BRST Comology 2006-2007

Tutorial Sheet 4

Operator product technology

The principal aim of this tutorial sheet is to attain fluency in manipulating operator product expansions in a way that makes it look a lot like Lie algebras. At the same time, I hope it will serve as revision of the axiomatics introduced in the lectures of Nils Scheithauer.

We will let V be the vector space underlying a conformal field theory. We will take V to be a $\mathbb{Z}_2$-graded vector space and if $A \in V$ is a homogeneous element, we denote its parity by $|A|$.

Given $A, B \in V$ we let $A(z)$ and $B(z)$ be the fields which create the vectors A and B acting on the vacuum. We define a set of bilinear brackets $[-,-]_n : V \otimes V \to V$ by

$$A(z)B(w) = \sum_{n \ll \infty} \frac{[A,B]_n(w)}{(z-w)^n} . \tag{1}$$

We assume that V has a Virasoro element T, whose field $T(z)$ obeys a Virasoro algebra with a fixed central charge c. We will assume that we can choose a basis for V composed of vectors $\phi \in V$ which obey $[T,\phi]_2 = h\phi$ and $[T,\phi]_1 = \partial\phi$, where the derivation $\partial : V \to V$ is defined by $(\partial\phi)(z) = \frac{d}{dz}\phi(z)$. A vector ϕ is said to have conformal weight h if $[T,\phi]_2 = h\phi$. If this is the case, we define linear operators $\phi_n : V \to V$ by the expansion $\phi(z) = \sum_n \phi_n z^{-n-h}$.

Notice that the moding convention differs from the one in Nils's lectures, in the appearance of h in the mode expansion. Changing conventions is simply a matter of shifting the moding of the field.

Problem 4.1. *Properties of the derivation* ∂

Prove the following properties of ∂:

a. $[\partial A, B]_n = (1-n)[A,B]_{n-1}$; hence $[\partial A, B]_1 = 0$

b. $[A, \partial B]_n = (n-1)[A,B]_{n-1} + \partial[A,B]_n$

c. $\partial[A,B]_n = [\partial A, B]_n + [A, \partial B]_n$

d. if A has conformal weight h_A, then ∂A has conformal weight $h_A + 1$.

Problem 4.2. *The identity*

Let $\mathbf{1}(z)$ denote the field with expansion $\mathbf{1}(z) = \mathrm{id}_V$, where $\mathrm{id}_V : V \to V$ is the identity map. Prove the following:

a. $\partial\mathbf{1} = 0$,

b. $[\mathbf{1}, A]_{n\neq 0} = 0$ and $[\mathbf{1}, A]_0 = A$, and

c. $\mathbf{1}$ has zero conformal weight.

Problem 4.3. *"Skew-symmetry"*

Prove that the brackets $[-,-]_n$ satisfy the following "skew-symmetry" condition:

$$[A,B]_n - (-)^{|A||B|+n}[B,A]_n = \sum_{\ell \geq 1} \frac{(-)^{1+\ell}}{\ell!} \partial^\ell [A,B]_{n+\ell} . \tag{2}$$

Problem 4.4. *"Jacobi identity"*
It follows from Problem 4.1 that the brackets $[-,-]_n$ for $n \geq 0$ determine the others.

a. Prove that these brackets satisfy the following Jacobi-like identity:

$$[A,[B,C]_n]_{m>0} = (-)^{|A||B|}[B,[A,C]_m]_n + \sum_{\ell=0}^{m-1} \binom{m-1}{\ell} [[A,B]_{m-\ell}, C]_{n+\ell} \,. \qquad (3)$$

b. Deduce that the operation $[A,-]_1$ is a derivation over all the $[-,-]_n$.

Problem 4.5. *Field and vectors*
Recall that there exists a vacuum $\Omega \in V$ such that $\lim_{z\to 0} \phi(z)\Omega = \phi$.

a. Prove that $\Omega = \mathbf{1}$.

b. Prove that if ϕ has conformal weight h, then $\phi_n \mathbf{1} = 0$ for $n > -h$, and that $\phi_{-h}\mathbf{1} = \phi$.

Problem 4.6. *The normal-ordered product*
Define the normal-ordered product $() : V \otimes V \to V$ by $(AB) := [A,B]_0$.

a. Prove that if A and B have conformal weights h_A and h_B respectively, then (AB) has conformal weight $h_A + h_B$.

b. Prove that $(AB)(z) = \sum_n (AB)_n z^{-n-h_A-h_B}$, where $(AB)_n$ is given by

$$(AB)_n := \sum_\ell :A_\ell B_{n-\ell} := \sum_{\ell \leq -h_A} A_\ell B_{n-\ell} + (-)^{|A||B|} \sum_{\ell > -h_A} B_{n-\ell} A_\ell \,, \qquad (4)$$

which defines the symbol $:\ :$.

c. Prove that the vector $(AB) \equiv \lim_{z\to 0} (AB)(z)\mathbf{1}$ is given by $(AB) = A_{-h_A} B$.

Problem 4.7. *An honest Lie bracket on* V
It follows from Problem 4.3 that the normal-ordered product is not commutative. In fact, let's introduce the symbol ([AB]) for the normal-ordered commutator: $([AB]) = (AB) - (-)^{|A||B|}(BA)$. Then it follows from Problem 4.3 that

$$([AB]) = \sum_{\ell \geq 1} \frac{(-)^{\ell+1}}{\ell!} \partial^\ell [AB]_\ell = (-)^{|A||B|} \sum_{\ell \geq 1} \frac{(-)^\ell}{\ell!} \partial^\ell [BA]_\ell$$

Prove that the normal-ordered product satisfies the following property:

$$(A(BC)) - (-)^{|A||B|}(B(AC)) = (([AB])C) \qquad (5)$$

and conclude from this that the normal-ordered commutator does satisfy the Jacobi identity.