Lecture V (EKC) Invariant connections

In the previous lecture we started to discuss homogeneous spaces G/H where G is a Lie group and $H \subset G$ a closed subgroup.
The projection $G \xrightarrow{\pi} M := G/H$ defines a principal H-bundle.
We recall the notation $\phi_g : M \to M$, $a \mapsto g \cdot a$, $\xi : \mathfrak{g} \to \mathfrak{X}(M)$, $X \mapsto \xi_X$
and $[\xi_X, \xi_Y] = \xi_{[X,Y]}$.
If $o \in M$ has stabilizer H, $\lambda_h := (\phi_h)_* : T_oM \to T_oM$, for $h \in H$, defines the linear isotropy representation. There is an isomorphism $T_oM \cong \mathfrak{g}/\mathfrak{h}$ which is H-equivariant. In other words, the following square commutes $\forall h \in H$:

$$\begin{tikzcd} T_oM \arrow[r, "\lambda_h"] \arrow[d, "\cong"'] & T_oM \arrow[d, "\cong"] \\ \mathfrak{g}/\mathfrak{h} \arrow[r, "\mathrm{Ad}(h)"] & \mathfrak{g}/\mathfrak{h} \end{tikzcd}$$

An affine connection ∇ on TM is G-invariant if $\nabla^{\phi_g} = \nabla \quad \forall g \in G$,
which can be written

$$\phi_g \cdot \nabla_\xi \eta = \nabla_{\phi_g \cdot \xi} \phi_g \cdot \eta .$$

We saw that if $H = \{e\}$ and hence $M = G$, with G acting via left transformations, then ∇ is left-invariant if and only if $\forall$ LIVF ξ_X, ξ_Y the VF $\nabla_{\xi_X} \xi_Y$ is also LI. Therefore ∇ is determined by a bilinear map $\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ where $\alpha(X,Y) := \nabla_{\xi_X} \xi_Y \big|_e$. This can be "curried" into a linear map $\Lambda : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, where $\lambda_X Y = \alpha(X,Y)$.

The torsion and curvature of a left-invariant connection are themselves left-invariant and determined by $T : \Lambda^2\mathfrak{g} \to \mathfrak{g}$ and $R : \Lambda^2 \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$:

$$T(X,Y) = \Lambda_X Y - \Lambda_Y X - [X,Y]$$
$$R(X,Y)Z = [\Lambda_X, \Lambda_Y]Z - \Lambda_{[X,Y]} Z$$

so the curvature measures the failure of Λ being a LA morphism.

In particular, there exists a flat connection, relative to which the LIVF are parallel ($\nabla \xi = 0$) having torsion $T(X,Y) = -[X,Y]$.

Now let $H \neq \{e\}$, but still G/H reductive. The usual yoga of homogeneous geometry (a.k.a. the holonomy principle) suggests that G-invariant geometric objects on G/H correspond to H-invariant algebraic objects at the 'origin' (any $a \in M$ with stabiliser H). Connections are not an exception to this rule. We will see that G-invariant connections on G/H are in one-to-one correspondence with H-invariant bilinear maps $\alpha : \mathfrak{m} \times \mathfrak{m} \longrightarrow \mathfrak{m}$ where $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is a reductive split. Equivalently, they are in bijective correspondence with H-equivariant linear maps $\Lambda : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{m})$ such that $\Lambda|_{\mathfrak{h}} = \lambda$, the linear isotropy representation. This result is due to Nomizu and α is known as the Nomizu map.

We let $\pi : G \longrightarrow G/H =: M$. Let $o \in G/H$ correspond to the coset H. Then $T_o M \cong \mathfrak{m}$ where $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ is the reductive split.

π is a diffeo from $U := \exp V$ (V a nbhd of $0 \in \mathfrak{m}$) to a nbhd of $o \in M$.

$\forall\, g \in U$, $\pi(g) = \phi_g \cdot o$ with $\phi_g \in \mathrm{Diff}(M)$ induced by left-multiplication on G.

