BRST Comology 2006

Tutorial Sheet 2

Symplectic reduction

Throughout this tutorial sheet, (M,ω) is a finite-dimensional symplectic manifold and $\mathfrak{g}$ is the Lie algebra of a Lie group G acting on M via symplectomorphisms.

Problem 2.1. Let (V,Ω) be a finite-dimensional symplectic vector space and let $W \subset V$ be a subspace. Show that $\dim V = \dim W + \dim W^{\perp}$, where $W^{\perp}$ is the symplectic perpendicular. Show further that the quotient $W/W\cap W^{\perp}$ inherits a unique symplectic structure $\widetilde{\Omega}$ such that

$$\pi^*\widetilde{\Omega} = i^*\Omega\,,$$

where $i : W \to V$ is the inclusion and $\pi : W \to W/W\cap W^{\perp}$ is the natural projection.

Problem 2.2. Prove that the Poisson bracket on $C^\infty(M)$ satisfies the Jacobi identity. (*Hint*: use that $d\omega = 0$.)

Problem 2.3. Show that the Lie bracket of two symplectic vector fields is hamiltonian. Hence show that if $H^1(\mathfrak{g}) = 0$, then a symplectic action of $\mathfrak{g}$ on (M,ω) is hamiltonian.
(*Hint*: If η,ξ are symplectic vector fields, show that $\imath_{[\eta,\xi]}\omega + d\omega(\eta,\xi) = 0$.)

Problem 2.4. Assume that the action of G on M is hamiltonian; whence there is a map $\mathfrak{g} \to C^\infty(M)$ taking $X \mapsto \phi_X$ where $\imath(\xi_X)\omega + d\phi_X = 0$. For every $X,Y \in \mathfrak{g}$, define the function

$$c(X,Y) = \phi_{[X,Y]} - \left\{\phi_X,\phi_Y\right\}\,.$$

Show that $dc(X,Y) = 0$ so that it is locally constant. This defines a map $c : \Lambda^2\mathfrak{g} \to H^0_{dR}(M)$. Show that c is a Lie algebra cocycle, where we interpret H^0_{dR} as a trivial $\mathfrak{g}$-module. Deduce that if and only if its cohomology class $[c] \in H^2\left(\mathfrak{g};H^0_{dR}(M)\right)$ is trivial, can one find functions $\widetilde{\phi}_X$ satisfying $\imath_{\xi_X}\omega + d\widetilde{\phi}(X) = 0$ and such that the map $\mathfrak{g} \to C^\infty(M)$ given by $X \mapsto \widetilde{\phi}_X$ is a Lie algebra homomorphism.

Problem 2.5. Show that if $\omega = d\theta$, where θ is G-invariant, then the action of G is Poisson.

Problem 2.6. Let the G-action on M be Poisson. Show that the components of the moment map are conserved quantities for any G-invariant hamiltonian.

Problem 2.7. Let $\Phi : M \to \mathfrak{g}^*$ be the moment mapping for the Poisson action of G on M. Let $p \in M$ be a given point. Then the differential of the moment mapping at p defines a linear map

$$d\Phi_p : T_pM \to \mathfrak{g}^*\,.$$

Let $G_p < G$ denote the stabilizer of p in G. Show that it is a closed subgroup of G. Let $\mathfrak{g}_p$ denote its Lie algebra. Show that $\operatorname{Im} d\Phi_p = \mathfrak{g}_p^0$, where

$$\mathfrak{g}_p^0 = \left\{\alpha \in \mathfrak{g}^* \,\middle|\, \alpha(X) = 0,\ \forall X \in \mathfrak{g}_p\right\}$$

is the annihilator of $\mathfrak{g}_p$ in $\mathfrak{g}^*$. Conclude that if $0 \in \mathfrak{g}^*$ is a regular value of the moment map, the group G acts with discrete stabilizers on $M_0 = \Phi^{-1}(0)$. Such actions are said to be **locally free** and the quotient M_0/G will generally be an orbifold.

Problem 2.8. Let N be a smooth manifold and let T^*N denote its cotangent bundle. We let $\pi : T^*N \to N$ denote the projection. Show that there is a one-form $\theta \in \Omega^1(T^*N)$ defined by either one of the following equivalent conditions:

a. $\gamma^*\theta = \gamma$, where $\gamma \in \Omega^1(N)$ thought of as a smooth map $N \to T^*N$ on the LHS;

b. $\theta_\alpha = \alpha \circ \pi_*$, where $\alpha \in T^*N$; or

c. $\theta = p_i dq^i$ relative to local coordinates (q^i, p_i) for T^*N.

