Lecture IV (EKC) Curvature

In the previous lectures we have studied Ehresmann connections on principal fibre bundles and Koszul connections on associated vector bundles. Before going on to the third kind of connection in these lectures (Cartan connections) we will discuss two further topics: curvature of Ehresmann and Koszul connections, and a small digression on homogeneous spaces and their invariant connections.

Let $P \xrightarrow{\pi} M$ be a ppal G-bundle and $\rho : G \to GL(V)$ a Lie group homomorphism. Let $E := P \times_G V \xrightarrow{\varpi} M$ be the associated vector bundle. We saw in the last lecture that we have a $C^\infty(M)$-module isomorphism

$$\underset{\{s : M \to E \mid \varpi \circ s = \mathrm{id}_M\}}{\underset{\shortparallel}{\Gamma(E)}} \cong \underset{\{\xi : P \to V \mid r_g^* \xi = \rho(g^{-1}) \circ \xi \quad \forall g \in G\}}{\underset{\shortparallel}{C^\infty_G(P,V)}}$$

where if $f \in C^\infty(M)$ & $\xi \in C^\infty_G(P,V)$, $f \cdot \xi := \pi^* f \, \xi$.

We wish to generalise this from functions to forms. We define $\Omega^k(P,V)$ to be the k-forms on P with values in V. If $p \in P$, $\omega \in \Omega^k(P,V)$ then $\omega_p : \Lambda^k T_p P \to V$ is linear. Let $\Omega^k_G(P,V) \subset \Omega^k(P,V)$ denote those V-valued k-forms ω which are both

(i) horizontal: $\forall$ vertical ξ, $\iota_\xi \omega = 0$

and

(ii) invariant: $r_g^* \omega = \rho(g^{-1}) \circ \omega \quad \forall g \in G$.

Forms $\omega \in \Omega^k_G(P,V)$ are said to be basic, since they come from bundle valued forms on the base. Indeed, we have

23. Proposition There is an isomorphism of $C^\infty(M)$-modules

$$\Omega^k_G(P,V) \cong \Omega^k(M; P\times_G V)$$

where if $f \in C^\infty(M)$ and $\omega \in \Omega^k_G(P,V)$, we define $f\cdot\omega = \pi^* f\, \omega$.

Proof Very similar to the $k=0$ case $\left(C^\infty_G(P,V) \cong \Gamma(P\times_G V)\right)$ which we proved last time. We describe $\sigma \in \Omega^k(M, P\times_G V)$ locally by $\{\sigma_\alpha \in \Omega^k(U_\alpha; V)\}$ obeying $\sigma_\alpha(a) = \rho(g_{\alpha\beta}(a))\sigma_\beta(a)$ $\forall a \in U_{\alpha\beta}$. Then $\xi_\alpha(p) = \rho(g_\alpha(p))^{-1}\circ \pi^*\sigma_\alpha$ is clearly horizontal. It can be shown to be invariant and that $\xi_\alpha(p) = \xi_\beta(p)$ $\forall$ $p \in \pi^{-1}U_{\alpha\beta}$. Conversely, if $\xi \in \Omega^k_G(P,V)$, we define $\sigma_\alpha = s^*_\alpha \xi$ and one can show that $\sigma_\alpha(a) = \rho(g_{\alpha\beta}(a))\,\sigma_\beta(a)$ for all $a \in U_{\alpha\beta}$. ■

If $\sigma \in \Gamma(P\times_G V)$, $d^\nabla\sigma_\alpha = \rho(g_{\alpha\beta})\, d^\nabla\sigma_\beta$ and hence $d^\nabla\sigma \in \Omega^1(M; P\times_G V)$.

24. Lemma Let $\alpha \in \Omega^k_G(P,V)$. Then $h^* d\alpha \in \Omega^{k+1}_G(P,V)$.

Proof $h^* d\alpha$ is horizontal by construction, so we check invariance:

$$r_g^* h^* d\alpha \overset{\mathcal{H}\text{ is invariant}}{=} h^* r_g^* d\alpha \overset{[d, r_g^*]=0}{=} h^* d\, r_g^*\alpha \overset{\omega\text{ is invariant}}{=} h^* d(\rho(g)^{-1}\circ\alpha) = \rho(g)^{-1}\circ h^* d\alpha \quad ■$$

25. Definition Let $\omega \in \Omega^1(P;\mathfrak{g})$ be the connection one-form of an Ehresmann connection $\mathcal{H} \subset TP$. Its curvature $\Omega := h^* d\omega$.