Let $\bar{V} = \{\phi_g \cdot o \mid g \in U\}$. For $X \in \mathfrak{m}$, let $\xi_X \in \mathfrak{X}(\bar{V})$ be defined by

$$\begin{aligned}
(\xi_X)_{\phi_g \cdot o} &= ((\phi_g)_*)_o \cdot (\pi_*)_e X \\
&= ((\phi_g \circ \pi)_*)_e X \\
&= ((\pi \circ L_g)_*)_e X \qquad (\text{since } \phi_g \circ \pi = \pi \circ L_g) \\
&= (\pi_*)_g X^L_g \qquad \text{where } X^L \text{ is the LI VF obeying } X^L|_e = X.
\end{aligned}$$

Let $h \in H$. Since $\phi_h \cdot o = o$, $(\phi_h)_* : T_o M \longrightarrow T_o M$ making the following square commute:

$$\begin{tikzcd}
T_o M \arrow[r, "(\phi_h)_*"] & T_o M \\
\mathfrak{m} \arrow[u, "\pi_*", "\cong"'] \arrow[r, "\mathrm{Ad}(h)"] & \mathfrak{m} \arrow[u, "\pi_*"', "\cong"]
\end{tikzcd} \qquad (*)$$

Let $W \subset V$ be such that $h W h^{-1} \subseteq V$ (which exists by reductivity) and let $\bar{W} = \{\phi_g \cdot o \mid g \in \exp W\}$. Then $\phi_h \cdot \bar{W} \subseteq \bar{V}$.

(Exercise!)

33. Lemma For all $g \in \exp W$, $(\phi_h)_* \xi_X = \xi_{Ad(h)X}$ at $\phi_g \cdot o$.

Proof
$$
\begin{aligned}
((\phi_h)_* \xi_X)_{\phi_h \cdot \phi_g \cdot o} &= (\phi_h)_* \cdot (\xi_X)_{\phi_g \cdot o} \\
&= (\phi_h)_* \cdot (\phi_g)_* \cdot \pi_* \cdot X \\
&= (\phi_{hg})_* \cdot \pi_* X \\
&= (\phi_{hgh^{-1}})_* \cdot (\phi_h \circ \pi)_* X \\
&= (\phi_{hgh^{-1}})_* \cdot \pi_* (Ad(h) X) \qquad \text{by } (*) \\
&= (\xi_{Ad(h)X})_{\phi_{hgh^{-1}} \cdot o} \\
&= (\xi_{Ad(h)X})_{\phi_h \cdot \phi_g \cdot o} \quad \blacksquare
\end{aligned}
$$

34. Lemma
Let $X, Y \in \mathfrak{m}$, and $\xi_X, \xi_Y \in \mathfrak{X}(\bar{\nabla})$. Then $[\xi_X, \xi_Y]|_o = \pi_* [X,Y]_{\mathfrak{m}}$.

Proof We saw above that ξ_X is π-related to X^L for all $X \in \mathfrak{g}$. Therefore $[\xi_X, \xi_Y]$ is π-related to $[X^L, Y^L] = [X,Y]^L$. And hence $[\xi_X, \xi_Y] = \xi_{[X,Y]}$. Evaluating at $o \in M$, and using that $\xi_{[X,Y]_{\mathfrak{h}}}|_o = 0$, $[\xi_X, \xi_Y]|_o = \xi_{[X,Y]_{\mathfrak{m}}}|_o = \pi_* [X,Y]_{\mathfrak{m}}$ $\blacksquare$

Recall that an affine connection ∇ on $M = G/H$ is G-invariant if $\nabla^{\phi_g} = \nabla$ $\forall g \in G$.

35. Theorem Let $M = G/H$ be a reductive homogeneous space with a fixed decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $Ad(h)\mathfrak{m} \subset \mathfrak{m}$. There is a bijective correspondence between the set of G-invariant affine connections on M and H-equivariant bilinear maps $\alpha : \mathfrak{m} \times \mathfrak{m} \longrightarrow \mathfrak{m}$, where the correspondence is given by $\alpha(X,Y) = \nabla_{\xi_X} \xi_Y|_o$ $\forall X, Y \in \mathfrak{m}$.

