Lecture 2: Symplectic reduction

In this lecture we discuss group actions on symplectic manifolds and symplectic reduction. We start with some generalities about group actions on manifolds.

2.1 Differentiable group actions

Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra. Suppose G acts smoothly on a differentiable manifold M. Letting $\mathscr{X}(M)$ denote the vector fields on M, we have a map

$$\begin{aligned} \mathfrak{g} &\to \mathscr{X}(M) \\ X &\mapsto \xi_X \end{aligned}$$

associating to each $X \in \mathfrak{g}$ a vector field ξ_X on M. This map is a Lie algebra homomorphism: $\xi_{[X,Y]} = [\xi_X, \xi_Y]$, where in the RHS we have the Lie bracket of vector fields. On a function $f \in C^\infty(M)$,

$$\xi_X f(m) = \frac{d}{dt} f(e^{-tX} \cdot m)\big|_{t=0} .$$

This is an example of the Lie derivative. If $\eta \in \mathscr{X}(M)$, then $\mathfrak{g}$ acts on it via

$$X \cdot \eta = [\xi_X, \eta] .$$

Similarly, if $\theta \in \Omega^1(M)$ in a one-form, then for all $\eta \in \mathscr{X}(M)$,

$$\begin{aligned} (X \cdot \theta)(\eta) &:= X \cdot \theta(\eta) - \theta(X \cdot \eta) \\ &= \xi_X \theta(\eta) - \theta([\xi_X, \eta]) . \end{aligned}$$

In general if $\omega \in \Omega^p(M)$ is a p-form,

$$X \cdot \omega := (d\imath(\xi_X) + \imath(\xi_X)d)\,\omega ,$$

where d is the exterior derivative and $\imath$ is the contraction operator defined by

$$(\imath(\xi)\omega)(\eta_1, \ldots, \eta_{p-1}) = \omega(\xi, \eta_1, \ldots, \eta_{p-1}) .$$

As a check of this formula, notice it agrees on functions and on one-forms.

Let ξ be a vector field and let $\mathscr{L}_\xi$ denote the Lie derivative on differential forms: $\mathscr{L}_\xi = d\imath(\xi) + \imath(\xi)d$. Then the following identities are easy to prove:

- $\imath(\xi)\imath(\eta) = -\imath(\eta)\imath(\xi)$,
- $\mathscr{L}_\xi \imath(\eta) - \imath(\eta)\mathscr{L}_\xi = \imath([\xi,\eta])$, and
- $\mathscr{L}_\xi \mathscr{L}_\eta - \mathscr{L}_\eta \mathscr{L}_\xi = \mathscr{L}_{[\xi,\eta]}$,

for all vector fields η, ξ.

2.2 Symplectic group actions

Now let (M,ω) be a symplectic manifold. That is, $\omega \in \Omega^2(M)$ is a closed non-degenerate 2-form. In other words, $d\omega = 0$ and the natural map

$$\begin{aligned} \flat : \mathscr{X}(M) &\to \Omega^1(M) \\ \xi &\mapsto \xi^\flat = \imath(\xi)\omega , \end{aligned}$$

is an isomorphism with inverse $\sharp : \Omega^1(M) \to \mathscr{X}(M)$. In local coordinates,

$$\omega = \tfrac{1}{2}\omega_{ij}\, dx^i \wedge dx^j ,$$

nondegeneracy means that $\det[\omega_{ij}] \neq 0$.

We now take a connected Lie group G acting on M via **symplectomorphisms**, i.e., diffeomorphisms which preserve ω. Infinitesimally, this means that if $X \in \mathfrak{g}$ then

$$\begin{aligned} 0 &= X \cdot \omega \\ &= d\imath(\xi_X)\omega + \imath(\xi_X) d\omega \\ &= d\imath(\xi_X)\omega , \end{aligned}$$

whence the one-form $\imath(\xi_X)\omega$ is closed. A vector field ξ such that $\imath(\xi)\omega$ is closed is said to be **symplectic**. Let $\mathfrak{sym}(M)$ denote the space of symplectic vector fields. It is clear that the symplectic vector fields are the image of the closed forms under $\sharp$:

$$\mathfrak{sym}(M) = \sharp\left(\Omega^1_{\text{closed}}(M)\right) .$$

If $\xi^\flat$ is actually exact, we say that ξ is a **hamiltonian vector field**. This means that there exists $\phi_\xi \in C^\infty(M)$ such that

$$\xi^\flat + d\phi_\xi = 0 .$$

This function is not unique because we can add to it a locally-constant function and still satisfy the above equation. We let $\mathfrak{ham}(M)$ denote the space of hamiltonian vector fields. Then we have that

$$\mathfrak{ham}(M) = \sharp\left(\Omega^1_{\text{exact}}(M)\right) .$$

We can summarise the preceding discussion with the following sequence of maps

$$0 \longrightarrow H^0_{dR}(M) \xrightarrow{\;i\;} C^\infty(M) \xrightarrow{\;\sharp\circ d\;} \mathfrak{sym}(M) \xrightarrow{\;\flat\;} H^1_{dR}(M) \longrightarrow 0 ,$$

where the kernel of each map is precisely the image of the preceding. Such sequences are called **exact**.

A G-action on M is said to be **hamiltonian** if to every $X \in \mathfrak{g}$ we can assign a function ϕ_X on M such that $\xi_X^\flat + d\phi_X = 0$. In this case we have a map $\mathfrak{g} \to C^\infty(M)$.

In a symplectic manifold, the functions define a **Poisson algebra**: if $f, g \in C^\infty(M)$ we define their **Poisson bracket** by

$$\{f, g\} = \omega(\xi_f, \xi_g) ,$$

where ξ_f is the hamiltonian vector field such that $\xi_f^\flat + df = 0$. The Poisson bracket is clearly skew-symmetric and obeys the Jacobi identity (since $d\omega = 0$) and moreover obeys

$$\{f, gh\} = \{f, g\}h + g\{f, h\} .$$

In particular it gives $C^\infty(M)$ the structure of a Lie algebra. A hamiltonian action is said to be **Poisson** if there is a Lie algebra homomorphism $\mathfrak{g} \to C^\infty(M)$ sending X to ϕ_X in such a way that $\xi_X^\flat + d\phi_X = 0$ and that

$$\phi_{[X,Y]} = \{\phi_X, \phi_y\} .$$

The obstruction for a symplectic group action to be Poisson can be measured cohomologically. Indeed, it is a mixture of the de Rham cohomology of M and the Chevalley–Eilenberg cohomology of $\mathfrak{g}$. For example, it is not hard to see that if $\mathfrak{g}$ is semisimple then the is no obstruction. In fact, the obstruction can be more succinctly expressed in terms of the **equivariant** cohomology of M.

2.3 Symplectic reduction

If the G-action on M is Poisson we can define the **moment(um) map(ping)**

$$\Phi : M \to \mathfrak{g}^*$$

by $\Phi(m)(X) = \phi_X(m)$ for every $X \in \mathfrak{g}$ and $m \in M$. In a sense, this map is dual to the map $\mathfrak{g} \to C^\infty(M)$ coming from the Poisson action. The group G acts both on M and on $\mathfrak{g}^*$ via the coadjoint representation and the momentum mapping Φ is G-equivariant, intertwining between the two actions. Indeed, since the group is connected, it suffices to prove equivariance under the action of the Lie algebra, but this is simply the fact that

$$\xi_X\phi_Y = \left\{\phi_X, \phi_Y\right\} = \phi_{[X,Y]} .$$

The equivariance of the moment map means that the G-action preserves the level set

$$M_0 := \{m \in M | \Phi(m) = 0\} ,$$

which is a closed embedded submanifold of M provided that $0 \in \mathfrak{g}^*$ is a regular value of Φ. In this case, we can take the quotient M_0/G, which, if the G-action is free and proper, will be a smooth manifold. In general, it may only be an orbifold. The following theorem is a centerpiece of this whole subject.

