BRST Cohomology

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These are the notes accompanying a PG minicourse on **BRST Cohomology** taught in Edinburgh in the Autumn Semester of 2006.
As usual, any statement which is not proved to your satisfaction is to be thought of as an exercise, even if not explicitly labelled as such!
These notes are still in a state of flux and I am happy to receive comments and suggestions either by email or in person.

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Lecture 1: Lie algebra cohomology

In this lecture we will introduce the Chevalley–Eilenberg cohomology of a Lie algebra, which will be morally one half of the BRST cohomology.

1.1 Cohomology

Let C be a vector space and $d : C \to C$ a linear transformation. If $d^2 = 0$ we say that (C, d) is a **(differential) complex**. We call C the **cochains** and d the **differential**. Vectors in the kernel $Z = \ker d$ are called **cocycles** and those in the image $B = \operatorname{Im} d$ are called **coboundaries**. Because $d^2 = 0$, $B \subset Z$ and we can define the **cohomology**

$$H(C, d) := Z/B\,.$$

It is an important observation that H is *not* a subspace of Z, but a quotient. It is a subquotient of C. Elements of H are equivalence classes of cocycles—two cocycles being equivalent if their difference is a coboundary.

Having said this, with additional structure it is often the case that we can choose a privileged representative cocycle for each cohomology class and in this way view H as a subspace of C. For example, if C has a (positive-definite) inner product and if d^* is the adjoint with respect to this inner product, then one can show that every cohomology class contains a unique cocycle which is annihilated also by d^*.

Most complexes we will meet will be **graded**. This means that $C = \bigoplus_n C^n$ and d has degree 1, so it breaks up into a sequence of maps $d_n : C^n \to C^{n+1}$, which satisfy $d_{n+1} \circ d_n = 0$. Such complexes are usually denoted $(C^\bullet, d)$ and depicted as a sequence of linear maps

$$\cdots \longrightarrow C^{n-1} \xrightarrow{d_{n-1}} C^n \xrightarrow{d_n} C^{n+1} \longrightarrow \cdots$$

the composition of any two being zero. The cohomology is now also a graded vector space $H(C^\bullet, d) = \bigoplus_n H^n$, where

$$H^n = Z_n/B_n\,,$$

with $Z_n = \ker d_n : C^n \to C^{n+1}$ and $B_n = \operatorname{Im} d_{n-1} : C^{n-1} \to C^n$.

The example most people meet for the first time is the de Rham complex of differential forms on a smooth m-dimensional manifold M, where $C^n = \Omega^n(M)$ and $d : \Omega^n(M) \to \Omega^{n+1}(M)$ is the exterior derivative. This example is special in that it has an additional structure, namely a graded commutative multiplication given by the wedge product of forms. Moreover the exterior derivative is a derivation over the wedge product, turning $(\Omega^\bullet(M), d)$ into a **differential graded algebra**. In particular the de Rham cohomology $H^\bullet(M)$ has a well-defined multiplication induced from the wedge product. If M is riemannian, compact and orientable one has the celebrated Hodge decomposition theorem stating that in every de Rham cohomology class there is a unique smooth harmonic form.

The second example most people meet is that of a Lie group G. The de Rham complex $\Omega^\bullet(G)$ has a subcomplex consisting of the left-invariant differential forms. (They form a subcomplex because the exterior derivative commutes with pull-backs.) A left-invariant p-form is uniquely determined by its value at the identity, where it defines a linear map $\Lambda^p\mathfrak{g} \to \mathbb{R}$, where we have identified the tangent space at the identity with the Lie algebra $\mathfrak{g}$—in other words, an element of $\Lambda^p\mathfrak{g}^*$. The exterior derivative then induces a linear map also called $d : \Lambda^p\mathfrak{g}^* \to \Lambda^{p+1}\mathfrak{g}^*$. When G is compact one can show that the cohomology of the left-invariant subcomplex is isomorphic to the de Rham cohomology of G, thus reducing in effect a topological calculation (the de Rham cohomology) to a linear algebra problem (the so-called Lie algebra cohomology). Indeed, one can show that every de Rham class has a unique bi-invariant representative and these are precisely the harmonic forms relative to a bi-invariant metric.

1.2 Lie algebra cohomology

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $\mathfrak{M}$ a representation, with $\varrho : \mathfrak{g} \to \operatorname{End}\mathfrak{M}$ the structure map:

$$\varrho(X)\varrho(Y) - \varrho(Y)\varrho(X) = \varrho([X, Y]) \tag{1}$$

for all $X, Y \in \mathfrak{g}$. We will refer to $\mathfrak{M}$ together with the map ϱ as a **$\mathfrak{g}$-module**. (The nomenclature stems from the fact that $\mathfrak{M}$ is an honest module over an honest ring: the universal enveloping algebra of $\mathfrak{g}$.)

Define the space of linear maps

$$C^p(\mathfrak{g};\mathfrak{M}) := \operatorname{Hom}(\Lambda^p\mathfrak{g}, \mathfrak{M}) \cong \Lambda^p\mathfrak{g}^* \otimes \mathfrak{M}$$

which we call the space of **p-forms on $\mathfrak{g}$ with values in $\mathfrak{M}$**.

We now define a differential $d : C^p(\mathfrak{g};\mathfrak{M}) \to C^{p+1}(\mathfrak{g};\mathfrak{M})$ as follows:

- for $m \in \mathfrak{M}$, let $dm(X) = \varrho(X)m$ for all $X \in \mathfrak{g}$;
- for $\alpha \in \mathfrak{g}^*$, let $d\alpha(X, Y) = -\alpha([X, Y])$ for all $X, Y \in \mathfrak{g}$;
- extend it to $\Lambda^\bullet\mathfrak{g}^*$ by

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d\beta , \tag{2}$$

- and extend it to $\Lambda^\bullet\mathfrak{g}^* \otimes \mathfrak{M}$ by

$$d(\omega \otimes m) = d\omega \otimes m + (-1)^{|\omega|}\omega \wedge dm . \tag{3}$$

We check that $d^2 m = 0$ for all $m \in \mathfrak{M}$ using (1) and that $d^2\alpha = 0$ for all $\alpha \in \mathfrak{g}^*$ because of the Jacobi identity. It then follows by induction using (2) and (3) that $d^2 = 0$ everywhere.

We have thus defined a graded differential complex

$$\cdots \longrightarrow \mathrm{C}^{p-1}(\mathfrak{g};\mathfrak{M}) \xrightarrow{d} \mathrm{C}^{p}(\mathfrak{g};\mathfrak{M}) \xrightarrow{d} \mathrm{C}^{p+1}(\mathfrak{g};\mathfrak{M}) \longrightarrow \cdots$$

called the **Chevalley–Eilenberg** complex of $\mathfrak{g}$ with values in $\mathfrak{M}$. Its cohomology

$$\mathrm{H}^p(\mathfrak{g};\mathfrak{M}) = \frac{\ker d : \mathrm{C}^p(\mathfrak{g};\mathfrak{M}) \to \mathrm{C}^{p+1}(\mathfrak{g};\mathfrak{M})}{\operatorname{Im} d : \mathrm{C}^{p-1}(\mathfrak{g};\mathfrak{M}) \to \mathrm{C}^{p}(\mathfrak{g};\mathfrak{M})}$$

is called the **Lie algebra cohomology of $\mathfrak{g}$ with values in $\mathfrak{M}$**.

It is easy to see that

$$\mathrm{H}^0(\mathfrak{g};\mathfrak{M}) = \mathfrak{M}^{\mathfrak{g}} := \left\{ m \in \mathfrak{M} \middle| \varrho(\mathrm{X})m = 0 \quad \forall \mathrm{X} \in \mathfrak{g} \right\} ;$$

that is, the invariants of $\mathfrak{M}$. This simple observation will be crucial to the aim of these lectures.

It is not hard to show that $\mathrm{H}^p(\mathfrak{g};\mathfrak{M} \oplus \mathfrak{N}) \cong \mathrm{H}^p(\mathfrak{g};\mathfrak{M}) \oplus \mathrm{H}^p(\mathfrak{g};\mathfrak{N})$.

We can take $\mathfrak{M}$ to be the trivial one-dimensional module, in which case we write simply $\mathrm{H}^\bullet(\mathfrak{g})$ for the cohomology. A simplified version of the **Whitehead lemmas** say that if $\mathfrak{g}$ is semisimple then $\mathrm{H}^1(\mathfrak{g}) = \mathrm{H}^2(\mathfrak{g}) = 0$. Indeed, it is not hard to show that

$$\mathrm{H}^1(\mathfrak{g}) \cong \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] ,$$

where $[\mathfrak{g},\mathfrak{g}]$ is the first derived ideal.

In general, the second cohomology $\mathrm{H}^2(\mathfrak{g})$ is isomorphic to the space of equivalence classes of central extensions of $\mathfrak{g}$.

We can take $\mathfrak{M} = \mathfrak{g}$ with the adjoint representation $\varrho = \mathrm{ad}$. The groups $\mathrm{H}^\bullet(\mathfrak{g};\mathfrak{g})$ contain structural information about $\mathfrak{g}$. It can be shown, for example, that $\mathrm{H}^1(\mathfrak{g};\mathfrak{g})$ is the space of outer derivations, whereas $\mathrm{H}^2(\mathfrak{g};\mathfrak{g})$ is the space of nontrivial infinitesimal deformations. Similarly the obstructions to integrating (formally) an infinitesimal deformation live in $\mathrm{H}^3(\mathfrak{g};\mathfrak{g})$.

One can also show that a Lie algebra $\mathfrak{g}$ is semisimple if and only if $\mathrm{H}^1(\mathfrak{g};\mathfrak{M}) = 0$ for every *finite-dimensional* module $\mathfrak{M}$.

Using Lie algebra cohomology one can give elementary algebraic proofs of important results such as Weyl's reducibility theorem, which states that every finite-dimensional module of a semisimple Lie algebra is isomorphic to a direct sum of irreducibles, and the Levi-Mal'čev theorem, which states that a finite-dimensional Lie algebra is isomorphic to the semidirect product of a semisimple and a solvable Lie algebra (the radical).

1.3 An operator expression for d

On $\Lambda^\bullet \mathfrak{g}^*$ we have two natural operations. If $\alpha \in \mathfrak{g}^*$ we define $\varepsilon(\alpha) : \Lambda^p \mathfrak{g}^* \to \Lambda^{p+1}\mathfrak{g}^*$ by wedging with α:

$$\varepsilon(\alpha)\omega = \alpha \wedge \omega \ .$$

Similarly, if $X \in \mathfrak{g}$, then we define $\imath(X) : \Lambda^p \mathfrak{g}^* \to \Lambda^{p-1}\mathfrak{g}^*$ by contracting with X:

$$\imath(X)\alpha = \alpha(X) \qquad \text{for } \alpha \in \mathfrak{g}^*$$

and extending it as an odd derivation

$$\imath(X)(\alpha \wedge \beta) = \imath(X)\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge \imath(X)\beta$$

to all of $\Lambda^\bullet \mathfrak{g}^*$. Notice that $\varepsilon(\alpha)\imath(X) + \imath(X)\varepsilon(\alpha) = \alpha(X)\,\mathrm{id}$.

Let $(\mathbf{X}_i)$ and $(\boldsymbol{\alpha}^i)$ be canonically dual bases for $\mathfrak{g}$ and $\mathfrak{g}^*$ respectively. In terms of these operations and the structure map of the $\mathfrak{g}$-module $\mathfrak{M}$, we can write the differential as

$$d = \varepsilon(\boldsymbol{\alpha}^i)\varrho(\mathbf{X}_i) - \tfrac{1}{2}\varepsilon(\boldsymbol{\alpha}^i)\varepsilon(\boldsymbol{\alpha}^j)\imath([\mathbf{X}_i,\mathbf{X}_j]) \ ,$$

where we here in the sequel we use the Einstein summation convention.

It is customary to introduce the **ghost** $c^i := \varepsilon(\boldsymbol{\alpha}^i)$ and the **antighost** $b_i := \imath(\mathbf{X}_i)$, in terms of which, and abstracting the structure map ϱ, we can rewrite the differential as

$$d = c^i \mathbf{X}_i - \tfrac{1}{2} f_{ij}^k c^i c^j b_k \ ,$$

where $[\mathbf{X}_i, \mathbf{X}_j] = f_{ij}^k \mathbf{X}_k$ are the structure functions in this basis. To show that the above operator is indeed the Chevalley–Eilenberg differential, one simply shows that it agrees with it on generators

$$dm = \boldsymbol{\alpha}^i \otimes \mathbf{X}_i m \qquad \text{and} \qquad d\boldsymbol{\alpha}^k = -\tfrac{1}{2} f_{ij}^k \boldsymbol{\alpha}^i \wedge \boldsymbol{\alpha}^j \ .$$

Finally, let us remark that using $c^i b_j + b_j c^i = \delta^i_j$ and $\mathbf{X}_i\mathbf{X}_j - \mathbf{X}_j\mathbf{X}_i = f_{ij}^k \mathbf{X}_k$, it is also possible to show directly that $d^2 = 0$.

1.4 Resolutions

We now come to a very useful technique in Lie algebra cohomology, which will prove decisive in our construction of the BRST complex; that of a resolution.