(The problem consists in showing that the definitions are equivalent and that they do define θ uniquely.) The one-form θ is called the **tautological one-form** on T^*N. Show that $\omega = -d\theta$ is a symplectic form. Let G be a group acting on N via diffeomorphisms. Show that the natural action of G on T^*N, under which π is equivariant, preserves the tautological one-form. Use Problem 2.5 to deduce that the G-action on T^*N is Poisson and write an expression for the moment mapping. Assuming that the action of G on N is free and proper so that N/G is a manifold, show that $T^*N//G$ is symplectomorphic to $T^*(N/G)$.
(*Hint:* For the moment mapping, show that at the point $(p,\alpha) \in T^*N$, the component in the direction $X \in \mathfrak{g}$ is given by $\phi_X(p,\alpha) = \alpha(\eta_X(p))$, where the $\eta_x \in \mathscr{X}(N)$ are the vector fields generating the G-action on N.)

Problem 2.9. Generalise the symplectic reduction in the second lecture to the case of nonzero momentum. In other words, let $\alpha \in \mathfrak{g}^*$ be a regular value of the moment map and let $M_\alpha = \Phi^{-1}(\alpha)$ be the submanifold of M consisting of points with momentum α. Then let

$$G_\alpha = \left\{ g \in G \middle| \mathrm{Ad}_g^* \alpha = \alpha \right\}$$

denote the stabilizer of α. Show that G_α acts on M_α with discrete stabilizers. Show that if the quotient M_α/G_α is a manifold it has a unique symplectic structure $\widetilde{\omega}$ such that $\pi^*\widetilde{\omega} = i^*\omega$, where $i : M_\alpha \to M$ and $\pi : M_\alpha \to M_\alpha/G_\alpha$ are the natural maps.

Problem 2.10. Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{g}^*$ its dual. The group G acts on $\mathfrak{g}$ via the adjoint representation and on $\mathfrak{g}^*$ via the coadjoint representation. Explicitly, if we identify $\mathfrak{g}$ with T_1G and $\mathfrak{g}^*$ with T_1^*G, then the adjoint representation is

$$\mathrm{Ad}_g = (L_g)_* \circ (R_{g^{-1}})_* : T_1G \to T_1G$$

and its dual is the coadjoint representation. If $\alpha \in \mathfrak{g}*$, then let $\mathscr{O}_\alpha$ denote the coadjoint orbit of α. In this problem we will show that $\mathscr{O}_\alpha$ is naturally a symplectic manifold. In particular, this will show that $\mathscr{O}_\alpha$ is even-dimensional.

a. Since $\mathfrak{g}^*$ is a vector space, we can identify the tangent spaces at each point with $\mathfrak{g}^*$ itself. We define a bivector B on $\mathfrak{g}^*$ as a map $\mathfrak{g}^* \to \Lambda^2\mathfrak{g}^*$ taking $\alpha \mapsto B_\alpha$, where $B_\alpha(X,Y) = \alpha([X,Y])$. Let $G_\alpha < G$ denote the stabilizer of α under the coadjoint representation and let $\mathfrak{g}_\alpha$ denote its Lie algebra. Show that the radical of B_α is precisely $\mathfrak{g}_\alpha$, and hence show that B_α induces a nondegenerate skew-symmetric bilinear form on $\mathfrak{g}/\mathfrak{g}_\alpha$.

b. Show that there is an exact sequence

$$0 \longrightarrow \mathfrak{g}_\alpha \longrightarrow \mathfrak{g} \xrightarrow{\sigma_\alpha} T_\alpha \mathscr{O}_\alpha \longrightarrow 0 ,$$

where the map $\sigma_\alpha : \mathfrak{g} \to T_\alpha \mathscr{O}_\alpha$ is given by $\sigma_\alpha(X) = \xi_X(\alpha)$, where ξ_X are the vector fields which generate the coadjoint action on $\mathfrak{g}^*$. Thus σ_α induces an isomorphism $T_\alpha \mathscr{O}_\alpha \cong \mathfrak{g}/\mathfrak{g}_\alpha$ via which B_α defines a nondegenerate 2-form ω on $\mathscr{O}_\alpha$:

$$\omega(\xi_X(\alpha), \xi_Y(\alpha)) = B_\alpha(X, Y) = \alpha([X, Y]) .$$

Check explicitly that ω is nondegenerate.

