BRST Comology 2006

Tutorial Sheet 1

Lie algebra cohomology

Let us introduce some notation. Let $\mathfrak{g}$ denote a real Lie algebra and let (e_i) denote a basis for $\mathfrak{g}$. The canonical dual basis for $\mathfrak{g}^*$ will be denoted (α^i). The Lie brackets in this basis are given in terms of the structure constants

$$[e_i, e_j] = f_{ij}^k e_k \,,$$

where here and also below we use the Einstein summation convention. The **Killing form** on $\mathfrak{g}$, which is defined by

$$\kappa(X, Y) = \operatorname{tr} \operatorname{ad}_X \operatorname{ad}_Y \,,$$

where for all $X \in \mathfrak{g}$, $\operatorname{ad}_X \in \operatorname{End} \mathfrak{g}$ is defined by $\operatorname{ad}_X Y = [X, Y]$, takes the following explicit expression in terms of the above basis:

$$\kappa(e_i, e_j) = f_{ik}^\ell f_{j\ell}^k \,.$$

You are allowed to use the fact that a Lie algebra is semisimple (defined as one having no abelian ideals) if and only if the Killing form is nondegenerate.

Problem 1.1. Let (E, d) be a finite-dimensional differential complex, where E has a euclidean inner product. Let d^* denote the adjoint of d. Prove that in each cohomology class there is a unique cocycle which is annihilated by d^* and which can be characterized by the fact that it is the cocycle with the smallest norm in its cohomology class. Prove that the cohomology is isomorphic as a vector space to the kernel of the "laplacian" $\triangle = dd^* + d^*d$; hence every cohomology class has a unique "harmonic" representative. The same is true for the de Rham cohomology of a compact orientable manifold, but the proof is more subtle due to the infinite dimensionality of the spaces of differential forms.

Problem 1.2. Let (C, d) be a differential complex and let $\langle -, - \rangle$ be a nondegenerate bilinear form on C relative to which d is (skew)symmetric: $\langle dc, c' \rangle = \pm \langle c, dc' \rangle$ for all $c, c' \in C$. Prove that the cohomology inherits a nondegenerate bilinear form from the restriction of the one on C to the cocycles.
Now assume that $(C = \bigoplus_n C^n, d)$ is a graded complex, and that the bilinear form $\langle -, - \rangle$ pairs up C^n with C^{-n}. Then show that $H^n(C) \cong H^{-n}(C)$ as vector spaces.

Problem 1.3. Let V be a real vector space, V^* its dual, and $\Lambda V^* = \bigoplus_p \Lambda^p V^*$ its exterior algebra. We can think of $\Lambda^p V^*$ as the space of antisymmetric linear p-forms on V. Let $d : V^* \to \Lambda^2 V^*$ be any linear map and extend it to a linear map $d : \Lambda^p V^* \to \Lambda^{p+1} V^*$ as a derivation; that is,

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-)^p \alpha \wedge d\beta$$

for $\alpha \in \Lambda^p V^*$. Prove the following:

a. If $d^2\alpha = 0$ for all $\alpha \in V^*$ then $d^2 = 0$ identically on ΛV^*.

b. Let $d^t : \Lambda^2 V \to V$ be the transpose of $d : V^* \to \Lambda^2 V^*$. Then (V, d^t) is a Lie algebra with Lie bracket d^t if and only if $d^2 = 0$.

Problem 1.4. Let b_i and c^i be the operators introduced in the lecture. Recall that $c^i : \Lambda^p \mathfrak{g}^* \to \Lambda^{p+1}\mathfrak{g}^*$ is defined by $c^i \omega = \alpha^i \wedge \omega$; and that $b_i : \Lambda^p \mathfrak{g}^* \to \Lambda^{p-1}\mathfrak{g}^*$ is the derivation defined by $b_i \alpha^j = \delta_i^j$. Prove the following identities:

a. $b_i c^j + c^j b_i = \delta_i^j$,

b. $b_i b_j + b_j b_i = 0$, and

c. $c^i c^j + c^j c^i = 0$.

Let $\mathfrak{M}$ be a $\mathfrak{g}$-module with representation $\rho : \mathfrak{g} \to \operatorname{End}\mathfrak{M}$. Then show that the differential d computing $H(\mathfrak{g};\mathfrak{M})$ is given by

$$d = c^i \rho(e_i) - \tfrac{1}{2} f_{ij}^k c^i c^j b_k \,.$$

Show by explicit computation that $d^2 = 0$.

Problem 1.5. A **perfect** Lie algebra is one in which every element can be written as a linear combination of Lie brackets; that is, $\mathfrak{g}$ is perfect when $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$. Prove that a Lie algebra is perfect if and only if $H^1(\mathfrak{g}) = 0$. Prove that semisimple Lie algebras are perfect. In fact, more generally, if $\mathfrak{g}$ has no center and has an invariant nondegenerate bilinear form, then it is perfect.