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{M}$ an $\mathfrak{g}$-module. By a **projective resolution** of $\mathfrak{M}$ we mean a complex $(K^\bullet, \delta)$ of $\mathfrak{g}$-modules

$$\cdots \xrightarrow{\delta} K^p \xrightarrow{\delta} K^{p-1} \xrightarrow{\delta} \cdots \xrightarrow{\delta} K^1 \xrightarrow{\delta} K^0 \xrightarrow{\delta} 0 \ ;$$

that is, $\delta : K^p \to K^{p-1}$ is a $\mathfrak{g}$-map and $\delta^2 = 0$, such that its homology is concentrated in zero degree:

$$H^p(K^\bullet) \cong \begin{cases} \mathfrak{M}\,, & p = 0 \\ 0\,, & \text{otherwise.} \end{cases}$$

In this way we may **augment** the complex to an exact sequence

$$\cdots \xrightarrow{\ \delta\ } K^1 \xrightarrow{\ \delta\ } K^0 \xrightarrow{\ \varepsilon\ } \mathfrak{M} \longrightarrow 0\,,$$

where $\varepsilon : K^0 \to K^0/\delta K^1 = \mathfrak{M}$ is the canonical projection. Tensoring this exact sequence with $\Lambda^p\mathfrak{g}^*$, we automatically obtain a projective resolution for the $\mathfrak{g}$-modules $C^p(\mathfrak{g};\mathfrak{M})$:

$$\cdots \xrightarrow{\ \delta\ } C^p(\mathfrak{g};K^1) \xrightarrow{\ \delta\ } C^p(\mathfrak{g};K^0) \xrightarrow{\ \varepsilon\ } C^p(\mathfrak{g};\mathfrak{M}) \longrightarrow 0\,,$$

where the maps δ and ε simply ignore $\Lambda^p\mathfrak{g}^*$.

Since δ is a $\mathfrak{g}$-map, Problem 1.13 tells us that it will induce chain maps $C^p(\mathfrak{g};K^q) \to C^p(\mathfrak{g};K^{q-1})$ with commuting squares:

$$\begin{array}{ccc} C^p(\mathfrak{g};K^q) & \xrightarrow{\ \delta\ } & C^p(\mathfrak{g};K^{q-1}) \\ {\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow \\ C^{p+1}(\mathfrak{g};K^q) & \xrightarrow{\ \delta\ } & C^{p+1}(\mathfrak{g};K^{q-1}) \end{array}$$

In other words, we have a bigraded complex $C^{p,q} := C^p(\mathfrak{g};K^q)$ and two commuting differentials:

- the Lie algebra differential $d : C^{p,q} \to C^{p+1,q}$, and
- the differential from the resolution $\delta : C^{p,q} \to C^{p,q-1}$.

Let $D = D' + D''$, where $D' = d$ and $D'' = (-1)^p\delta$ on $C^{p,q}$, where we introduced the alternating signs so that D' and D'' anticommute. In other words, $D^2 = 0$, so it is a differential. However, $D : C^{p,q} \to C^{p+1,q} \oplus C^{p,q-1}$ and hence does not respect the bidegree, but only the **total degree**, which assigns $p-q$ to $C^{p,q}$. Therefore we have a graded complex $(\mathscr{C}^\bullet, D)$, where $\mathscr{C}^n = \bigoplus_{p-q=n} C^{p,q}$ and $D : \mathscr{C}^n \to \mathscr{C}^{n+1}$, called the **total complex**.

The basic result is now the following:

Theorem 1.1.

$$H^n(\mathscr{C}^\bullet) \cong H^n(\mathfrak{g};\mathfrak{M})\,.$$

Proof. We will prove this by first interpreting cocycles and coboundaries in $C^n(\mathfrak{g};\mathfrak{M})$ in terms of objects in the bigraded complex $C^{\bullet,\bullet}$, and then exhibiting two maps between the cochains which send cocycles to cocycles and coboundaries to coboundaries and which are mutual inverses in cohomology.

First of all notice that since $\mathfrak{M} \cong K^0/\delta K^1$,

$$C^n(\mathfrak{g};\mathfrak{M}) \cong \frac{C^n(\mathfrak{g};K^0)}{C^n(\mathfrak{g};\delta K^1)} \cong \frac{C^{n,0}}{\delta C^{n,1}} .$$

Furthermore, the differential $d : C^n(\mathfrak{g};\mathfrak{M}) \to C^{n+1}(\mathfrak{g};\mathfrak{M})$ is induced from the differential $d : C^{n,0} \to C^{n+1,0}$ and projecting modulo $\delta C^{n+1,1}$. This implies that

$$\begin{aligned} Z^n(\mathfrak{g};\mathfrak{M}) &\cong \frac{\left\{\omega \in C^{n,0} \middle| d\omega \in \delta C^{n+1,1}\right\}}{\delta C^{n,1}} \\ B^n(\mathfrak{g};\mathfrak{M}) &\cong \frac{\left(dC^{n-1,0} + \delta C^{n,1}\right)}{\delta C^{n,1}} . \end{aligned}$$

From now on we will identify $Z^n(\mathfrak{g};\mathfrak{M})$ and $B^n(\mathfrak{g};\mathfrak{M})$ with the spaces in the right-hand sides of the above isomorphisms.

We now proceed to exhibit the maps between the cohomologies in the theorem. Let $\omega \in \mathscr{C}^n$; that is, $\omega = \omega_0 + \omega_1 + \omega_2 + \cdots$, where $\omega_i \in C^{n+i,i}$. The cocycle condition $D\omega = 0$ decomposes, according to the bidegree, into a number of equations:

$$\begin{aligned} D''\omega_0 &= 0 \\ D'\omega_0 + D''\omega_1 &= 0 \\ D'\omega_1 + D''\omega_2 &= 0 \\ &\vdots \\ D'\omega_{\text{top}} &= 0 . \end{aligned}$$

In particular, we see that ω_0 is such that $d\omega_0 \in \delta C^{n+1,1}$, whence it defines a cocycle in $Z^n(\mathfrak{g};\mathfrak{M})$. If this cocycle is a coboundary, so that $\omega = D\varphi$, for some $\varphi = \varphi_0 + \varphi_1 + \cdots \in \mathscr{C}^{n-1}$ with $\varphi_i \in C^{n-1+i,i}$, we have that, in particular,

$$\omega_0 = D'\varphi_0 + D''\varphi_1 \in dC^{n-1,0} + \delta C^{n,1} ,$$

whence it defines a coboundary in $B^n(\mathfrak{g};\mathfrak{M})$. In other words, the map $\omega \mapsto \omega_0$, which projects onto the $C^{n,0}$ component, gives rise to a map in cohomology

$$H^n(\mathscr{C}^\bullet) \to H^n(\mathfrak{g};\mathfrak{M}) . \tag{4}$$

Conversely, suppose that $\omega_0 \in C^{n,0}$ defines a cocycle in $Z^n(\mathfrak{g};\mathfrak{M})$. This means that $D'\omega_0 \in D''C^{n+1,1}$, whence there is $\omega_1 \in C^{n+1,1}$ such that

$$D'\omega_0 + D''\omega_1 = 0 .$$

Now,

$$D''D'\omega_1 = -D'D''\omega_1 = D'D'\omega_0 = 0 ,$$

whence $D'\omega_1 \in C^{n+2,1}$ is a D''-cocycle. But D'' has no cohomology there, since it is the differential of a resolution, hence $D'\omega_1$ has to be a D''-coboundary. In other words, there is some $\omega_2 \in C^{n+2,2}$ such that

$$D'\omega_1 + D''\omega_2 = 0 .$$

Continuing in this way we arrive at $\omega = \omega_0 + \omega_1 + \cdots \in \mathscr{C}^n$ which is a a D-cocycle. However, if ω_0 is a coboundary, then there are $\varphi_0 \in C^{n-1,0}$ and $\varphi_1 \in C^{n,1}$ such that $\omega_0 = D'\varphi_0 + D''\varphi_1$. Hence we see that

$$D''D'\varphi_1 = -D'D''\varphi_1 = -D'(\omega_0 - D'\varphi_0) = -D'\omega_0 = D''\omega_1 ,$$

whence $D''(\omega_1 - D'\varphi_1) = 0$ and by the acyclicity there of D'', there exists $\varphi_2 \in C^{n+1,2}$ such that $\omega_1 - D'\varphi_1 = D''\varphi_2$, and in this way we continue building $\varphi = \varphi_0 + \varphi_1 + \cdots \in \mathscr{C}^{n-1}$ such that $\omega = D\varphi$, and hence is a D-coboundary. In other words, we have defined a map in cohomology

$$H^n(\mathfrak{g};\mathfrak{M}) \to H^n(\mathscr{C}^\bullet) ,$$

which is easily seen to be inverse to the one in (4), thus proving the theorem. □

1.5 Problems

Problem 1.1. *An analogue of the Hodge decomposition theorem.*
Let (E, d) be a finite-dimensional differential complex, where E has a euclidean inner product. Let d^* denote the adjoint of d. Prove that in each cohomology class there is a unique cocycle which is annihilated by d^* and which can be characterized by the fact that it is the cocycle with the smallest norm in its cohomology class. Prove that the cohomology is isomorphic as a vector space to the kernel of the “laplacian” $\triangle = dd^* + d^*d$; hence every cohomology class has a unique “harmonic” representative. The same is true for the de Rham cohomology of a compact orientable manifold, but the proof is more subtle due to the infinite dimensionality of the spaces of differential forms.

Problem 1.2.
Let (C, d) be a differential complex and let $\langle -,-\rangle$ be a nondegenerate bilinear form on C relative to which d is (skew)symmetric: $\langle dc, c'\rangle = \pm\langle c, dc'\rangle$ for all $c, c' \in C$. Prove that the cohomology inherits a nondegenerate bilinear form from the restriction of the one on C to the cocycles.
Now assume that $(C = \bigoplus_n C^n, d)$ is a graded complex, and that the bilinear form $\langle -,-\rangle$ pairs up C^n with C^{-n}. Then show that $H^n(C) \cong H^{-n}(C)$ as vector spaces.

Problem 1.3. *A dual formulation of a Lie algebra.*
Let V be a real vector space, V^* its dual, and $\Lambda V^* = \bigoplus_p \Lambda^p V^*$ its exterior algebra.

We can think of $\Lambda^p V^*$ as the space of antisymmetric linear p-forms on V. Let $d : V^* \to \Lambda^2 V^*$ be any linear map and extend it to a linear map $d : \Lambda^p V^* \to \Lambda^{p+1} V^*$ as a derivation; that is,

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-)^p \alpha \wedge d\beta$$

for $\alpha \in \Lambda^p V^*$. Prove the following:

1. If $d^2\alpha = 0$ for all $\alpha \in V^*$ then $d^2 = 0$ identically on ΛV^*.
2. Let $d^t : \Lambda^2 V \to V$ be the transpose of $d : V^* \to \Lambda^2 V^*$. Then (V, d^t) is a Lie algebra with Lie bracket d^t if and only if $d^2 = 0$.

Problem 1.4. *(Anti-)ghosts and the Chevalley–Eilenberg differential.*
Let b_i and c^i be the operators introduced in Section 1.3. Recall that $c^i : \Lambda^p \mathfrak{g}^* \to \Lambda^{p+1} \mathfrak{g}^*$ is defined by $c^i \omega = \boldsymbol{\alpha}^i \wedge \omega$; and that $b_i : \Lambda^p \mathfrak{g}^* \to \Lambda^{p-1} \mathfrak{g}^*$ is the derivation defined by $b_i \boldsymbol{\alpha}^j = \delta_i^j$. Prove the following identities:

1. $b_i c^j + c^j b_i = \delta_i^j$,
2. $b_i b_j + b_j b_i = 0$, and
3. $c^i c^j + c^j c^i = 0$.

Let $\mathfrak{M}$ be a $\mathfrak{g}$-module with representation $\rho : \mathfrak{g} \to \operatorname{End} \mathfrak{M}$. Then show that the differential d computing $H(\mathfrak{g}; \mathfrak{M})$ is given by

$$d = c^i \rho(e_i) - \tfrac{1}{2} f_{ij}^k c^i c^j b_k \,.$$

Show by explicit computation that $d^2 = 0$.

Problem 1.5. *An explicit formula for the Chevalley–Eilenberg differential.*
Show that if $\phi \in C^p(\mathfrak{g}; \mathfrak{M})$, then

$$d\phi(\mathbf{X}_1, \dots, x_{p+1}) = \sum_{1 \le i < j \le p+1} (-1)^{i+j} \phi([\mathbf{X}_i, \mathbf{X}_j], \mathbf{X}_1, \dots, \widehat{\mathbf{X}_i}, \dots, \widehat{\mathbf{X}_j}, \dots, \mathbf{X}_{p+1}) \\ - \sum_{i=1}^{p+1} (-1)^i \varrho(\mathbf{X}_i) \phi(\mathbf{X}_1, \dots, \widehat{\mathbf{X}_i}, \dots, \mathbf{X}_{p+1}) \,,$$

where a hat over a Lie algebra element indicates its omission. Verify that it agrees with the expressions for $p = 0, 1$ derived in lecture:

$$dm(X) = \varrho(X) m \qquad \text{and} \qquad d\alpha(X, Y) = \varrho(X)\alpha(Y) - \varrho(Y)\alpha(X) - \alpha([X, Y]) \,,$$

for $m \in \mathfrak{M}$ and $\alpha : \mathfrak{g} \to \mathfrak{M}$.