c. Every $X \in \mathfrak{g}$ defines a linear function on $\mathfrak{g}^*$ and, by restriction, on any coadjoint orbit. We will let $\phi_X \in C^\infty(\mathscr{O}_\alpha)$ denote this function; that is, $\phi_X(\alpha) = \alpha(X)$. Show that $\xi_X \phi_Y = \phi_{[X,Y]}$ and that

$$\imath_{\xi_X} \omega = -d\phi_X . \tag{1}$$

Use this to show that ω is G-invariant; that is, $\mathscr{L}_{\xi_X} \omega = 0$ and hence conclude that ω is closed.
(*Hint*: For the first statement, compute $\mathscr{L}_{\xi_X} d\phi_Y$.)

d. Notice that equation (1) shows that the action of G on $\mathscr{O}_\alpha$ is hamiltonian. Show that this action is actually Poisson and prove that the moment map is simply the inclusion $\mathscr{O}_\alpha \to \mathfrak{g}^*$.

The above procedure is called the **Kirillov–Kostant–Souriau** construction.

Problem 2.11. In this problem you will show that the symplectic structure on a coadjoint orbit constructed in Problem 2.10 arises from a symplectic quotient of T^*G, where the G-action is induced by left multiplication on G. Since left multiplication is a diffeomorphism, the canonical one-form on T^*G is invariant and hence the G-action is Poisson. The point of this problem is to work out the moment map explicitly and show that the symplectic quotients are the coadjoint orbits.

a. Let G act on itself via left multiplication. Show that the vector fields generating this action are the right-invariant vector fields on G.

b. From Problem 2.8 we know that this action preserves the canonical symplectic structure on T^*G and moreover that the action is Poisson with an equivariant moment map $\Phi : T^*G \to \mathfrak{g}^*$. Show that $\Phi(g, \mu) = R_g^* \mu$, where $\mu \in T_g^* G$; that is, Φ is the map which trivialises the cotangent bundle via right multiplication.

c. Let $M_\alpha = \Phi^{-1}(\alpha)$ denote the level set of momentum $\alpha \in \mathfrak{g}^*$. Show that M_α is the graph of the right-invariant 1-form with value α at the identity and hence diffeomorphic to the group G itself. Conclude that $M_\alpha \subset T^*N$ is a submanifold.

d. Let $G_\alpha < G$ denote the stabilizer of α under the coadjoint representation. Then G_α acts on M_α. Show that the quotient M_α / G_α is symplectomorphic to the coadjoint orbit $\mathscr{O}_\alpha$.

Problem 2.12. Let $\phi_a \in C^\infty(M)$, for $a = 1, \ldots, k$, be smooth functions on M which we will think of as **constraints**. We will assume that $0 \in \mathbb{R}^k$ is a regular value of the map $\Phi : M \to \mathbb{R}^k$ whose components are the ϕ_a. Let $\mathscr{I}$ denote the ideal in $C^\infty(M)$ generated by the $\{\phi_a\}$; that is, $\mathscr{I}$ consists of linear combinations

$$f_1\phi_1 + \cdots + f_k\phi_k\ ,$$

where $f_a \in C^\infty(M)$. Let Ψ denote the vector space of linear combinations

$$c_1\phi_1 + \cdots + c_k\phi_k\ ,$$

where $c_a \in \mathbb{R}$. Then let $F \subset \Psi$ be a maximal subspace with the property that $\{F, \Psi\} \subset \mathscr{I}$ and let (ψ_i) denote a basis for F and complete it to a basis for Ψ by adding $\{\chi_\alpha\}$. Following Dirac, let us call the $\{\psi_i\}$ **first-class constraints** and the $\{\chi_\alpha\}$ **second-class constraints**. Show that the matrix of Poisson brackets $P_{\alpha\beta} := \{\chi_\alpha, \chi_\beta\}$ is nondegenerate on the zero locus S of the second-class constraints and hence show that S is a symplectic submanifold. Write down an explicit expression for the Poisson bracket on S in terms of the Poisson bracket on M and the matrix $P_{\alpha\beta}$. This is called the **Dirac bracket**. Finally show that the zero locus of the first-class constraints $\{\psi_i\}$ define a coistropic submanifold of S. In this way we have reduced the general situation to the one of coisotropic reduction. This, in a nutshell, is Dirac's theory of constraints.