T_p M_0)^\perp$ does not change as we move $p$. In this case, the subspaces $(T_p M_0)^\perp \subset T_p M_0$ define a distribution (in the sense of Frobenius) called the **characteristic distribution** of $\omega_0$ and denoted $TM_0^\perp$. We claim that it is integrable.

Let $v, w$ be local sections of $TM_0^\perp$, we want to show that so is their Lie bracket $[v, w]$. This follows from the fact that $\omega_0$ is closed. Indeed, if $u$ is any vector field tangent to $M_0$, then

$$0 = d\omega_0(u, v, w)$$

$$= u \omega_0(v, w) - v \omega_0(u, w) + w \omega_0(u, v)$$

$$- \omega_0([u, v], w) + \omega_0([u, w], v) - \omega_0([v, w], u).$$

All terms but the last vanish because of the fact that $v, w \in TM_0^\perp$, leaving

$$\omega_0([v, w], u) = 0 \quad \text{for all } u \in TM_0,$$

whence $[v, w] \in TM_0^\perp$.

By the Frobenius integrability theorem, $M_0$ is foliated by connected submanifolds whose tangent spaces make up $TM_0^\perp$. Let $\tilde{M}$ denote the space of leaves of this foliation and let $\pi : M_0 \rightarrow \tilde{M}$ denote the natural surjection taking a point of $M_0$ to the unique leaf containing it. Then locally (and also globally if the foliation ‘fibers’) $\tilde{M}$ is a manifold whose tangent space at a leaf is isomorphic to $T_p M_0 / T_p M_0^\perp$ for any point $p$ lying in that leaf. We then give $\tilde{M}$ a symplectic structure $\tilde{\omega}$ by demanding that $\pi^* \tilde{\omega} = \omega_0$. In other words, if $\tilde{v}, \tilde{w}$ are vectors tangent to a leaf, we define $\tilde{\omega}(\tilde{v}, \tilde{w})$ by choosing a point $p$ in the leaf and lifting $\tilde{v}, \tilde{w}$ to vectors $v, w \in T_p M_0$ and declaring $\tilde{\omega}(\tilde{v}, \tilde{w}) = \omega_0(v, w)$. We have to show that this is well-defined, so that it does not depend neither on the choice of $p$ nor on the choice of lifts. That it does not depend on the choice of lifts is
basically the algebraic result that since $T_pM_0 \subset T_pM$ is a coisotropic subspace, $T_pM_0/(T_pM_0)^\perp$ inherits a symplectic structure. To show independence on the point it is enough, since the leaves are connected, to show that $\omega_0$ is invariant under the flow of vector fields in $TM_0^\perp$. So let $v \in TM_0^\perp$ and consider

$$\mathcal{L}_v \omega_0 = d\iota(v)\omega_0 + \iota(v)d\omega_0,$$

which vanishes because $\omega_0$ is closed and $\iota(v)\omega_0 = 0$.

Finally, we show that $(\tilde{M}, \tilde{\omega})$ is symplectic by showing that $\tilde{\omega}$ is smooth and closed. Smoothness follows from the fact that $\pi^*\tilde{\omega}$ is smooth. To show that it is closed, we simply notice that

$$\pi^*d\tilde{\omega} = d\pi^*\tilde{\omega} = d\omega_0 = 0,$$

and then that $\pi_*$ is surjective.

In summary we have proved\footnote{modulo the bit about $TM_0^\perp$ having constant rank, but we only used this in order to use Frobenius's Theorem. There is another integrability theorem due to Sussmann, which does not require that $TM_0^\perp$ have constant rank.} the following:

**Theorem 2.2.** Let $(M, \omega)$ be a symplectic manifold and $i : M_0 \hookrightarrow M$ be a coisotropic submanifold. Then the space of leaves $\tilde{M}$ of the characteristic foliation of $i^*\omega$ inherits locally (and globally, if the foliation fibers) a unique symplectic form $\tilde{\omega}$ such that $\pi^*\tilde{\omega} = i^*\omega$, where $\pi : M_0 \to \tilde{M}$ is the natural surjection.

Notice that the passage from $M$ to $\tilde{M}$ is a subquotient: one passes to the coisotropic submanifold $M_0$ and then to a quotient. This is to be compared with the cohomology of a complex which is also a subquotient: one passes to a subspace (the cocycles) and then projects out the coboundaries. It therefore would seem possible (or even plausible) that there is a cohomology theory underlying symplectic reduction. Happily there is and is the topic to which we now turn.