Problem 1.6. *The Chevalley–Eilenberg complex as a* $\mathfrak{g}$*-module.*
The Chevalley–Eilenberg complex is itself a $\mathfrak{g}$-module, where the action of $\mathfrak{g}$ on $\Lambda^p\mathfrak{g}^*$ is the p-exterior power of the coadjoint representation. For every $X \in \mathfrak{g}$, define the linear map $\imath_X : \Lambda^p\mathfrak{g}^* \to \Lambda^{p-1}\mathfrak{g}^*$ by

$$\imath_X \alpha = \alpha(X) \quad \text{for } \alpha \in \mathfrak{g}^*$$

and extending as an odd derivation:

$$\imath_X(\alpha \wedge \beta) = \imath_X\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge \imath_X\beta \,.$$

Show that the action of $\mathfrak{g}$ on the Chevalley–Eilenberg complex is given by

$$X \cdot \phi = (\imath_X d + d\imath_X)\phi \,,$$

where $\phi \in \Lambda^p\mathfrak{g}^* \otimes \mathfrak{M}$. Deduce that d is a $\mathfrak{g}$-map and hence that the action of $\mathfrak{g}$ on the cohomology is trivial.

Problem 1.7.
A **perfect** Lie algebra is one in which every element can be written as a linear combination of Lie brackets; that is, $\mathfrak{g}$ is perfect when $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$. Prove that a Lie algebra is perfect if and only if $H^1(\mathfrak{g}) = 0$. Prove that semisimple Lie algebras are perfect. In fact, more generally, if $\mathfrak{g}$ has no center and has an invariant nondegenerate bilinear form, then it is perfect.

Problem 1.8. *Cohomology and central extensions.*
By a (real) **central extension** of a Lie algebra $\mathfrak{g}$ we mean a Lie algebra structure on the vector space $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$, which has the following form. Let (e_i, k) be a basis for $\tilde{\mathfrak{g}}$. Then k is central in $\tilde{\mathfrak{g}}$ (that is, it commutes with everything) and the bracket $[e_i, e_j]$ develops an extra term:

$$[e_i, e_j] = f_{ij}^k e_k + c_{ij} k \,,$$

where f_{ij}^k are the structure constants of $\mathfrak{g}$. Let $c = \frac{1}{2}c_{ij}\boldsymbol{\alpha}^i \wedge \boldsymbol{\alpha}^j \in \Lambda^2\mathfrak{g}^*$. Prove that c is a 2-cocycle.
A central extension $\tilde{\mathfrak{g}}$ is called **trivial** if it is isomorphic as a Lie algebra to $\mathfrak{g} \times \mathbb{R}$. Show that the central extension defined by a 2-cocycle is trivial if and only if the cocycle is also a coboundary. Hence $H^2(\mathfrak{g})$ is in one-to-one correspondence with nontrivial central extensions of $\mathfrak{g}$. Prove that a semisimple Lie algebra has no nontrivial central extensions. In other words, $H^2(\mathfrak{g}) = 0$ for $\mathfrak{g}$ semisimple.

Problem 1.9. *Cohomology and derivations.*
Let $\delta : \mathfrak{g} \to \mathfrak{g}$ be a linear map. It is called a **derivation** if $\delta[X, Y] = [\delta X, Y] + [X, \delta Y]$. A derivation is called **inner**, if for all $X \in \mathfrak{g}$, $\delta X = [Z, X]$ for some $Z \in \mathfrak{g}$. Prove that δ is a derivation if and only if $\boldsymbol{\alpha}^i \otimes \delta(e_i) \in \mathfrak{g}^* \otimes \mathfrak{g}$ is a 1-cocycle; and that it is an

inner derivation when it is also a coboundary. The quotient $H^1(\mathfrak{g};\mathfrak{g})$ of all derivations by the inner derivations is the space of **outer** derivations. Prove that in a semisimple Lie algebra, all derivations are inner. Notice that derivations form a Lie algebra in which the inner derivations constitute an ideal. Therefore $H^1(\mathfrak{g};\mathfrak{g})$ becomes a Lie algebra. More generally, one can show that $H(\mathfrak{g};\mathfrak{g})$ is a Lie superalgebra (with the degree offset by one from the natural one).

Let $\mathfrak{g}$ possess an invariant inner product. We call such $\mathfrak{g}$ **self-dual**. Prove that if all derivations of $\mathfrak{g}$ are inner, then $\mathfrak{g}$ doesn't admit any nontrivial central extensions. Conversely, prove that if $\mathfrak{g}$ doesn't admit any nontrivial central extensions, then all derivations which preserve the inner product (i.e., the antisymmetric derivations) are inner.

Problem 1.10. *Cohomology and deformations.*

Given a vector space V, how many different Lie brackets can we define on it?[1] A Lie bracket is a map $\Lambda^2\mathfrak{g} \to \mathfrak{g}$ subject to the Jacobi identity. Therefore Lie algebras on V are in one-to-one correspondence with the intersection of certain quadrics (the Jacobi identity) on $\Lambda^2 V^* \otimes V$. Let $J(V) \subset \Lambda^2 V^* \otimes V$ denote the space of solutions of the Jacobi identity.

Clearly not all points on $J(V)$ correspond to different Lie algebras—Lie brackets related by a change of basis in V yield the same Lie algebra. Therefore we define the moduli space $L(V)$ of Lie algebras on V as the quotient of $J(V)$ by the action of $GL(V)$. $L(V)$ may be a complicated object, but it is easy to probe its local structure by looking in the neighbourhood of a point. In other words, given a Lie algebra $\mathfrak{g}$ with underlying vector space V, one can study the infinitesimal deformations of the Lie bracket on $\mathfrak{g}$. Prove that the tangent space to $J(V)$ at $\mathfrak{g}$ is given by the cocycles $Z^2(\mathfrak{g};\mathfrak{g})$. Prove that those cocycles which are also coboundaries are tangent to the $GL(V)$ orbit through $\mathfrak{g}$. Conclude that the tangent space to $L(V)$ at $\mathfrak{g}$ is precisely $H^2(\mathfrak{g};\mathfrak{g})$. Prove that a semisimple Lie algebra is rigid; that is, it admits no nontrivial infinitesimal deformations.

It's not hard to show (Nijenhuis–Richardson) that there are an infinite set of obstructions to integrating (at least formally) a given infinitesimal deformation. Each obstruction is a class in $H^3(\mathfrak{g};\mathfrak{g})$.

Problem 1.11. *Cohomological criterion for semisimplicity.*

Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{M}$ denote a finite-dimensional $\mathfrak{g}$-module. Prove the following:

1. If $H^1(\mathfrak{g};\mathfrak{M}) = 0$ for all $\mathfrak{M}$, then every finite-dimensional $\mathfrak{g}$-module is fully reducible.
2. If every $\mathfrak{g}$-module is fully reducible, then $\mathfrak{g}$ is semisimple.
3. Conclude that $\mathfrak{g}$ is semisimple if and only if $H^1(\mathfrak{g};\mathfrak{M}) = 0$ for all $\mathfrak{M}$.

[1] This is a rhetorical question, not part of the problem!

Problem 1.12. *Chain maps.*
Let (C, d) and (C', d') be two differential complexes. Let $\varphi : C \to C'$ be a linear map which commutes with the action of the differentials: $\varphi \circ d = d' \circ \varphi$. Such a φ is called a **chain map**. Prove that φ induces a map in cohomology $\varphi^* : H(C) \to H(C')$. (Hint: Prove that φ sends cocycles to cocycles and coboundaries to coboundaries and argue from there.)

Problem 1.13. *Functorial properties of cohomology.*
This is boring to do in class—but it ought to be done. Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras and let $\varphi : \mathfrak{h} \to \mathfrak{g}$ be a homomorphism. Then let $\varphi^* : \Lambda\mathfrak{g}^* \to \Lambda\mathfrak{h}^*$ denote the natural map induced by φ. Also notice that if $\mathfrak{M}$ is a $\mathfrak{g}$-module, then it becomes an $\mathfrak{h}$-module via φ. Putting this together we find a map also denoted $\varphi^* : \Lambda\mathfrak{g}^* \otimes \mathfrak{M} \to \Lambda\mathfrak{h}^* \otimes \mathfrak{M}$. Prove that this map commutes with d. Therefore it induces a map in cohomology $\varphi^* : H(\mathfrak{g}; \mathfrak{M}) \to H(\mathfrak{h}; \mathfrak{M})$.
Now let $\mathfrak{M}$ and $\mathfrak{N}$ be $\mathfrak{g}$-modules. Prove that any linear map $f : \mathfrak{M} \to \mathfrak{N}$ commuting with the action of $\mathfrak{g}$ induces a map $f_* : H(\mathfrak{g}; \mathfrak{M}) \to H(\mathfrak{g}; \mathfrak{N})$.
Finally prove the following isomorphism:

$$H(\mathfrak{g}; \mathfrak{M} \oplus \mathfrak{N}) \cong H(\mathfrak{g}; \mathfrak{M}) \oplus H(\mathfrak{g}; \mathfrak{N}) .$$

(*Hint*: Abuse Problem 1.12.) If you only do one part of this problem, do the last one!

Problem 1.14. *The unavoidable snake lemma.*
Let $(A^\bullet, d_A)$, $(B^\bullet, d_B)$ and $(C^\bullet, d_C)$ be graded complexes. Exact sequences

$$0 \longrightarrow A^p \xrightarrow{\lambda_p} B^p \xrightarrow{\mu_p} C^p \longrightarrow 0 ,$$

for every p, where λ_p and μ_p are chain maps is called a **(short) exact sequence of graded complexes**. Show that such a sequence induces a long exact sequence in cohomology:

$$\cdots \longrightarrow H^p(A) \longrightarrow H^p(B) \longrightarrow H^p(C) \longrightarrow H^{p+1}(A) \longrightarrow \cdots$$

Make sure you understand the map $H^p(C) \to H^{p+1}(A)$ and the fact that it is induced by the differential.

Problem 1.15. *Another snake.*
Let $\mathfrak{M}$ and $\mathfrak{N}$ be $\mathfrak{g}$-modules and let $\varphi : \mathfrak{M} \to \mathfrak{N}$ be a $\mathfrak{g}$**-map**; that is, a linear map commuting with the action of $\mathfrak{g}$. Show that φ induces a chain map $C^\bullet(\mathfrak{g}; \mathfrak{M}) \to C^\bullet(\mathfrak{g}; \mathfrak{N})$ and hence maps $\varphi_* : H^p(\mathfrak{g}; \mathfrak{M}) \to H^p(\mathfrak{g}; \mathfrak{N})$ for all p.
Now let

$$0 \longrightarrow \mathfrak{M} \xrightarrow{\lambda} \mathfrak{N} \xrightarrow{\mu} \mathfrak{P} \longrightarrow 0$$

be a short exact sequence of $\mathfrak{g}$-modules. Show that this induces an exact sequence of the corresponding Chevalley–Eilenberg complexes:

$$0 \longrightarrow C^\bullet(\mathfrak{g};\mathfrak{M}) \xrightarrow{\lambda_\bullet} C^\bullet(\mathfrak{g};\mathfrak{N}) \xrightarrow{\mu_\bullet} C^\bullet(\mathfrak{g};\mathfrak{P}) \longrightarrow 0\,,$$

and hence a long exact sequence in cohomology:

$$\cdots \longrightarrow H^p(\mathfrak{g};\mathfrak{M}) \longrightarrow H^p(\mathfrak{g};\mathfrak{N}) \longrightarrow H^p(\mathfrak{g};\mathfrak{P}) \longrightarrow H^{p+1}(\mathfrak{g};\mathfrak{M}) \longrightarrow \cdots$$

Lecture 2: Symplectic reduction

In this lecture we discuss group actions on symplectic manifolds and symplectic reduction. We start with some generalities about group actions on manifolds.

2.1 Differentiable group actions

Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra. Suppose G acts smoothly on a differentiable manifold M. Letting $\mathscr{X}(M)$ denote the vector fields on M, we have a map

$$\begin{aligned} \mathfrak{g} &\to \mathscr{X}(M) \\ X &\mapsto \xi_X \end{aligned}$$

associating to each $X \in \mathfrak{g}$ a vector field ξ_X on M. This map is a Lie algebra homomorphism: $\xi_{[X,Y]} = [\xi_X, \xi_Y]$, where in the RHS we have the Lie bracket of vector fields. On a function $f \in C^\infty(M)$,

$$\xi_X f(m) = \frac{d}{dt} f(e^{-tX} \cdot m)\big|_{t=0} .$$

This is an example of the Lie derivative. If $\eta \in \mathscr{X}(M)$, then $\mathfrak{g}$ acts on it via

$$X \cdot \eta = [\xi_X, \eta] .$$

Similarly, if $\theta \in \Omega^1(M)$ in a one-form, then for all $\eta \in \mathscr{X}(M)$,

$$\begin{aligned} (X\cdot\theta)(\eta) &:= X\cdot\theta(\eta) - \theta(X\cdot\eta) \\ &= \xi_X\theta(\eta) - \theta([\xi_X, \eta]) . \end{aligned}$$

In general if $\omega \in \Omega^p(M)$ is a p-form,

$$X \cdot \omega := (d\imath(\xi_X) + \imath(\xi_X)d)\,\omega ,$$

where d is the exterior derivative and $\imath$ is the contraction operator defined by

$$(\imath(\xi)\omega)(\eta_1, \dots, \eta_{p-1}) = \omega(\xi, \eta_1, \dots, \eta_{p-1}) .$$

As a check of this formula, notice it agrees on functions and on one-forms.