Problem 1.6. By a (real) **central extension** of a Lie algebra $\mathfrak{g}$ we mean a Lie algebra structure on the vector space $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$, which has the following form. Let (e_i, k) be a basis for $\tilde{\mathfrak{g}}$. Then k is central in $\tilde{\mathfrak{g}}$ (that is, it commutes with everything) and the bracket $[e_i, e_j]$ develops an extra term:

$$[e_i, e_j] = f_{ij}^k e_k + c_{ij} k \,,$$

where f_{ij}^k are the structure constants of $\mathfrak{g}$. Let $c = \tfrac{1}{2} c_{ij} \alpha^i \wedge \alpha^j \in \Lambda^2 \mathfrak{g}^*$. Prove that c is a 2-cocycle.

A central extension $\tilde{\mathfrak{g}}$ is called **trivial** if it is isomorphic as a Lie algebra to $\mathfrak{g} \times \mathbb{R}$. Show that the central extension defined by a 2-cocycle is trivial if and only if the cocycle is also a coboundary. Hence $H^2(\mathfrak{g})$ is in one-to-one correspondence with nontrivial central extensions of $\mathfrak{g}$. Prove that a semisimple Lie algebra has no nontrivial central extensions. In other words, $H^2(\mathfrak{g}) = 0$ for $\mathfrak{g}$ semisimple.

Problem 1.7. Let $\delta : \mathfrak{g} \to \mathfrak{g}$ be a linear map. It is called a **derivation** if $\delta[X,Y] = [\delta X, Y] + [X, \delta Y]$. A derivation is called **inner**, if for all $X \in \mathfrak{g}$, $\delta X = [Z, X]$ for some $Z \in \mathfrak{g}$. Prove that δ is a derivation if and only if $\alpha^i \otimes \delta(e_i) \in \mathfrak{g}^* \otimes \mathfrak{g}$ is a 1-cocycle; and that it is an inner derivation when it is also a coboundary. The quotient $H^1(\mathfrak{g};\mathfrak{g})$ of all derivations by the inner derivations is the space of **outer** derivations. Prove that in a semisimple Lie algebra, all derivations are inner. Notice that derivations form a Lie algebra in which the inner derivations constitute an ideal. Therefore $H^1(\mathfrak{g};\mathfrak{g})$ becomes a Lie algebra. More generally, one can show that $H(\mathfrak{g};\mathfrak{g})$ is a Lie superalgebra (with the degree offset by one from the natural one).

Let $\mathfrak{g}$ possess an invariant inner product. We call such $\mathfrak{g}$ **self-dual**. Prove that if all derivations of $\mathfrak{g}$ are inner, then $\mathfrak{g}$ doesn't admit any nontrivial central extensions. Conversely, prove that if $\mathfrak{g}$ doesn't admit any nontrivial central extensions, then all derivations which preserve the inner product (i.e., the antisymmetric derivations) are inner.

Problem 1.8. Given a vector space V, how many different Lie brackets can we define on it? A Lie bracket is a map $\Lambda^2\mathfrak{g} \to \mathfrak{g}$ subject to the Jacobi identity. Therefore Lie algebras on V are in one-to-one correspondence with the intersection of certain quadrics (the Jacobi identity) on $\Lambda^2 V^* \otimes V$. Let $J(V) \subset \Lambda^2 V^* \otimes V$ denote the space of solutions of the Jacobi identity.

Clearly not all points on J(V) correspond to different Lie algebras—Lie brackets related by a change of basis in V yield the same Lie algebra. Therefore we define the moduli space L(V) of Lie algebras on V as the quotient of J(V) by the action of GL(V). L(V) may be a complicated object, but it is easy to probe its local structure by looking in the neighbourhood of a point. In other words, given a Lie algebra $\mathfrak{g}$ with underlying vector space V, one can study the infinitesimal deformations of the Lie bracket on $\mathfrak{g}$. Prove that the tangent space to J(V) at $\mathfrak{g}$ is given by the cocycles $Z^2(\mathfrak{g};\mathfrak{g})$. Prove that those cocycles which are also coboundaries are tangent to the GL(V) orbit through $\mathfrak{g}$. Conclude that the tangent space to L(V) at $\mathfrak{g}$ is precisely $H^2(\mathfrak{g};\mathfrak{g})$. Prove that a semisimple Lie algebra is rigid; that is, it admits no nontrivial infinitesimal deformations.

It's not hard to show (Nijenhuis–Richardson) that there are an infinite set of obstructions to integrating (at least formally) a given infinitesimal deformation. Each obstruction is a class in $H^3(\mathfrak{g};\mathfrak{g})$.