Theorem 2.1 (Marsden–Weinstein)**.** *Let* (M,ω) *be a symplectic manifold and let* G *be a connected Lie group acting on* M *with an equivariant momentum mapping* $\Phi : M \to \mathfrak{g}^*$. *Let* $M_0 = \Phi^{-1}(0)$ *and let* $\widetilde{M} := M_0/G$. *If* $\widetilde{M}$ *is a manifold, then it is symplectic and the symplectic form is uniquely defined as follows. Let* $i : M_0 \to M$ *and* $\pi : M_0 \to \widetilde{M}$ *the natural maps:* i *is the inclusion and* π *sends every point in* M_0 *to the orbit it lies in. Then there exists a unique symplectic form* $\widetilde{\omega} \in \Omega^2(\widetilde{M})$ *such that* $i^*\omega = \pi^*\widetilde{\omega}$.

A common notation for $\widetilde{M}$ is $M//G$.

We will actually sketch the proof of a more general result, but before doing so we need to introduce some notation.

2.4 Coisotropic reduction

A symplectic vector space (V,ω) is a vector space V together with a nondegenerate skew-symmetric bilinear form ω. Nondegeneracy means that the linear map $\flat : V \to V^*$ defined by $v \mapsto \omega(v,-)$ is an isomorphism. The tangent space T_pM at any point p in a symplectic manifold is a symplectic vector space relative to the restriction to p of the symplectic form.

If $W \subset V$ is a linear subspace of a symplectic vector space, we let

$$W^\perp := \{v \in V | \omega(v,w) = 0 \quad \forall\ w \in W\}$$

denote the **symplectic perpendicular**. Unlike the case of a positive-definite inner product, W and $W^\perp$ need not be disjoint. Nevertheless, one can show that $\dim W^\perp = \dim V - \dim W$. A subspace $W \subset V$ is said to be

- **isotropic**, if $W \subset W^\perp$;
- **coisotropic**, if $W^\perp \subset W$;
- **lagrangian**, if $W^\perp = W$; and
- **symplectic**, if $W^\perp \cap W = \{0\}$.

It is easy to see that if $W \subset V$ is isotropic, then $\dim W \leq \frac{1}{2}\dim V$, whereas if it is coisotropic, then $\dim W \geq \frac{1}{2}\dim V$. Lagrangian subspaces are both isotropic and coisotropic, whence they are middle-dimensional. Notice that the restriction of the symplectic structure to an isotropic subspace is identically zero, whereas if W is coisotropic, the quotient $W/W^\perp$ inherits a symplectic structure from that of V.

Now let (M,ω) be a symplectic manifold and let $N \subset M$ be a (closed, embedded) sumbanifold. We say that N is **isotropic** (resp. **coisotropic**, **lagrangian**, **symplectic**) if for every $p \in N$, $T_p \subset T_pM$ is isotropic (resp. coisitropic, lagrangian, symplectic).

If G acts on (M,ω) giving rise to an equivariant moment mapping $\Phi : M \to \mathfrak{g}^*$, then the zero locus M_0 of the moment mapping turns out to be a coisotropic submanifold. To prove this we need to show that $(T_pM_0)^\perp \subset T_pM_0$ for all $p \in M_0$. This will follow from the following observation. A vector $v \in T_pM$, $p \in M_0$, is tangent to M_0 if and only if $d\Phi(v) = 0$. However, for all $X \in \mathfrak{g}$,

$$d\Phi(v)(X) = d\phi_X(v) = \omega(v,\xi_X) ,$$

which shows that $(T_pM_0)^\perp$ is the subspace of T_pM spanned by the $\xi_X(p)$; in other words, the tanget space of the G-orbit $\mathscr{O}$ through p. Now G preserves M_0, whence $\mathscr{O} \subset M_0$ and hence $(T_pM_0)^\perp = T_p\mathscr{O} \subset T_pM_0$.

We will now leave the case of a G-action and consider a general coisotropic submanifold $M_0 \subset M$ and let $i : M_0 \to M$ denote the inclusion. Let $\omega_0 = i^*\omega$ denote the pull-back of the symplectic form to M_0. It is not a symplectic form, because it is degenerate. Indeed, its kernel at p is $(T_pM_0)^\perp \subset T_pM_0$. We will assume that $\dim(T_pM_0)^\perp$ does not change as we move p. In this case, the subspaces $(T_pM_0)^\perp \subset T_pM_0$ define a distribution (in the sense of Frobenius) called the **characteristic distribution of** ω_0 and denoted $TM_0^\perp$. We claim that it is integrable.