Let ξ be a vector field and let $\mathscr{L}_\xi$ denote the Lie derivative on differential forms: $\mathscr{L}_\xi = d\imath(\xi) + \imath(\xi)d$. Then the following identities are easy to prove:

- $\imath(\xi)\imath(\eta) = -\imath(\eta)\imath(\xi)$,
- $\mathscr{L}_\xi\imath(\eta) - \imath(\eta)\mathscr{L}_\xi = \imath([\xi,\eta])$, and
- $\mathscr{L}_\xi\mathscr{L}_\eta - \mathscr{L}_\eta\mathscr{L}_\xi = \mathscr{L}_{[\xi,\eta]}$,

for all vector fields η, ξ. Notice in particular that the Lie derivative of a closed form is always exact. Hence a connected Lie group acting on M induces a trivial action on the de Rham cohomology.

2.2 Symplectic group actions

Now let (M,ω) be a symplectic manifold. That is, $\omega \in \Omega^2(M)$ is a closed nondegenerate 2-form. In other words, $d\omega = 0$ and the natural map

$$\begin{aligned} \flat : \mathscr{X}(M) &\to \Omega^1(M) \\ \xi &\mapsto \xi^\flat = \imath(\xi)\omega \,, \end{aligned}$$

is an isomorphism with inverse $\sharp : \Omega^1(M) \to \mathscr{X}(M)$. In local coordinates,

$$\omega = \tfrac{1}{2}\omega_{ij}\, dx^i \wedge dx^j \,,$$

nondegeneracy means that $\det[\omega_{ij}] \neq 0$.

We now take a connected Lie group G acting on M via **symplectomorphisms**, i.e., diffeomorphisms which preserve ω. Infinitesimally, this means that if $X \in \mathfrak{g}$ then

$$\begin{aligned} 0 &= X \cdot \omega \\ &= d\imath(\xi_X)\omega + \imath(\xi_X)d\omega \\ &= d\imath(\xi_X)\omega \,, \end{aligned}$$

whence the one-form $\imath(\xi_X)\omega$ is closed. A vector field ξ such that $\imath(\xi)\omega$ is closed is said to be **symplectic**. Let $\mathfrak{sym}(M)$ denote the space of symplectic vector fields. It is clear that the symplectic vector fields are the image of the closed forms under $\sharp$:

$$\mathfrak{sym}(M) = \sharp\left(\Omega^1_{\text{closed}}(M)\right) \,.$$

If $\xi^\flat$ is actually exact, we say that ξ is a **hamiltonian vector field**. This means that there exists $\phi_\xi \in C^\infty(M)$ such that

$$\xi^\flat + d\phi_\xi = 0 \,.$$

This function is not unique because we can add to it a locally-constant function and still satisfy the above equation. We let $\mathfrak{ham}(M)$ denote the space of hamiltonian vector fields. Then we have that

$$\mathfrak{ham}(M) = \sharp\left(\Omega^1_{\text{exact}}(M)\right) \,.$$

We can summarise the preceding discussion with the following sequence of maps

$$0 \longrightarrow H^0_{dR}(M) \xrightarrow{\;i\;} C^\infty(M) \xrightarrow{\;\sharp\circ d\;} \mathfrak{sym}(M) \xrightarrow{\;\flat\;} H^1_{dR}(M) \longrightarrow 0 \,,$$

where the kernel of each map is precisely the image of the preceding. Such sequences are called **exact**.

A G-action on M is said to be **hamiltonian** if to every $X \in \mathfrak{g}$ we can assign a function ϕ_X on M such that $\xi_X^\flat + d\phi_X = 0$. In this case we have a map $\mathfrak{g} \to C^\infty(M)$.

In a symplectic manifold, the functions define a **Poisson algebra**: if $f, g \in C^\infty(M)$ we define their **Poisson bracket** by

$$\{f, g\} := \xi_f g = \omega(\xi_f, \xi_g) ,$$

where ξ_f is the hamiltonian vector field such that $\xi_f^\flat + df = 0$. The Poisson bracket is clearly skew-symmetric and obeys the Jacobi identity (since $d\omega = 0$) and moreover obeys

$$\{f, gh\} = \{f, g\}h + g\{f, h\} ,$$

since ξ_f is a derivation on functions. In particular it gives $C^\infty(M)$ the structure of a Lie algebra. A hamiltonian action is said to be **Poisson** if there is a Lie algebra homomorphism $\mathfrak{g} \to C^\infty(M)$ sending X to ϕ_X in such a way that $\xi_X^\flat + d\phi_X = 0$ and that

$$\phi_{[X,Y]} = \{\phi_X, \phi_y\} .$$

The obstruction for a symplectic group action to be Poisson can be measured cohomologically. Indeed, it is a mixture of the de Rham cohomology of M and the Chevalley–Eilenberg cohomology of $\mathfrak{g}$. For example, it is not hard to see that if $\mathfrak{g}$ is semisimple then the is no obstruction. In fact, the obstruction can be more succinctly expressed in terms of the **equivariant** cohomology of M.

2.3 Symplectic reduction

If the G-action on M is Poisson we can define the **moment(um) map(ping)**

$$\Phi : M \to \mathfrak{g}^*$$

by $\Phi(m)(X) = \phi_X(m)$ for every $X \in \mathfrak{g}$ and $m \in M$. In a sense, this map is dual to the map $\mathfrak{g} \to C^\infty(M)$ coming from the Poisson action. The group G acts both on M and on $\mathfrak{g}^*$ via the coadjoint representation and the momentum mapping Φ is G-equivariant, intertwining between the two actions. Indeed, since the group is connected, it suffices to prove equivariance under the action of the Lie algebra, but this is simply the fact that

$$\xi_X \phi_Y = \left\{\phi_X, \phi_Y\right\} = \phi_{[X,Y]} .$$

The equivariance of the moment map means that the G-action preserves the level set

$$M_0 := \{m \in M | \Phi(m) = 0\} ,$$

which is a closed embedded submanifold of M provided that $0 \in \mathfrak{g}^*$ is a regular value of Φ. In this case, we can take the quotient M_0/G, which, if the G-action is free and proper, will be a smooth manifold. In general, it may only be an orbifold. The following theorem is a centerpiece of this whole subject.[2]

[2] The subject of symplectic reduction is very old and many people have contributed to it. The theorem here is but one statement which emphasises the rôle of the equivariant moment mapping. For a brief history of this subject you can consult [MRS00].

Theorem 2.1 (Marsden–Weinstein)**.** *Let* (M,ω) *be a symplectic manifold and let* G *be a connected Lie group acting on* M *with an equivariant momentum mapping* $\Phi : M \to \mathfrak{g}^*$. *Let* $M_0 = \Phi^{-1}(0)$ *and let* $\widetilde{M} := M_0/G$. *If* $\widetilde{M}$ *is a manifold, then it is symplectic and the symplectic form is uniquely defined as follows. Let* $i : M_0 \to M$ *and* $\pi : M_0 \to \widetilde{M}$ *the natural maps: i is the inclusion and* π *sends every point in* M_0 *to the orbit it lies in. Then there exists a unique symplectic form* $\widetilde{\omega} \in \Omega^2(\widetilde{M})$ *such that* $i^*\omega = \pi^*\widetilde{\omega}$.

A common notation for $\widetilde{M}$ is $M//G$.

We will actually sketch the proof of a more general result, but before doing so we need to introduce some notation.

2.4 Coisotropic reduction

A symplectic vector space (V,ω) is a vector space V together with a nondegenerate skew-symmetric bilinear form ω. Nondegeneracy means that the linear map $\flat : V \to V^*$ defined by $v \mapsto \omega(v,-)$ is an isomorphism. The tangent space T_pM at any point p in a symplectic manifold is a symplectic vector space relative to the restriction to p of the symplectic form.

If $W \subset V$ is a linear subspace of a symplectic vector space, we let

$$W^\perp := \{v \in V | \omega(v,w) = 0 \quad \forall\ w \in W\}$$

denote the **symplectic perpendicular**. Unlike the case of a positive-definite inner product, W and $W^\perp$ need not be disjoint. Nevertheless, one can show that $\dim W^\perp = \dim V - \dim W$. A subspace $W \subset V$ is said to be

- **isotropic**, if $W \subset W^\perp$;
- **coisotropic**, if $W^\perp \subset W$;
- **lagrangian**, if $W^\perp = W$; and
- **symplectic**, if $W^\perp \cap W = \{0\}$.

It is easy to see that if $W \subset V$ is isotropic, then $\dim W \leq \frac{1}{2}\dim V$, whereas if it is coisotropic, then $\dim W \geq \frac{1}{2}\dim V$. Lagrangian subspaces are both isotropic and coisotropic, whence they are middle-dimensional. Notice that the restriction of the symplectic structure to an isotropic subspace is identically zero, whereas if W is coisotropic, the quotient $W/W^\perp$ inherits a symplectic structure from that of V.

Now let (M,ω) be a symplectic manifold and let $N \subset M$ be a (closed, embedded) sumbanifold. We say that N is **isotropic** (resp. **coisotropic**, **lagrangian**, **symplectic**) if for every $p \in N$, $T_p \subset T_pM$ is isotropic (resp. coisitropic, lagrangian, symplectic).

If G acts on (M, ω) giving rise to an equivariant moment mapping $\Phi : M \to \mathfrak{g}^*$, then the zero locus M_0 of the moment mapping turns out to be a coisotropic submanifold. To prove this we need to show that $(T_pM_0)^\perp \subset T_pM_0$ for all $p \in M_0$. This will follow from the following observation. A vector $v \in T_pM$, $p \in M_0$, is tangent to M_0 if and only if $d\Phi(v) = 0$. However, for all $X \in \mathfrak{g}$,

$$d\Phi(v)(X) = d\phi_X(v) = \omega(v, \xi_X) ,$$

which shows that $(T_pM_0)^\perp$ is the subspace of T_pM spanned by the $\xi_X(p)$; in other words, the tangent space of the G-orbit $\mathscr{O}$ through p. Now G preserves M_0, whence $\mathscr{O} \subset M_0$ and hence $(T_pM_0)^\perp = T_p\mathscr{O} \subset T_pM_0$.

We will now leave the case of a G-action and consider a general coisotropic submanifold $M_0 \subset M$ and let $i : M_0 \to M$ denote the inclusion. Let $\omega_0 = i^*\omega$ denote the pull-back of the symplectic form to M_0. It is not a symplectic form, because it is degenerate. Indeed, its kernel at p is $(T_pM_0)^\perp \subset T_pM_0$. We will assume that $\dim(T_pM_0)^\perp$ does not change as we move p. In this case, the subspaces $(T_pM_0)^\perp \subset T_pM_0$ define a distribution (in the sense of Frobenius) called the **characteristic distribution of** ω_0 and denoted $TM_0^\perp$. We claim that it is integrable.

Let v, w be local sections of $TM_0^\perp$, we want to show that so is their Lie bracket $[v, w]$. This follows from the fact that ω_0 is closed. Indeed, if u is any vector field tangent to M_0, then

$$\begin{aligned} 0 &= d\omega_0(u, v, w) \\ &= u\omega_0(v, w) - v\omega_0(u, w) + w\omega_0(u, v) \\ &\quad - \omega_0([u, v], w) + \omega_0([u, w], v) - \omega_0([v, w], u) . \end{aligned}$$

All terms but the last vanish because of the fact that $v, w \in TM_0^\perp$, leaving

$$\omega_0([v, w], u) = 0 \qquad \text{for all } u \in TM_0,$$

whence $[v, w] \in TM_0^\perp$.

By the Frobenius integrability theorem, M_0 is foliated by connected submanifolds whose tangent spaces make up $TM_0^\perp$. Let $\widetilde{M}$ denote the space of leaves of this foliation and let $\pi : M_0 \to \widetilde{M}$ denote the natural surjection taking a point of M_0 to the unique leaf containing it. Then locally (and also globally if the foliation 'fibers') $\widetilde{M}$ is a manifold whose tangent space at a leaf is isomorphic to $T_pM_0/T_pM_0^\perp$ for any point p lying in that leaf. We then give $\widetilde{M}$ a symplectic structure $\widetilde{\omega}$ by demanding that $\pi^*\widetilde{\omega} = \omega_0$. In other words, if $\widetilde{v}, \widetilde{w}$ are vectors tangent to a leaf, we define $\widetilde{\omega}(\widetilde{v}, \widetilde{w})$ by choosing a point p in the leaf and lifting $\widetilde{v}, \widetilde{w}$ to vectors $v, w \in T_pM_0$ and declaring $\widetilde{\omega}(\widetilde{v}, \widetilde{w}) = \omega_0(v, w)$. We have to show that this is well-defined, so that it does not depend either on the choice of p or on the choice of lifts. That it does not depend on the choice of lifts is basically the algebraic result that since $T_pM_0 \subset T_pM$ is a coisotropic subspace,

$T_pM_0/(T_pM_0)^\perp$ inherits a symplectic structure. To show independence on the point it is enough, since the leaves are connected, to show that ω_0 is invariant under the flow of vector fields in $TM_0^\perp$. So let $v \in TM_0^\perp$ and consider

$$\mathscr{L}_v\omega_0 = d\imath(v)\omega_0 + \imath(v)d\omega_0 \ ,$$

which vanishes because ω_0 is closed and $\imath(v)\omega_0 = 0$.