26. Lemma $\Omega \in \Omega^2_G(P,\mathfrak{g})$.

Proof Ω is horizontal by definition and by the same calculation as in the proof of lemma 24, Ω is invariant because ω is. ■

27. Proposition $\Omega = 0$ iff $\mathcal{H} \subset TP$ is (Frobenius) integrable.

Proof

$$\Omega(\xi,\eta) = d\omega(h\xi, h\eta) = h\xi\ \cancel{\omega(h\eta)}^{\,0} - h\eta\ \cancel{\omega(h\xi)}^{\,0} - \omega([h\xi, h\eta])$$
$$= -\omega([h\xi, h\eta])$$

$\therefore\ \Omega = 0 \iff \forall \xi,\eta \quad [h\xi, h\eta]$ is horizontal

$\iff [\mathcal{H},\mathcal{H}] \subset \mathcal{H}$

$\iff \mathcal{H} \subset TP$ is integrable. ■

28. Proposition (The structure equation)

$$\Omega = d\omega + \tfrac{1}{2}[\omega,\omega]$$

Proof

We need to show that $\Omega(\xi,\eta) = d\omega(\xi,\eta) + [\omega(\xi),\omega(\eta)]$.

(1) Let ξ,η be horizontal. Then $h\xi = \xi$ and $h\eta = \eta$ and $\omega(\xi) = \omega(\eta) = 0$, so $\Omega(\xi,\eta) = d\omega(\xi,\eta)$ holds.

(2) Let η be horizontal and let $\xi = \xi_X$ is vertical. Then $h\xi = 0$, $h\eta = \eta$ and $\omega(\eta) = 0$. So we need to show that

$$0 = -\eta\, \omega(\xi_X) - \omega([\xi_X, \eta]) = -\cancel{\eta X}^{\,0} - \omega([\xi_X,\eta])$$

so that $[\xi_X, \text{horizontal}] \subset \text{horizontal}$. But this is the case because $\mathcal{H}$ is invariant.

(3) $\xi = \xi_X$, $\eta = \xi_Y$ vertical $\exists X,Y \in \mathfrak{g}$. Then $h\xi_X = h\xi_Y = 0$ and $\omega(\xi_X) = X$, $\omega(\xi_Y) = Y$. So we must show that

$$0 = d\omega(\xi_X,\xi_Y) + [\omega(\xi_X),\omega(\xi_Y)]$$
$$= \xi_X \cdot Y - \xi_Y \cdot X - \omega([\xi_X,\xi_Y]) + [X,Y]$$

but $[\xi_X,\xi_Y] = \xi_{[X,Y]}$ and $\omega(\xi_{[X,Y]}) = [X,Y]$ ✓ ■

29. Corollary (Bianchi identity)

$$h^* d\Omega = 0.$$

Proof

$$h^* d\Omega = h^* d\left(d\omega + \tfrac{1}{2}[\omega,\omega]\right) = h^*[d\omega,\omega] = [h^* d\omega, h^*\omega] = 0,$$

since $h^*\omega = 0$. ■

Let's define $d^\nabla : \Omega^k_G(P,V) \to \Omega^{k+1}_G(P,V)$ by $d^\nabla := h^* d$.
Then unlike d, d^∇ need not be a differential. The obstruction is the curvature.

30. Proposition For all $\alpha \in \Omega^k_G(P,V)$, $d^\nabla(d^\nabla \alpha) = \rho(\Omega)\wedge\alpha$.

Proof

$$d^\nabla \alpha = d\alpha + \rho(\omega)\wedge\alpha$$

$$\begin{aligned}
\therefore\ d^\nabla d^\nabla \alpha &= d\left(d\alpha + \rho(\omega)\wedge\alpha\right) + \rho(\omega)\wedge\left(d\alpha + \rho(\omega)\wedge\alpha\right) \\
&= \rho(d\omega)\wedge\alpha - \rho(\omega)\wedge d\alpha + \rho(\omega)\wedge d\alpha + \rho(\omega)\wedge\rho(\omega)\wedge\alpha \\
&= \rho(d\omega)\wedge\alpha + \tfrac{1}{2}[\rho(\omega),\rho(\omega)]\wedge\alpha \\
&= \rho\left(d\omega + \tfrac{1}{2}[\omega,\omega]\right)\wedge\alpha = \rho(\Omega)\wedge\alpha. \ \blacksquare
\end{aligned}$$