Let v, w be local sections of $TM_0^\perp$, we want to show that so is their Lie bracket $[v,w]$. This follows from the fact that ω_0 is closed. Indeed, if u is any vector field tangent to M_0, then

$$\begin{aligned}
0 &= d\omega_0(u,v,w)\\
&= u\omega_0(v,w) - v\omega_0(u,w) + w\omega_0(u,v)\\
&\quad - \omega_0([u,v],w) + \omega_0([u,w],v) - \omega_0([v,w],u) .
\end{aligned}$$

All terms but the last vanish because of the fact that $v, w \in TM_0^\perp$, leaving

$$\omega_0([v,w],u) = 0 \qquad \text{for all } u \in TM_0,$$

whence $[v,w] \in TM_0^\perp$.

By the Frobenius integrability theorem, M_0 is foliated by connected submanifolds whose tangent spaces make up $TM_0^\perp$. Let $\widetilde{M}$ denote the space of leaves of this foliation and let $\pi : M_0 \to \widetilde{M}$ denote the natural surjection taking a point of M_0 to the unique leaf containing it. Then locally (and also globally if the foliation 'fibers') $\widetilde{M}$ is a manifold whose tangent space at a leaf is isomorphic to $T_pM_0/T_pM_0^\perp$ for any point p lying in that leaf. We then give $\widetilde{M}$ a symplectic structure $\widetilde{\omega}$ by demanding that $\pi^*\widetilde{\omega} = \omega_0$. In other words, if $\widetilde{v}, \widetilde{w}$ are vectors tangent to a leaf, we define $\widetilde{\omega}(\widetilde{v},\widetilde{w})$ by choosing a point p in the leaf and lifting $\widetilde{v}, \widetilde{w}$ to vectors $v, w \in T_pM_0$ and declaring $\widetilde{\omega}(\widetilde{v},\widetilde{w}) = \omega_0(v,w)$. We have to show that this is well-defined, so that it does not depend neither on the choice of p nor on the choice of lifts. That it does not depend on the choice of lifts is basically the algebraic result that since $T_pM_0 \subset T_pM$ is a coisotropic subspace, $T_pM_0/(T_pM_0)^\perp$ inherits a symplectic structure. To show independence on the point it is enough, since the leaves are connected, to show that ω_0 is invariant under the flow of vector fields in $TM_0^\perp$. So let $v \in TM_0^\perp$ and consider

$$\mathscr{L}_v \omega_0 = d\imath(v)\omega_0 + \imath(v) d\omega_0 ,$$

which vanishes because ω_0 is closed and $\imath(v)\omega_0 = 0$.

Finally, we show that $(\widetilde{M}, \widetilde{\omega})$ is symplectic by showing that $\widetilde{\omega}$ is smooth and closed. Smoothness follows from the fact that $\pi^*\widetilde{\omega}$ is smooth. To show that it is closed, we simply notice that

$$\pi^* d\widetilde{\omega} = d\pi^*\widetilde{\omega} = d\omega_0 = 0 ,$$

and then that π_* is surjective.

In summary we have proved[1] the following:

Theorem 2.2. *Let* (M, ω) *be a symplectic manifold and* $i : M_0 \hookrightarrow M$ *be a coisotropic submanifold. Then the space of leaves* $\widetilde{M}$ *of the characteristic foliation of* $i^*\omega$ *inherits locally (and globally, if the foliation fibers) a unique symplectic form* $\widetilde{\omega}$ *such that* $\pi^*\widetilde{\omega} = i^*\omega$*, where* $\pi : M_0 \to \widetilde{M}$ *is the natural surjection.*

Notice that the passage from M to $\widetilde{M}$ is a subquotient: one passes to the coisotropic submanifold M_0 and then to a quotient. This is to be compared with the cohomology of a complex which is also a subquotient: one passes to a subspace (the cocycles) and then projects out the coboundaries. It therefore would seem possible (or even plausible) that there is a cohomology theory underlying symplectic reduction. Happily there is and is the topic to which we now turn.

[1] modulo the bit about $TM_0^\perp$ having constant rank, but we only used this in order to use Frobenius's Theorem. There is another integrability theorem due to Sussmann, which does not require that $TM_0^\perp$ have constant rank.