Finally, we show that $(\widetilde{M}, \widetilde{\omega})$ is symplectic by showing that $\widetilde{\omega}$ is smooth and closed. Smoothness follows from the fact that $\pi^*\widetilde{\omega}$ is smooth. To show that it is closed, we simply notice that

$$\pi^* d\widetilde{\omega} = d\pi^*\widetilde{\omega} = d\omega_0 = 0 \ ,$$

and then that π_* is surjective.

In summary we have proved[3] the following:

Theorem 2.2. *Let* (M, ω) *be a symplectic manifold and* $i : M_0 \hookrightarrow M$ *be a coisotropic submanifold. Then the space of leaves* $\widetilde{M}$ *of the characteristic foliation of* $i^*\omega$ *inherits locally (and globally, if the foliation fibers) a unique symplectic form* $\widetilde{\omega}$ *such that* $\pi^*\widetilde{\omega} = i^*\omega$*, where* $\pi : M_0 \to \widetilde{M}$ *is the natural surjection.*

Notice that the passage from M to $\widetilde{M}$ is a subquotient: one passes to the coisotropic submanifold M_0 and then to a quotient. This is to be compared with the cohomology of a complex which is also a subquotient: one passes to a subspace (the cocycles) and then projects out the coboundaries. It therefore would seem possible (or even plausible) that there is a cohomology theory underlying symplectic reduction. Happily there is and is the topic to which we now turn.

2.5 Problems

Problem 2.1. *Reduction of symplectic vector spaces.*
Let (V, Ω) be a finite-dimensional symplectic vector space and let $W \subset V$ be a subspace. Show that $\dim V = \dim W + \dim W^\perp$, where $W^\perp$ is the symplectic perpendicular. Show further that the quotient $W/W \cap W^\perp$ inherits a unique symplectic structure $\widetilde{\Omega}$ such that

$$\pi^*\widetilde{\Omega} = i^*\Omega \ ,$$

where $i : W \to V$ is the inclusion and $\pi : W \to W/W\cap W^\perp$ is the natural projection.

Problem 2.2.
Prove that the Poisson bracket on $C^\infty(M)$ satisfies the Jacobi identity. (*Hint*: use that $d\omega = 0$.)

[3] modulo the bit about $TM_0^\perp$ having constant rank, but we only used this in order to use Frobenius's Theorem. There is another integrability theorem due to Sussmann, which does not require that $TM_0^\perp$ have constant rank.

Problem 2.3.
Show that if $\omega = d\theta$, where θ is G-invariant, then the action of G is Poisson.

Problem 2.4. *Obstructions to a hamiltonian action.*
Show that the Lie bracket of two symplectic vector fields is hamiltonian. Hence show that if $H^1(\mathfrak{g}) = 0$, then a symplectic action of $\mathfrak{g}$ on (M,ω) is hamiltonian. (*Hint*: If η,ξ are symplectic vector fields, show that $\imath_{[\eta,\xi]}\omega + d\omega(\eta,\xi) = 0$.)

Problem 2.5. *Obstructions to a Poisson action.*
Assume that the action of G on M is hamiltonian; whence there is a map $\mathfrak{g} \to C^\infty(M)$ taking $X \mapsto \phi_X$ where $\imath(\xi_X)\omega + d\phi_X = 0$. For every $X,Y \in \mathfrak{g}$, define the function

$$c(X,Y) = \phi_{[X,Y]} - \{\phi_X, \phi_Y\} \; .$$

Show that $dc(X,Y) = 0$ so that it is locally constant. This defines a map $c : \Lambda^2\mathfrak{g} \to H^0_{dR}(M)$. Show that c is a Lie algebra cocycle, where we interpret H^0_{dR} as a trivial $\mathfrak{g}$-module. Deduce that if and only if its cohomology class $[c] \in H^2\left(\mathfrak{g}; H^0_{dR}(M)\right)$ is trivial, can one find functions $\widetilde{\phi}_X$ satisfying $\imath_{\xi_X}\omega + d\widetilde{\phi}(X) = 0$ and such that the map $\mathfrak{g} \to C^\infty(M)$ given by $X \mapsto \widetilde{\phi}_X$ is a Lie algebra homomorphism.

Problem 2.6. *Another view on the obstructions.*
Let G act on (M,ω) preserving the symplectic form. Let $\varphi : G \times M \to M$ denote the G-action. Define an action $\psi : G \times G \times M \to G \times M$ of G on $G \times M$ by $\psi(g_1, g_2, m) = (g_1 g_2, m)$. Show that this makes φ into an equivariant map. Let $\mathrm{pr}_M : G \times M \to M$ denote the cartesian projection and define the following 2-form on $G \times M$:

$$\omega_\varphi := \varphi^*\omega - \mathrm{pr}_M^*\omega \; .$$

Show that ω_φ is closed and G-invariant, whence it defines a class in $H^2(G \times M)^G$, the cohomology of the subcomplex of $\Omega^\bullet(G \times M)$ consisting of G-invariant forms. The equivariant Künneth formula gives an isomorphism

$$H^2(G \times M)^G \cong H^2\left(\mathfrak{g}; H^0(M)\right) \oplus H^1\left(\mathfrak{g}; H^1(M)\right) \oplus H^0\left(\mathfrak{g}; H^2(M)\right)$$

by breaking up ω_φ into forms of different bidegree. Show that the third component vanishes in cohomology, whereas the first two are precisely the obstructions to defining a Poisson action we encountered above.

Problem 2.7. *Momentum conservation.*
Let the G-action on M be Poisson. Show that the components of the moment map are conserved quantities for any G-invariant hamiltonian.

Problem 2.8. *Poisson actions and discrete stabilizers.*
Let $\Phi : M \to \mathfrak{g}^*$ be the moment mapping for the Poisson action of G on M. Let $p \in$

M be a given point. Then the differential of the moment mapping at p defines a linear map

$$d\Phi_p : T_pM \to \mathfrak{g}^* .$$

Let $G_p < G$ denote the stabilizer of p in G. Show that it is a closed subgroup of G. Let $\mathfrak{g}_p$ denote its Lie algebra. Show that $\operatorname{Im} d\Phi_p = \mathfrak{g}_p^0$, where

$$\mathfrak{g}_p^0 = \left\{\alpha \in \mathfrak{g}^* \middle| \alpha(X) = 0,\ \forall X \in \mathfrak{g}_p\right\}$$

is the annihilator of $\mathfrak{g}_p$ in $\mathfrak{g}^*$. Conclude that if $0 \in \mathfrak{g}^*$ is a regular value of the moment map, the group G acts with discrete stabilizers on $M_0 = \Phi^{-1}(0)$. Such actions are said to be **locally free** and the quotient M_0/G will generally be an orbifold.

Problem 2.9. *Cotangent bundles.*
Let N be a smooth manifold and let T^*N denote its cotangent bundle. We let $\pi : T^*N \to N$ denote the projection. Show that there is a one-form $\theta \in \Omega^1(T^*N)$ defined by either one of the following equivalent conditions:

1. $\gamma^*\theta = \gamma$, where $\gamma \in \Omega^1(N)$ thought of as a smooth map $N \to T^*N$ on the LHS;
2. $\theta_\alpha = \alpha \circ \pi_*$, where $\alpha \in T^*N$; or
3. $\theta = p_i dq^i$ relative to local coordinates (q^i, p_i) for T^*N.

(The problem consists in showing that the definitions are equivalent and that they do define θ uniquely.) The one-form θ is called the **tautological one-form** on T^*N. Show that $\omega = -d\theta$ is a symplectic form. Let G be a group acting on N via diffeomorphisms. Show that the natural action of G on T^*N, under which π is equivariant, preserves the tautological one-form. Use Problem 2.3 to deduce that the G-action on T^*N is Poisson and write an expression for the moment mapping. Assuming that the action of G on N is free and proper so that N/G is a manifold, show that $T^*N//G$ is symplectomorphic to $T^*(N/G)$.
(*Hint:* For the moment mapping, show that at the point $(p,\alpha) \in T^*N$, the component in the direction $X \in \mathfrak{g}$ is given by $\phi_X(p,\alpha) = \alpha(\eta_X(p))$, where the $\eta_x \in \mathscr{X}(N)$ are the vector fields generating the G-action on N.)

Problem 2.10. *Reduction at nonzero momentum.*
Generalise the symplectic reduction in Section 2.3 to the case of nonzero momentum. In other words, let $\alpha \in \mathfrak{g}^*$ be a regular value of the moment map and let $M_\alpha = \Phi^{-1}(\alpha)$ be the submanifold of M consisting of points with momentum α. Then let

$$G_\alpha = \left\{g \in G \middle| \operatorname{Ad}_g^* \alpha = \alpha\right\}$$

denote the stabilizer of α. Show that G_α acts on M_α with discrete stabilizers. Show that if the quotient M_α/G_α is a manifold it has a unique symplectic structure $\tilde{\omega}$ such that $\pi^*\tilde{\omega} = i^*\omega$, where $i : M_\alpha \to M$ and $\pi : M_\alpha \to M_\alpha/G_\alpha$ are the natural maps. Show further that M_α/G_α is diffeomorphic to $\Phi^{-1}(\mathscr{O}_\alpha)/G$, where $\mathscr{O}_\alpha$ is the coadjoint orbit of α.

Problem 2.11. *Coadjoint orbits and the KKS construction.*
Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{g}^*$ its dual. The group G acts on $\mathfrak{g}$ via the adjoint representation and on $\mathfrak{g}^*$ via the coadjoint representation. Explicitly, if we identify $\mathfrak{g}$ with T_1G and $\mathfrak{g}^*$ with T_1^*G, then the adjoint representation is

$$\mathrm{Ad}_g = (L_g)_* \circ (R_{g^{-1}})_* : T_1G \to T_1G$$

and its dual is the coadjoint representation. If $\alpha \in \mathfrak{g}*$, then let $\mathscr{O}_\alpha$ denote the coadjoint orbit of α. In this problem we will show that $\mathscr{O}_\alpha$ is naturally a symplectic manifold. In particular, this will show that $\mathscr{O}_\alpha$ is even-dimensional.

1. Since $\mathfrak{g}^*$ is a vector space, we can identify the tangent spaces at each point with $\mathfrak{g}^*$ itself. We define a bivector B on $\mathfrak{g}^*$ as a map $\mathfrak{g}^* \to \Lambda^2\mathfrak{g}^*$ taking $\alpha \mapsto B_\alpha$, where $B_\alpha(X,Y) = \alpha([X,Y])$. Let $G_\alpha < G$ denote the stabilizer of α under the coadjoint representation and let $\mathfrak{g}_\alpha$ denote its Lie algebra. Show that the radical of B_α is precisely $\mathfrak{g}_\alpha$, and hence show that B_α induces a nondegenerate skew-symmetric bilinear form on $\mathfrak{g}/\mathfrak{g}_\alpha$.

2. Show that there is an exact sequence

$$0 \longrightarrow \mathfrak{g}_\alpha \longrightarrow \mathfrak{g} \xrightarrow{\sigma_\alpha} T_\alpha\mathscr{O}_\alpha \longrightarrow 0\,,$$

   where the map $\sigma_\alpha : \mathfrak{g} \to T_\alpha\mathscr{O}_\alpha$ is given by $\sigma_\alpha(X) = \xi_X(\alpha)$, where ξ_X are the vector fields which generate the coadjoint action on $\mathfrak{g}^*$. Thus σ_α induces an isomorphism $T_\alpha\mathscr{O}_\alpha \cong \mathfrak{g}/\mathfrak{g}_\alpha$ via which B_α defines a nondegenerate 2-form ω on $\mathscr{O}_\alpha$:

$$\omega(\xi_X(\alpha), \xi_Y(\alpha)) = B_\alpha(X,Y) = \alpha([X,Y])\,.$$

   Check explicitly that ω is nondegenerate.

3. Every $X \in \mathfrak{g}$ defines a linear function on $\mathfrak{g}^*$ and, by restriction, on any coadjoint orbit. We will let $\phi_X \in C^\infty(\mathscr{O}_\alpha)$ denote this function; that is, $\phi_X(\alpha) = \alpha(X)$. Show that $\xi_X\phi_Y = \phi_{[X,Y]}$ and that

$$\imath_{\xi_X}\omega = -d\phi_X\,. \tag{5}$$

   Use this to show that ω is G-invariant; that is, $\mathscr{L}_{\xi_X}\omega = 0$ and hence conclude that ω is closed.
   (*Hint*: For the first statement, compute $\mathscr{L}_{\xi_X} d\phi_Y$.)

4. Notice that equation (5) shows that the action of G on $\mathscr{O}_\alpha$ is hamiltonian. Show that this action is actually Poisson and prove that the moment map is simply the inclusion $\mathscr{O}_\alpha \to \mathfrak{g}^*$.

The above procedure is called the **Kirillov–Kostant–Souriau** construction.