Exercise Write $F_\alpha = s_\alpha^*\Omega$. Express F_α in terms of $A_\alpha = s_\alpha^*\omega$, and relate F_α and F_β on $U_{\alpha\beta} \neq \emptyset$.

We now change topics temporarily to discuss homogeneous spaces. Let G be a Lie group acting transitively on a manifold M. Pick $a \in M$ & let $H < G$ denote the stabilizer subgroup $H = \{g \in G \mid g\cdot a = a\}$. It is a closed subgroup of G. Then $M \cong G/H$, where the diffeomorphism is G-equivariant and $G \curvearrowright G/H$ is induced by left-multiplication in G. If $g \in G$, we will let $\phi_g : M \to M$ denote the corresponding diffeomorphism. If $X \in \mathfrak{g}$, we define a vector field $\xi_X \in \mathfrak{X}(M)$ by $(\xi_X f)(m) = \frac{d}{dt} f(\phi_{\exp(-tX)}(m))\Big|_{t=0}$ $\forall m \in M$. Then $[\xi_X, \xi_Y] = \xi_{[X,Y]}$.

Since H stabilises $a \in M$, $(\phi_h)_* : T_aM \to T_aM$ and we get a Lie group homomorphism $\lambda : H \to GL(T_aM)$, called the linear isotropy representation. We will use the same notation for the induced Lie algebra representation $\lambda : \mathfrak{h} \to \mathfrak{gl}(T_aM)$.

Evaluating at $a \in M$, we get a surjective linear map $\mathfrak{g} \xrightarrow{X \mapsto \xi_X|_a} T_aM$ whose kernel is $\mathfrak{h}$. We say that G/H is reductive if the short exact sequence

$$0 \to \mathfrak{h} \to \mathfrak{g} \to T_aM \to 0$$

splits as H-modules. In other words, if $\exists\, \mathfrak{m} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $\forall h \in H$, $\mathrm{Ad}(h) : \mathfrak{m} \to \mathfrak{m}$. In that case $T_aM \cong \mathfrak{m}$ as H-modules.

If $g \in G$ and $\phi_g \in \mathrm{Diff}(M)$, we define $\phi_g \cdot f = f \circ \phi_{g^{-1}}$ $\quad \forall f \in C^\infty(M)$
and $\phi_g \cdot \xi = (\phi_g)_* \xi$, where $((\phi_g)_* \xi)_a = ((\phi_g)_*)_{\phi_g^{-1}(a)}\, \xi_{\phi_g^{-1}(a)}$, $\forall \xi \in \mathfrak{X}(M)$.
It follows that $\forall g \in G$, $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$,

$$\phi_g \cdot (Xf) = (\phi_g \cdot X)(\phi_g \cdot f) \quad \text{and} \quad \phi_g \cdot (fX) = (\phi_g \cdot f)(\phi_g \cdot X).$$

Now let ∇ be a an affine connection : $\forall X, Y \in \mathfrak{X}(M)$, $\nabla_X Y \in \mathfrak{X}(M)$.
Let $\phi \in \mathrm{Diff}(M)$. Let us define ∇^ϕ by

$$\nabla^\phi_X Y := \phi \cdot \nabla_{\phi^{-1} \cdot X} \phi^{-1} \cdot Y$$

31. Lemma ∇^ϕ is an affine connection.

Proof ①
$$\begin{aligned}
\nabla^\phi_{fX} Y &= \phi \cdot \nabla_{\phi^{-1}(fX)} (\phi^{-1} Y) && \text{by definition of } \nabla^\phi \\
&= \phi \cdot \nabla_{(\phi^{-1} f)(\phi^{-1} X)} (\phi^{-1} Y) && \text{by } \phi \cdot (fX) = (\phi \cdot f)(\phi \cdot X) \\
&= \phi \cdot \left(\phi^{-1} \cdot f\, \nabla_{\phi^{-1} X} (\phi^{-1} Y) \right) && \text{since } \nabla \text{ is an affine connection} \\
&= \phi \cdot (\phi^{-1} f)\ \phi \cdot (\nabla_{\phi^{-1}} (\phi^{-1} Y)) && \text{by } \phi \cdot (fX) = (\phi \cdot f)(\phi \cdot X) \\
&= f\ \nabla^\phi_X Y && \phi \cdot \phi^{-1} = \mathrm{id} \ \& \text{ definition of } \nabla^\phi
\end{aligned}$$