Problem 1.9. Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{M}$ denote a finite-dimensional $\mathfrak{g}$-module. Prove the following:

a. If $H^1(\mathfrak{g};\mathfrak{M}) = 0$ for all $\mathfrak{M}$, then every finite-dimensional $\mathfrak{g}$-module is fully reducible.

b. If every $\mathfrak{g}$-module is fully reducible, then $\mathfrak{g}$ is semisimple.

c. Conclude that $\mathfrak{g}$ is semisimple if and only if $H^1(\mathfrak{g};\mathfrak{M}) = 0$ for all $\mathfrak{M}$.

Problem 1.10. Let (C, d) and (C', d') be two differential complexes. Let $\varphi : C \to C'$ be a linear map which commutes with the action of the differentials: $\varphi \circ d = d' \circ \varphi$. Such a φ is called a **chain map**. Prove that φ induces a map in cohomology $\varphi^* : H(C) \to H(C')$. (Hint: Prove that φ sends cocycles to cocycles and coboundaries to coboundaries and argue from there.)

Problem 1.11. This is boring to do in class—but it ought to be done. Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras and let $\varphi : \mathfrak{h} \to \mathfrak{g}$ be a homomorphism. Then let $\varphi^* : \Lambda\mathfrak{g}^* \to \Lambda\mathfrak{h}^*$ denote the natural map induced by φ. Also notice that if $\mathfrak{M}$ is a $\mathfrak{g}$-module, then it becomes an $\mathfrak{h}$-module via φ. Putting this together we find a map also denoted $\varphi^* : \Lambda\mathfrak{g}^* \otimes \mathfrak{M} \to \Lambda\mathfrak{h}^* \otimes \mathfrak{M}$. Prove that this map commutes with d. Therefore it induces a map in cohomology $\varphi^* : H(\mathfrak{g};\mathfrak{M}) \to H(\mathfrak{h};\mathfrak{M})$.

Now let $\mathfrak{M}$ and $\mathfrak{N}$ be $\mathfrak{g}$-modules. Prove that any linear map $f : \mathfrak{M} \to \mathfrak{N}$ commuting with the action of $\mathfrak{g}$ induces a map $f_* : H(\mathfrak{g};\mathfrak{M}) \to H(\mathfrak{g};\mathfrak{N})$.

Finally prove the following isomorphism:

$$H(\mathfrak{g};\mathfrak{M} \oplus \mathfrak{N}) \cong H(\mathfrak{g};\mathfrak{M}) \oplus H(\mathfrak{g};\mathfrak{N}) .$$

(*Hint*: Abuse Problem 1.10.) If you only do one part of this problem, do the last one!

Problem 1.12. Let $(A^\bullet, d_A)$, $(B^\bullet, d_B)$ and $(C^\bullet, d_C)$ be graded complexes. Exact sequences

$$0 \longrightarrow A^p \xrightarrow{\lambda_p} B^p \xrightarrow{\mu_p} C^p \longrightarrow 0\,,$$

for every p, where λ_p and μ_p are chain maps is called a **(short) exact sequence of graded complexes**. Show that such a sequence induces a long exact sequence in cohomology:

$$\cdots \longrightarrow H^p(A) \longrightarrow H^p(B) \longrightarrow H^p(C) \longrightarrow H^{p+1}(C) \longrightarrow \cdots$$

Make sure you understand the map $H^p(C) \to H^{p+1}(A)$ and the fact that it is induced by the differential.

Problem 1.13. Let $\mathfrak{M}$ and $\mathfrak{N}$ be $\mathfrak{g}$-modules and let $\varphi : \mathfrak{M} \to \mathfrak{N}$ be a $\mathfrak{g}$-**map**; that is, a linear map commuting with the action of $\mathfrak{g}$. Show that φ induces a chain map $C^\bullet(\mathfrak{g};\mathfrak{M}) \to C^\bullet(\mathfrak{g};\mathfrak{N})$ and hence maps $\varphi_* : H^p(\mathfrak{g};\mathfrak{M}) \to H^p(\mathfrak{g};\mathfrak{N})$ for all p.
Now let

$$0 \longrightarrow \mathfrak{M} \xrightarrow{\lambda} \mathfrak{N} \xrightarrow{\mu} \mathfrak{P} \longrightarrow 0$$

be a short exact sequence of $\mathfrak{g}$-modules. Show that this induces an exact sequence of the corresponding Chevalley–Eilenberg complexes:

$$0 \longrightarrow C^\bullet(\mathfrak{g};\mathfrak{M}) \xrightarrow{\lambda_\bullet} C^\bullet(\mathfrak{g};\mathfrak{N}) \xrightarrow{\mu_\bullet} C^\bullet(\mathfrak{g};\mathfrak{P}) \longrightarrow 0\,,$$

and hence a long exact sequence in cohomology:

$$\cdots \longrightarrow H^p(\mathfrak{g};\mathfrak{M}) \longrightarrow H^p(\mathfrak{g};\mathfrak{N}) \longrightarrow H^p(\mathfrak{g};\mathfrak{P}) \longrightarrow H^{p+1}(\mathfrak{g};\mathfrak{M}) \longrightarrow \cdots$$