Problem 2.12. *The KKS construction as a symplectic quotient.*
In this problem you will show that the symplectic structure on a coadjoint orbit constructed in Problem 2.11 arises from a symplectic quotient of T^*G, where the G-action is induced by left multiplication on G. Since left multiplication is a diffeomorphism, the canonical one-form on T^*G is invariant and hence the G-action is Poisson. The point of this problem is to work out the moment map explicitly and show that the symplectic quotients are the coadjoint orbits.

1. Let G act on itself via left multiplication. Show that the vector fields generating this action are the right-invariant vector fields on G.

2. From Problem 2.9 we know that this action preserves the canonical symplectic structure on T^*G and moreover that the action is Poisson with an equivariant moment map $\Phi : T^*G \to \mathfrak{g}^*$. Show that $\Phi(g, \mu) = R_g^* \mu$, where $\mu \in T_g^*G$; that is, Φ is the map which trivialises the cotangent bundle via right multiplication.

3. Let $M_\alpha = \Phi^{-1}(\alpha)$ denote the level set of momentum $\alpha \in \mathfrak{g}^*$. Show that M_α is the graph of the right-invariant 1-form with value α at the identity and hence diffeomorphic to the group G itself. Conclude that $M_\alpha \subset T^*N$ is a submanifold.

4. Let $G_\alpha < G$ denote the stabilizer of α under the coadjoint representation. Show that the action of G_α on M_α is simply left translations on the group and conclude that the quotient M_α/G_α is diffeomorphic to the coadjoint orbit $\mathscr{O}_\alpha$. Finally, show that this diffeomorphism is a symplectomorphism.

Problem 2.13. *Dirac's theory of constraints.*
Let $\phi_a \in C^\infty(M)$, for $a = 1, \ldots, k$, be smooth functions on M which we will think of as **constraints**. We will assume that $0 \in \mathbb{R}^k$ is a regular value of the map $\Phi : M \to \mathbb{R}^k$ whose components are the ϕ_a. Let $\mathscr{I}$ denote the ideal in $C^\infty(M)$ generated by the $\{\phi_a\}$; that is, $\mathscr{I}$ consists of linear combinations

$$f_1\phi_1 + \cdots + f_k\phi_k ,$$

where $f_a \in C^\infty(M)$. Let Ψ denote the vector space of linear combinations

$$c_1\phi_1 + \cdots + c_k\phi_k ,$$

where $c_a \in \mathbb{R}$. Then let $F \subset \Psi$ be a maximal subspace with the property that $\{F, \Psi\} \subset \mathscr{I}$ and let (ψ_i) denote a basis for F and complete it to a basis for Ψ by adding $\{\chi_\alpha\}$. Following Dirac, let us call the $\{\psi_i\}$ **first-class constraints** and the $\{\chi_\alpha\}$ **second-class constraints**. Show that the matrix of Poisson brackets $P_{\alpha\beta} := \{\chi_\alpha, \chi_\beta\}$ is nondegenerate on the zero locus S of the second-class constraints and hence show that S is a symplectic submanifold. Write down an explicit expression for the Poisson bracket on S in terms of the Poisson bracket on M and the matrix $P_{\alpha\beta}$. This is called the **Dirac bracket**. Finally show that the zero locus of the first-class constraints $\{\psi_i\}$ define a coistropic submanifold of S. In this way we have reduced the general situation to the one of coisotropic reduction.

Lecture 3: The BRST complex

In this lecture we will present the homological approach to coisotropic reduction by first studying the case where the coisotropic submanifold is the zero locus of an equivariant moment map coming from a group action. We will then outline the general case of coisotropic reduction.

(Co)homology is algebraic by its very nature, whereas coisotropic reduction as described above is geometric. This means that before we can relate them, they must be phrased in a common language. In this case, and as much as it may hurt one's sensibilities, the simplest thing to do is to translate geometry into algebra.

3.1 An algebraic interlude

The natural algebraic structure associated to a smooth manifold M is its algebra $C^\infty(M)$ of smooth functions. It is a commutative associative unital algebra which encodes a lot of information on M and from which in many cases on can reconstruct M. A symplectic structure on M lends $C^\infty(M)$ additional structure. The Poisson bracket turns $C^\infty(M)$ into a Lie algebra and moreover, for any $f \in C^\infty(M)$, $\{f,-\}$ is a derivation over the commutative multiplication. This turns $C^\infty(M)$ into a **Poisson algebra**.

Any closed embedded submanifold M_0 of M defines an ideal $\mathscr{I} \subset C^\infty(M)$ consisting of those functions which vanish on M_0. We call this the **vanishing ideal of** M_0. If $M_0 = \Phi^{-1}(0)$ is the zero locus of a smooth function $\Phi : M \to \mathbb{R}^k$ where $0 \in \mathbb{R}^k$ is a regular value, then the ideal $\mathscr{I}$ is precisely the ideal generated by the components ϕ_i of Φ relative to any basis for $\mathbb{R}^k$.

Every smooth function on M restricts to a smooth function on M_0 and two such functions restrict to the same function if and only if their difference belongs to the ideal $\mathscr{I}$. Conversely every smooth function on M_0 can be extended (not uniquely) to a smooth function on M. In other words, there is an isomorphism

$$C^\infty(M_0) \cong C^\infty(M)/\mathscr{I} . \tag{6}$$

We must now algebraize the fact that M_0 is coisotropic. We start by recalling that vector fields are derivations of the algebra of functions: $\mathscr{X}(M) = \operatorname{Der} C^\infty(M)$. From the isomorphism in (6), a derivation of $C^\infty(M)$ gives rise to a derivation of $C^\infty(M_0)$ if and only if it preserves the ideal $\mathscr{I}$. Indeed, it is not hard to show that

$$\operatorname{Der} C^\infty(M_0) = \left\{\xi \in \operatorname{Der} C^\infty(M) \middle| \xi(\mathscr{I}) \subset \mathscr{I}\right\} .$$

As we saw above, vector fields in $TM_0^\perp$ are precisely the hamiltonian vector fields which arise from functions in $\mathscr{I}$, whence the coisotropy condition $TM_0^\perp \subset TM_0$ becomes the condition that the vanishing ideal be closed under the Poisson bracket: $\{\mathscr{I},\mathscr{I}\} \subset \mathscr{I}$. Such ideals are called **coisotropic** for good reason.

Finally, the functions on $\widetilde{M}$ are those functions on M_0 which are constant on the leaves of the foliation. Since the leaves are connected and the tangent vectors to the leaves are the hamiltonian vector fields of functions in $\mathscr{I}$, we have an isomorphism

$$C^\infty(\widetilde{M}) = \left\{ f \in C^\infty(M_0) \middle| \left\{ f, \mathscr{I} \right\} = 0 \right\} ,$$

where $\{f, \mathscr{I}\} = 0$ on M_0. Extending f to a function on M, the isomorphism becomes

$$C^\infty(\widetilde{M}) = \left\{ f \in C^\infty(M) \middle| \left\{ f, \mathscr{I} \right\} \subset \mathscr{I} \right\} \big/ \mathscr{I} ,$$

which does not depend on the extension because $\mathscr{I}$ is closed under the Poisson bracket. In other words,

$$C^\infty(\widetilde{M}) = N(\mathscr{I})/\mathscr{I} ,$$

where $N(\mathscr{I})$ is the Poisson normalizer of $\mathscr{I}$ in $C^\infty(M)$.

Notice that $N(\mathscr{I})$ is a Poisson subalgebra of $C^\infty(M)$ having $\mathscr{I}$ as a Poisson ideal. This means that the quotient $N(\mathscr{I})/\mathscr{I}$ inherits the structure of a Poisson algebra.

The power of the algebraic formalism is that it continues to make sense in situations where the geometry might be singular. Indeed, it is possible now to rephrase this reduction purely in the category of Poisson algebras. Let P be a Poisson algebra and $I \subset P$ a coisotropic ideal. Then the normalizer $N(I) \subset P$ of I in P is a Poisson subalgebra containing I as a Poisson ideal, and hence the quotient $N(I)/I$ is a Poisson algebra, which we can think of as the reduced Poisson algebra of P by I. The aim of the BRST construction is to construct a complex of Poisson (super)algebras and a differential which is a Poisson (super)derivation so that its cohomology (at least in zero degree) is isomorphic as a Poisson algebra to $N(I)/I$. This turns out to be possible in huge generality, but the details of the construction depend on the 'regularity' of the ideal I. To keep things simple we will make a regularity assumption along the way.

3.2 The BRST complex of a group action

As a warmup we will construct the BRST complex for the case of a group action with an equivariant moment map $\Phi : M \to \mathfrak{g}^*$ which has $0 \in \mathfrak{g}^*$ as a regular value. Let $M_0 = \Phi^{-1}(0)$ be the coisotropic submanifold of zero momentum and let $\mathscr{I}$ denote its vanishing ideal. Let $\pi : M_0 \to \widetilde{M}$ denote the projection onto the quotient $\widetilde{M} = M_0/G$. The pull-back $\pi^* : C^\infty(\widetilde{M}) \to C^\infty(M_0)$ allows us to view functions on the quotient as functions on M_0. Indeed, a function on M_0 comes from a function on $\widetilde{M}$ if and only if it is constant on the fibres, which are the G-orbits. Since G is connected, this is the same thing as being constant along the flows of the vector fields ξ_X, which is the same thing as Poisson-commuting with (the restriction to M_0 of) ϕ_X, or in more algebraic terms,

$$C^\infty(\widetilde{M}) \cong C^\infty(M_0)^{\mathfrak{g}} = H^0\left(\mathfrak{g}; C^\infty(M_0)\right) .$$

This is not satisfactory because $C^\infty(M_0)$ is a quotient of $C^\infty(M)$, whereas we would like to work directly with $C^\infty(M)$. This suggests, as discussed in 1.4, to introduce a resolution for $C^\infty(M_0)$ in terms of (modules over) $C^\infty(M)$.

3.2.1 The Koszul resolution

To understand why such a resolution might exist, let us recall that $C^\infty(M_0) \cong C^\infty(M)/\mathscr{I}$ is already a projection of $C^\infty(M)$, where $\mathscr{I}$ is the vanishing ideal of M_0. The first fact we will need is that the vanishing ideal $\mathscr{I}$ coincides with the ideal $I[\Phi]$ generated by (the components of) the moment map.

Lemma 3.1. *The ideal* $I[\Phi]$ *generated by the components of the moment map is precisely the vanishing ideal* $\mathscr{I}$ *of* M_0.

Proof. Since the components of the moment map vanish on M_0, it is clear that $I[\Phi] \subseteq \mathscr{I}$. What we have to show is that if a function vanishes on M_0 it is contained in the ideal generated by the components of the moment map. We only prove this locally, leaving the globalisation to a standard argument using partitions of unity.

Let $N = \dim M$ and $k = \dim \mathfrak{g}$ for definiteness. We will choose a basis $\mathbf{X}_i$ for $\mathfrak{g}$ and let $\phi_i = \Phi(\mathbf{X}_i)$ be the components of the moment map relative to this basis. Since M_0 is an embedded submanifold, around every point of M_0 there is an open set $U \subset M$ and local coordinates $(\boldsymbol{x}, \boldsymbol{y}) : U \to \mathbb{R}^{N-k} \times \mathbb{R}^k$ where $y^i = \phi_i$ for $i = 1, \dots, k$. Suppose now that a function f vanishes on M_0. Restricting to U, and thinking of it as a function on $\mathbb{R}^N$, we have that $f(\boldsymbol{x}, \mathbf{0}) = 0$ for all $\boldsymbol{x}$. Then

$$\begin{aligned} f(\boldsymbol{x}, \boldsymbol{y}) &= \int_0^1 \tfrac{d}{dt} f(\boldsymbol{x}, t\boldsymbol{y}) dt \\ &= \int_0^1 \sum_{i=1}^k y^i (D_i f)(\boldsymbol{x}, t\boldsymbol{y}) dt \\ &= \sum_{i=1}^k \phi_i \int_0^1 (D_i f)(\boldsymbol{x}, t\boldsymbol{y}) dt , \end{aligned}$$

whence the restriction of f to U belongs to the ideal generated by the ϕ_i. In other words, there are functions $h^i_U \in C^\infty(U)$ such that $f|_U = \sum_i h^i_U \phi_i|_U$. We now cover M with such charts and patch things up with a partition of unity subordinate to this cover, which ensures that $f = \sum_i h^i \phi_i$ for some $h^i \in C^\infty(M)$. □

This means that we can build a two-step complex

$$\mathfrak{g} \otimes C^\infty(M) \xrightarrow{\ \delta\ } C^\infty(M) \xrightarrow{\ \delta\ } 0 ,$$

where $\delta f = 0$ for $f \in C^\infty(M)$ and $\delta X = \phi_X$, so that if $(\mathbf{X}_i)$ is a basis for $\mathfrak{g}$, then

$$\delta(\sum_i \mathbf{X}_i \otimes f_i) = \sum_i f_i \phi_i ,$$

where $\phi_i = \Phi(\mathbf{X}_i)$, whence its image is the ideal $I[\Phi] = I$. It is clear that δ so defined obeys $\delta^2 = 0$, and its cohomology is

$$H^0_\delta = \frac{C^\infty(M)}{I} \cong C^\infty(M_0) \ .$$

This is not yet a resolution because there is cohomology in positive degrees. Indeed, the map $\mathfrak{g} \otimes C^\infty(M) \to C^\infty(M)$ has kernel:

$$\delta(\sum_{i,j} \mathbf{X}_i \otimes h_{ij}\phi_j) = \sum_{i,j} h_{ij}\phi_i\phi_j \ ,$$

whence if $h_{ij} = -h_{ji}$, then it is a cocycle. This suggests extending the complex to the left

$$\Lambda^2\mathfrak{g} \otimes C^\infty(M) \xrightarrow{\ \delta\ } \mathfrak{g} \otimes C^\infty(M) \xrightarrow{\ \delta\ } C^\infty(M) \xrightarrow{\ \delta\ } 0 \ ,$$

where δ is extended as an odd derivation; that is,

$$\delta(X \wedge Y \otimes f) = Y \otimes \phi_X f - X \otimes \phi_Y f \ .$$

In this way, we may kill the cocycle we found before, namely

$$\sum_{i,j} \mathbf{X}_i \otimes h_{ij}\phi_j = \delta\left(-\tfrac{1}{2}\sum_{i,j} \mathbf{X}_i \wedge \mathbf{X}_j \otimes h_{ij}\right) \ .$$

The general idea is now clear. Consider the graded vector space $K^\bullet = \Lambda^\bullet\mathfrak{g} \otimes C^\infty(M)$ and define $\delta : K^q \to K^{q-1}$ by extending δ as a derivation. This defines a complex $(K^\bullet, \delta)$

$$\cdots \longrightarrow \Lambda^2\mathfrak{g} \otimes C^\infty(M) \xrightarrow{\ \delta\ } \mathfrak{g} \otimes C^\infty(M) \xrightarrow{\ \delta\ } C^\infty(M) \longrightarrow 0$$

called the **Koszul complex**.