②

$$\begin{aligned}
\nabla^\phi_X (f \cdot Y) &= \phi \cdot \left(\nabla_{\phi^{-1} \cdot X}\, \phi^{-1} (f \cdot Y) \right) && \text{by definition of } \nabla^\phi \\
&= \phi \cdot \left(\nabla_{\phi^{-1} \cdot X} (\phi^{-1} \cdot f)(\phi^{-1} \cdot Y) \right) && \text{by } \phi \cdot (fX) = (\phi \cdot f)(\phi \cdot X) \\
&= \phi \cdot \left((\phi^{-1} \cdot X)(\phi^{-1} \cdot f)\ \phi^{-1} \cdot Y + (\phi^{-1} \cdot f) \nabla_{\phi^{-1} X} (\phi^{-1} Y) \right) && \text{since } \nabla \text{ is a conn.} \\
&= \phi \cdot \phi^{-1} \cdot X(f)\ \phi \cdot \phi^{-1} \cdot Y + \phi \cdot \phi^{-1} f\ \phi \cdot \nabla_{\phi^{-1} X} (\phi^{-1} Y) && \text{by } \phi \cdot (fX) = (\phi \cdot f)(\phi \cdot X) \\
&= X(f) Y + f\ \nabla^\phi_X Y && \text{by } \phi \cdot \phi^{-1} = \mathrm{id} \ \& \text{ def}^n \text{ of } \nabla^\phi
\end{aligned}$$

$\blacksquare$

32. Definition An affine connection ∇ on a reductive homogeneous $M = G/H$ is said to be G-invariant if $\nabla^{\phi_g} = \nabla$ for all $g \in G$. The invariance condition can also be written as

$$\phi_g \cdot \nabla_X Y = \nabla_{\phi_g X} \phi_g Y \qquad \forall\, X, Y \in \mathfrak{X}(M) \text{ and } \forall\, g \in G.$$

Suppose for a moment that $H = \{e\}$ and that $M = G$. Then an affine connection ∇ on G is left-invariant if $\forall g \in G$, $\forall X, Y \in \mathfrak{X}(G)$,

$$L_g \cdot \nabla_X Y = \nabla_{L_g \cdot X} (L_g \cdot Y)$$

Suppose that X, Y are left-invariant, so that $L_g \cdot X = X$ and $L_g \cdot Y = Y$. In that case, the left-invariance of ∇ implies that $\nabla_X Y$ is also left-invariant. Now, on a Lie group we may trivialise the tangent bundle via left translations. That means that we have a global frame $(X_1, \dots, X_n)$ consisting of left invariant vector fields. The connection is therefore uniquely determined by n^3 numbers Γ_{ij}^k defined by $\nabla_{X_i} X_j = \sum_k \Gamma_{ij}^k X_k$. These are the components relative to the basis $\{X_i\}$ of a linear map $\Lambda : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. The torsion and curvature tensors are also left-invariant and are given in terms of Λ by

$T(X,Y) = \Lambda_X Y - \Lambda_Y X - [X,Y]$

& $R(X,Y)Z = [\Lambda_X, \Lambda_Y]Z - \Lambda_{[X,Y]} Z$ for LI $X, Y, Z \in \mathfrak{X}(G)$.

We see that the curvature measures the failure of Λ being a Lie algebra homomorphism.

In particular, taking $\Lambda \equiv 0$, we see that there exists a flat connection with torsion given by $T(X,Y) = -[X,Y]$ relative to which the left-invariant vector fields on G are parallel. (That is, $\nabla X = 0$ for X left invariant.) Of course, there exists another flat connection with torsion annihilating the right-invariant vector fields.