Proposition 3.2. *The homology of the Koszul complex is given by*

$$H^p(K^\bullet) \cong \begin{cases} C^\infty(M_0) \ , & p = 0 \\ 0 \ , & \textit{otherwise.} \end{cases}$$

Proof. We sketch the proof. A sequence of functions $(\phi_1, \dots, \phi_k)$ is said to be **regular** if the following property holds for every j: if for some function f, $\phi_j f$ belongs to the ideal I_{j-1} generated by $(\phi_1, \dots, \phi_{j-1})$, then $f \in I_{j-1}$ already. Letting $I_0 = 0$, this implies that ϕ_1 is not identically zero. Now from the Lemma it follows that the sequence $(\phi_1, \dots, \phi_k)$, where $\phi_i = \Phi(\mathbf{X}_i)$, is regular. Finally it is a straight-forward result in homological algebra (see, for example, [Lan84, Ch.XIV,§10,Theorem 10.5]) that the Koszul complex of a regular sequence is acyclic in positive dimension. □

Augmenting the complex by the homology, we obtain an exact sequence

$$\cdots \longrightarrow K^2 \xrightarrow{\ \delta\ } K^1 \xrightarrow{\ \delta\ } C^\infty(M) \longrightarrow C^\infty(M_0) \longrightarrow 0 \ ,$$

which is a (projective) resolution of $C^\infty(M_0)$ in terms of (free) $C^\infty(M)$-modules, called the **Koszul resolution**.

3.2.2 The BRST complex

As was done in Section 1.4, we now construct a double complex

$$C^{p,q} = C^p(\mathfrak{g};K^q) = \Lambda^p\mathfrak{g}^* \otimes \Lambda^q\mathfrak{g} \otimes C^\infty(M)$$

and an associated total complex $(\mathscr{C}^\bullet, D)$ whose cohomology is given by Theorem 1.1. In particular, we see that in degree zero, the cohomology of the total complex is

$$H^0(\mathscr{C}^\bullet) \cong H^0(\mathfrak{g}; C^\infty(M_0)) \cong C^\infty(\tilde{M}) ,$$

which are the functions on the symplectic quotient. The total complex $(\mathscr{C}^\bullet, D)$ is called the **BRST complex** and D is called the **BRST differential**. The total degree is called **ghost number**.

One can actually prove [FO90] that the classical BRST cohomology is given by

$$H^n(\mathscr{C}^\bullet) \cong H^n(\mathfrak{g}) \otimes C^\infty(\tilde{M}) .$$

The isomorphism $H^0(\mathscr{C}^\bullet) \cong C^\infty(\tilde{M})$ in this theorem is one of vector spaces. However we know that $C^\infty(\tilde{M})$ is a Poisson algebra and it is therefore a natural question to ask whether we can strengthen this theorem by showing that the isomorphism is one of Poisson algebras. This requires defining a Poisson algebra structure on the BRST cohomology, which is the task we turn to now.

3.2.3 The classical BRST operator and the Poisson structure

We will now show that the total complex $\mathscr{C}^\bullet$ can be given the structure of a graded Poisson superalgebra in such a way that the total differential $D = \{Q, -\}$ is an inner derivation by an element $Q \in \mathscr{C}^1$ called the **classical BRST operator**. Since the differential acts by Poisson derivations, the cocycles are Poisson subsuperalgebras of which the coboundaries are Poisson ideals, thus making the cohomology into a Poisson superalgebra. In particular, the cohomology in dimension zero is a Poisson algebra.

The total complex is $\Lambda(\mathfrak{g} \oplus \mathfrak{g}^*) \otimes C^\infty(M)$. We will show that it admits the structure of a Poisson superalgebra, but first we will recall the relevant notions.

A **Poisson superalgebra** is a $\mathbb{Z}_2$-graded vector space $P = P_0 \oplus P_1$ together with two bilinear operations preserving the grading:

$$\begin{aligned} P \times P &\to P \\ (a,b) &\mapsto ab \end{aligned} \qquad \text{and} \qquad \begin{aligned} P \times P &\to P \\ (a,b) &\mapsto \{a,b\} , \end{aligned}$$

obeying the following properties:

- P is an associative supercommutative superalgebra under multiplication:

$$a(bc) = (ab)c \qquad \text{and} \qquad ab = (-1)^{|a||b|} ba ,$$

- P is a Lie superalgebra under Poisson bracket:

  $$\{a,b\} = (-1)^{|a||b|}\{b,a\} \qquad \text{and} \qquad \{a,\{b,c\}\} = \{\{a,b\}\,c\} + (-1)^{|a||b|}\{b,\{a,c\}\}\ ,$$

- the Poisson bracket is a derivation over multiplication:

  $$\{a,bc\} = \{a,b\}c + (-1)^{|a||b|}b\{a,c\}\ ,$$

  for all $a,b,c \in \mathrm{P}$ and where $|a|$ equals 0 or 1 according to whether a is even or odd, respectively.

The algebra $\mathrm{C}^\infty(\mathrm{M})$ is clearly an example of a Poisson superalgebra without odd part. On the other hand, the exterior algebra $\Lambda(\mathfrak{g}\oplus\mathfrak{g}^*)$ posseses a Poisson superalgebra structure. The associative multiplication is given by the wedge product and the Poisson bracket is defined for $\mathrm{X},\mathrm{Y} \in \mathfrak{g}$ and $\alpha,\beta \in \mathfrak{g}^*$ by

$$\{\alpha,\mathrm{X}\} = \alpha(\mathrm{X}) = \{\mathrm{X},\alpha\} \qquad \{\mathrm{X},\mathrm{Y}\} = 0 = \{\alpha,\beta\}\ .$$

We then extend it to all of $\Lambda(\mathfrak{g}\oplus\mathfrak{g}^*)$ as an odd derivation.

To show that the total complex $\mathscr{C}^\bullet$ is a Poisson superalgebra we need to discuss tensor products. Given two Poisson superalgebras P and Q, their tensor product $\mathrm{P}\otimes\mathrm{Q}$ can be given the structure of a Poisson superalgebra as follows. For $a,b \in \mathrm{P}$ and $u,v \in \mathrm{Q}$ we define

$$\begin{aligned}(a\otimes u)(b\otimes v) &= (-1)^{|u||b|}ab\otimes uv\\ \{a\otimes u, b\otimes v\} &= (-1)^{|u||b|}\left(\{a,b\}\otimes uv + ab\otimes\{u,v\}\right)\end{aligned}$$

One can easily show that these operations satisfy the axioms of a Poisson superalgebra.

Now let P be a Poisson superalgebra which, in addition, is $\mathbb{Z}$-graded, that is, $\mathrm{P} = \bigoplus_n \mathrm{P}^n$ and $\mathrm{P}^n\mathrm{P}^m \subseteq \mathrm{P}^{m+n}$ and $\{\mathrm{P}^n,\mathrm{P}^m\} \subseteq \mathrm{P}^{m+n}$; and such that the $\mathbb{Z}_2$-grading is the reduction modulo 2 of the $\mathbb{Z}$-grading, that is, $\mathrm{P}_0 = \bigoplus_n \mathrm{P}^{2n}$ and $\mathrm{P}_1 = \bigoplus_n \mathrm{P}^{2n+1}$. We call such an algebra a **graded Poisson superalgebra**. Notice that P^0 is a Poisson subalgebra of P.

For example, $\mathscr{C} = \Lambda(\mathfrak{g}\oplus\mathfrak{g}^*)\otimes\mathrm{C}^\infty(\mathrm{M})$, with the grading described above becomes a $\mathbb{Z}$-graded Poisson superalgebra. Although the bigrading is preserved by the exterior product, the Poisson bracket does not preserve it. In fact, the Poisson bracket obeys

$$\{\mathrm{C}^{i,j},\mathrm{C}^{k,l}\} \subseteq \mathrm{C}^{i+k,j+l}\oplus\mathrm{C}^{i+k-1,j+l-1}\ ,$$

but the total degree is preserved.

By a **Poisson derivation** of degree k we will mean a linear map $\mathrm{D}:\mathrm{P}^n \to \mathrm{P}^{n+k}$ such that

$$\begin{aligned}\mathrm{D}(ab) &= (\mathrm{D}a)b + (-1)^{k|a|}a(\mathrm{D}b)\\ \mathrm{D}\{a,b\} &= \{\mathrm{D}a,b\} + (-1)^{k|a|}\{a,\mathrm{D}b\}\ .\end{aligned}$$

The map $a \mapsto \{\mathrm{Q},a\}$ for some $\mathrm{Q}\in\mathrm{P}^k$ is an **inner Poisson derivation**.

Proposition 3.3. *The total differential* $D = \{Q, -\}$, *where* $Q \in \mathscr{C}^1$ *is given explicitly by the following expression*

$$Q = \boldsymbol{\alpha}^i \phi_i - \tfrac{1}{2} f_{jk}^i \boldsymbol{\alpha}^j \wedge \boldsymbol{\alpha}^k \wedge \mathbf{X}_i ,$$

where we have introduced a basis $(\mathbf{X}_i)$ *for* $\mathfrak{g}$, *relative to which* $[\mathbf{X}_i, \mathbf{X}_j] = f_{ij}^k \mathbf{X}_k$ *and a dual basis* $(\boldsymbol{\alpha}^i)$ *for* $\mathfrak{g}^*$ *and where we have used the summation convention.*

Proof. Being a derivation, it is enough to show that $\{Q, -\}$ acts as it should on the generators; that is, on functions $f \in C^\infty(M)$, and elements $Y \in \mathfrak{g}$ and $\beta \in \mathfrak{g}^*$. Clearly,

$$\{Q, f\} = \boldsymbol{\alpha}^i \{\phi_i, f\} \in \mathfrak{g}^* \otimes C^\infty(M)$$

agrees with the Chevalley–Eilenberg differential df. On $\beta \in \mathfrak{g}^*$,

$$\{Q, \beta\} = -\tfrac{1}{2} \beta_i f_{jk}^i \boldsymbol{\alpha}^j \wedge \boldsymbol{\alpha}^k \in \Lambda^2 \mathfrak{g}^*$$

which again agrees with $d\beta$. Finally on $Y \in \mathfrak{g}$ we have

$$\{Q, Y\} = Y^i \phi_i + f_{jk}^i Y^k \boldsymbol{\alpha}^j \wedge \mathbf{X}_i ,$$

where the first term agrees with $\delta Y = \phi_Y$ and the second term agrees with $dY \in \mathfrak{g}^* \otimes \mathfrak{g}$ defined by $dY(Z) = [Z, Y]$. □

One can show that the classical BRST operator Q satisfies $\{Q, Q\} = 0$, which is not immediate because the Poisson bracket on odd elements is symmetric.

Notation. It is customary in the Physics literature to denote the image of $\mathbf{X}_i$ and $\boldsymbol{\alpha}^i$ in $\Lambda(\mathfrak{g} \oplus \mathfrak{g}^*)$ by b_i and c^i, respectively, and to also drop the explicit mention of the wedge product. In this notation, which we shall adopt from now on, the classical BRST operator can be rewritten

$$Q = c^i \phi_i - \tfrac{1}{2} f_{jk}^i c^j c^k b_i .$$

The c^i and b_i are the classical **ghosts** and **antighosts**, respectively.

3.3 The general BRST complex

We briefly indicate the construction of the BRST complex in the more general case where M_0 is a coisotropic submanifold which is not necessarily the zero level set of an equivariant moment mapping. We will make the simplifying assumption that M_0 has trivial normal bundle, so that it is given as the zero set of a smooth function $\Phi : M \to V$, where V is a $(k = \operatorname{codim} M_0)$-dimensional vector space. Choose a basis $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k)$ for V and let $\Phi = \sum_{i=1}^k \phi_i \boldsymbol{e}_i$ for some smooth functions $\phi_i : M \to \mathbb{R}$. Since M_0 is a submanifold (equivalently, $\mathbf{0} \in V$ is a regular value of Φ), the ϕ_i generate the vanishing ideal of M_0:

$$\mathscr{I} = \left\{ \sum_{i=1}^{k} f_i \phi_i \middle| f_i \in C^\infty(M) \right\} .$$

Furthermore, since M_0 is coisotropic, the vanishing ideal is closed under the Poisson bracket: $\{\mathscr{I}, \mathscr{I}\} \subset \mathscr{I}$. At the level of the ϕ_i, this means that

$$\{\phi_i, \phi_j\} = \sum_{\ell=1}^{k} f_{ij}^{\ell} \phi_\ell ,$$

for some *functions* $f_{ij}^{\ell} \in C^\infty(M)$. In the case of the group action, these functions are constant and agree with the structure constants of the Lie algebra in the chosen basis.

We will now mimic the construction of the BRST complex in the group case. Let

$$C^{p,q} := \Lambda^p V^* \otimes \Lambda^q V \otimes C^\infty(M) ,$$

and let $\mathscr{C}^n := \bigoplus_{p-q=n} C^{p,q}$. Then $\mathscr{C} := \bigoplus_n \mathscr{C}^n = \Lambda(V \oplus V^*) \otimes C^\infty(M)$ is a graded Poisson superalgebra, where $\Lambda(V \oplus V^*)$ inherits the Poisson superalgebra structure from the dual pairing between V and V^*; that is, on generators,

$$\{\alpha, v\} = \alpha(v) = \{v, \alpha\} \qquad \{\alpha, \beta\} = 0 = \{v, w\} ,$$

for all $v, w \in V$ and $\alpha, \beta \in V^*$. Let $(\boldsymbol{e}_i)$ and $(\boldsymbol{\theta}^i)$ be canonical dual bases for V and V^*, respectively. Their images in the Poisson superalgebra $\Lambda(V \oplus V^*)$ will be denoted by b_i and c^i respectively. Furthermore we will not write the wedge product explicitly.

Theorem 3.4. *Let* $Q \in \mathscr{C}^1$ *satisfy* $\{Q, Q\} = 0$. *Let* $Q = Q_0 + Q_1 + \cdots$, *with* $Q_i \in C^{i+1,i}$, *and* $Q_0 = c^i \phi_i$. *Then* $D := \{Q, -\} : \mathscr{C}^\bullet \to \mathscr{C}^{\bullet+1}$ *is a differential and the cohomology of the graded complex* $(\mathscr{C}^\bullet, D)$ *in degree zero is given by*

$$H^0(\mathscr{C}^\bullet) \cong \frac{N(\mathscr{I})}{\mathscr{I}} ,$$

where $N(\mathscr{I})$ *is the normalizer of* $\mathscr{I}$ *in* $C^\infty(M)$ *and the isomorphism is one of Poisson algebras.*

In other words, the existence of such a Q allows us to construct a BRST complex computing in zero degree the Poisson algebra of functions on the coisotropic reduction of M by M_0.

It remains to show that such a Q exists. This can be proven by induction on the obvious filtration degree in $\mathscr{C}$, using the acyclicity of the Koszul differential. The first thing we notice is that the general BRST differential $D = \{Q, -\}$ will have terms of different bidegrees and hence it will not be the total differential of a double complex, as in the case of a group action. Nevertheless, D has ghost number 1, and moreover it will not decrease the p degree:

$$D : C^{p,q} \to C^{p,q-1} \oplus C^{p+1,q} \oplus C^{p+2,q+1} \oplus \cdots .$$

Moreover, the part of D which maps $C^{p,q} \to C^{p,q-1}$ only depends on the part $Q_0 = c^i \phi_i$ of Q in $C^{1,0}$, which can be seen to agree with the Koszul differential. What we have to show is that there are $Q_i \in C^{i+1,i}$ such that $Q = Q_0 + \sum_{i>0} Q_i$ satisfies $\{Q, Q\} = 0$.

This suggests filtering the complex C by the number of ghosts; that is, defining

$$F^n C := \bigoplus_{\substack{p \geq n \\ q}} C^{p,q} ,$$

so that

$$C = F^0 C \supset F^1 C \supset \cdots \supset F^k C \supset F^{k+1} C = 0 .$$

We will build Q inductively. Let $R_j = Q_0 + Q_1 + \cdots + Q_j$. The inductive hypothesis is that $\{R_j, R_j\} \in F^{j+2}C$. Notice that $R_0 = Q_0$ and that

$$\{Q_0, Q_0\} = c^i c^j \{\phi_i, \phi_j\} \in F^2 C ,$$

whence the zeroth step in the induction is satisfied. Before proving that the inductive hypothesis propagates, let us do the first case "by hand" to see what is involved. The Jacobi identity for the Poisson bracket,

$$\{Q_0, \{Q_0, Q_0\}\} = 0 ,$$

implies that $\{Q_0, Q_0\} = 0$ is a Koszul cocycle. In fact, it is also a coboundary:

$$\{Q_0, Q_0\} = c^i c^j \{\phi_i, \phi_j\} = c^i c^j f_{ij}^{\ell} \phi_\ell = \delta\left(c^i c^j b_\ell f_{ij}^{\ell}\right) .$$

This suggests defining $Q_1 = -\frac{1}{2} c^i c^j b_\ell f_{ij}^{\ell}$, so that, $R_1 = Q_0 + Q_1$ obeys $\{R_1, R_1\} \in F^3 C$, extending the hypothesis.

Lemma 3.5. *With the above notation and with the above induction hypothesis,* $\delta\{R_j, R_j\} = 0$.

Proof. By induction, $\{R_j, R_j\} \in F^{j+2}C$. We want to show that $\delta\{R_j, R_j\} = \{Q_0, \{R_j, R_j\}\}$ mod $F^{j+3}C = 0$. In other words, we want to show that $\{Q_0, \{R_j, R_j\}\} \in F^{j+3}C$. Indeed, by the Jacobi identity $\{R_j, \{R_j, R_j\}\} = 0$, it follows that

$$\{Q_0, \{R_j, R_j\}\} = -\{Q_1 + \cdots + Q_j, \{R_j, R_j\}\} ,$$

which is clearly in $F^{j+3}C$, since $\{R_j, R_j\} \in F^{j+2}C$ and $\{Q_{i>0}, -\}$ has positive filtration degree. □

By the acyclicity of the Koszul differential in nonzero degree, there exists Q_{j+1} such that $2\delta Q_{j+1} + \{R_j, R_j\} = 0$, whence $2\{R_j, Q_{j+1}\} + \{R_j, R_j\} \in F^{j+3}C$. Letting $R_{j+1} = R_j + Q_{j+1}$, we see that this implies that

$$\{R_{j+1}, R_{j+1}\} = \{R_j, R_j\} + 2\{R_j, Q_{j+1}\} + \{Q_{j+1}, Q_{j+1}\} \in F^{j+3}C\,.$$

This propagates the induction hypothesis and completes the proof of the existence of Q.

3.4 A quantum fantasy

The main uses of BRST cohomology are in the quantization of constrained systems. It is often that one is faced with a constrained phase space defined by a symplectic manifold M and a coisotropic submanifold M_0 and, hence, with the coisotropic reduction $M \to \widetilde{M}$. Furthermore it is often the case that one wishes to quantize the system whose physical phase space is defined by $\widetilde{M}$. This is often difficult in practice, e.g., due to the absence of natural coordinates on $\widetilde{M}$, or undesirable due to perhaps the loss of important properties such as 'manifest covariance' or 'locality'. If, however, we were able to quantize M, we may *define* the quantization of $\widetilde{M}$ by the seemingly circuitous route of exhibiting $C^\infty(\widetilde{M})$ as the classical BRST cohomology of $C^\infty(M)$, quantizing the classical BRST complex, which is no harder to quantize than $C^\infty(M)$ itself, and then computing quantum BRST cohomology. In other words, we have a diagram

$$\begin{array}{ccc} C^\infty(M) & \xrightarrow{\text{quantization}} & \mathscr{H} \\ {\scriptstyle\text{classical BRST}}\big\downarrow & & \big\downarrow{\scriptstyle\text{quantum BRST}} \\ C^\infty(M) & \xrightarrow{\text{quantization}} & \widetilde{\mathscr{H}}\,, \end{array}$$

and one then defines the bottom horizontal arrow, which need not 'exist' in practice, by going around the other way.

3.5 Problems

Problem 3.1. *A Poisson algebra as an algebra with only one operation*
Let P be a Poisson algebra; that is, P has a commutative associative multiplication $(a, b) \mapsto ab$ and a Lie bracket $(a, b) \mapsto \{a, b\}$ satisfying the condition $\{a, bc\} = \{a, b\}c + \{a, c\}b$. Define a new multiplication on P by

$$(a, b) \mapsto a \bullet b := \frac{1}{\sqrt{2}} (ab + \{a, b\})\ .$$

Show that the new operation satisfies the condition

$$(a \bullet c) \bullet b + (b \bullet c) \bullet a - (b \bullet a) \bullet c - (c \bullet a) \bullet b = 3A(a, b, c)\ , \tag{7}$$

where A is the associator: $A(a,b,c) = a \bullet (b \bullet c) - (a \bullet b) \bullet c$.
Conversely, if P is a vector space with a multiplication $(a,b) \mapsto a \bullet b$ obeying equation (7), show that

$$ab := \frac{1}{\sqrt{2}}(a \bullet b + b \bullet a) \qquad \text{and} \qquad \{a,b\} := \frac{1}{\sqrt{2}}(a \bullet b - b \bullet a)$$

turn P into a Poisson algebra.
For extra credit, formulate and prove a 'super' version of these results.

Problem 3.2. *Poisson algebras have a tensor product*
Show that the tensor product of two Poisson superalgebras is naturally a Poisson superalgebra.

Problem 3.3. *Inner Poisson derivations*
Let $P = \bigoplus_n P^n$ be a graded Poisson superalgebra and let $\nu : P \to P$ denote the **degree derivation** such that $\nu(a) = pa$ if $a \in P^p$. Show that if the degree derivation is inner, then so is any other Poisson derivation of nonzero degree.

Problem 3.4. *The cohomology of a Poisson derivation*
Let P be a Poisson superalgebra and $Q \in P$ an odd element satisfying $\{Q,Q\} = 0$. Show that $D := \{Q, -\}$ is a Poisson derivation and that $D^2 = 0$. Then show that the kernel of D is a Poisson sub-superalgebra containing the image of D as a Poisson ideal. Conclude that the cohomology $\ker D / \operatorname{Im} D$ is a Poisson superalgebra.

Problem 3.5. *The square-zero property of the BRST operator*
Show that the classical BRST operator Q in Proposition 3.3 satisfies $\{Q,Q\} = 0$.

Problem 3.6.
Show that a submanifold $M_0 \subset M$ is given by the zero locus of a smooth function $\Phi : M \to \mathbb{R}^k$, where $k = \operatorname{codim} M_0$, if and only if its normal bundle is trivial.

Problem 3.7. *The general BRST cohomology in zero degree*
Prove Theorem 3.4.
(*Hint*: Use 'tic-tac-toe' as in the proof of Theorem 1.1, exploiting the acyclicity of the Koszul complex in positive degree. Where is the Koszul differential in D?)

Problem 3.8. *BRST cohomology for general coisotropic* M_0
Let us try to extend the construction of the general BRST complex to the case when M_0 has nontrivial normal bundle. Cover M by open sets $\{U_\alpha\}$ such that either $M_0 \cap U_\alpha = \varnothing$ or else the normal bundle of M_0 is trivial on $U_\alpha \cap M_0$. From now on we will consider only those α for which $U_\alpha \cap M_0 \neq \varnothing$. On each such U_α, the ideal $I_\alpha \subset C^\infty(U_\alpha)$ of functions vanishing on $U_\alpha \cap M_0$ is generated by

k functions ϕ_i^α. By the results in the lecture there is on U_α a local BRST operator $Q_\alpha \in \Lambda(V \oplus V^*) \otimes C^\infty(U_\alpha)$ obeying $\{Q_\alpha, Q_\alpha\} = 0$ and the BRST cohomology in zero degree is isomorphic as a Poisson algebra to $N(I_\alpha)/I_\alpha$, where $N(I_\alpha)$ is the normalizer of I_α in $C^\infty(I_\alpha)$. Now consider two overlapping open sets U_α and U_β with $U_\alpha \cap U_\beta \cap M_0 \neq \varnothing$. Show that whereas the complexes need not agree in the overlap $U_\alpha \cap U_\beta$, the BRST cohomologies are isomorphic (at least in zero degree, although it can be shown that they agree in general). Conclude that to each U_α intersecting M_0, we can assign a Poisson algebra $P_\alpha := N(I_\alpha)/I_\alpha$ and isomorphisms $\psi_{\alpha\beta} := P_\alpha|_{U_\alpha \cap U_\beta} \to P_\beta|_{U_\alpha \cap U_\beta}$. Show that this defines a sheaf of Poisson algebras, whose space of global sections is precisely $N(\mathscr{I})/\mathscr{I}$.

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