Proof Let ∇ be an invariant connection and define $\alpha: \mathfrak{m}\times\mathfrak{m} \longrightarrow \mathfrak{m}$ via $\alpha(X,Y) = \nabla_{\xi_X}\xi_Y|_o$ for all $X,Y\in\mathfrak{m}$. We need to check that α is $Ad(h)$-invariant $\forall h\in H$. Since ∇ is G-invariant, it is in particular H-invariant:

$$\phi_h\cdot\nabla_{\xi_X}\xi_Y = \nabla_{\phi_h\cdot\xi_X}(\phi_h\cdot\xi_Y)$$

Evaluate at o:

$$LHS = \phi_h\cdot\alpha(X,Y) = \phi_h\cdot\xi_{\alpha(X,Y)} = \xi_{Ad(h)\cdot\alpha(X,Y)} \quad \text{(by Lemma 33)}$$

and

$$RHS = \nabla_{\xi_{Ad(h)\cdot X}}\xi_{Ad(h)\cdot Y} = \alpha(Ad(h)X, Ad(h)Y)$$

$$\Rightarrow \alpha \text{ is } Ad(h)\text{-invariant.}$$

Conversely, let $\alpha: \mathfrak{m}\times\mathfrak{m}\longrightarrow\mathfrak{m}$ be $Ad(H)$-invariant. We define a connection on $\bar{V}$ as follows. Let $(X_1,\dots,X_m)$ be a basis for $\mathfrak{m}$. Then $(\xi_1 := \xi_{X_1},\dots,\xi_m)$ is a local frame on $\bar{V}$ and we define $\bar{\nabla}$ on $\bar{V}$ by $\bar{\nabla}_{\xi_i}\xi_j = \xi_{\alpha(X_i,X_j)}$.

36. Lemma $\bar{\nabla}_{\xi_i}\xi_j|_o$ is H-invariant.

Proof of the lemma Since α is $Ad(H)$-invariant:

$$\begin{aligned}\phi_h\cdot\bar{\nabla}_{\xi_i}\xi_j|_o &= \phi_h\cdot\xi_{\alpha(X_i,X_j)}|_o = \xi_{Ad(h)\alpha(X_i,X_j)}|_o\\ &= \xi_{\alpha(Ad(h)X_i, Ad(h)X_j)}|_o\\ &= \bar{\nabla}_{\xi_{Ad(h)\cdot X_i}}\xi_{Ad(h)\cdot X_j}|_o\\ &= \bar{\nabla}_{\phi_h\cdot\xi_i}\phi_h\cdot\xi_j|_o. \quad\blacksquare\end{aligned}$$

More generally,

37. Lemma If $\zeta,\eta\in\mathfrak{X}(\bar{V})$ then $\bar{\nabla}_\zeta\eta|_o$ is $Ad(H)$-invariant.

Proof Assume WLOG that $\zeta|_o = \xi_X|_o$ $\exists X\in\mathfrak{m}$, and $\eta = \sum f_i\xi_i$ $\exists f_i\in C^\infty(\bar{V})$.

$$\bar{\nabla}_\zeta\eta|_o = \bar{\nabla}_{\xi_X}(\textstyle\sum_i f_i\xi_i)|_o = \sum_i\bar{\nabla}_{\xi_X}(f_i\xi_i)|_o = \sum_i\xi_X(f_i)\,\xi_i|_o + \sum_i f_i(o)\,\bar{\nabla}_{\xi_X}\xi_i|_o$$

$$\begin{aligned}\therefore\ \phi_h\cdot\bar{\nabla}_\zeta\eta|_o &= \sum_i(\phi_h\cdot\xi_X)(\phi_h\cdot f_i)\,\phi_h\cdot\xi_i|_o + \sum_i f_i(o)\,\phi_h\cdot\bar{\nabla}_{\xi_X}\xi_i|_o\\ &= \sum_i\xi_{Ad(h)X}(\phi_h\cdot f_i)\,\xi_{Ad(h)X_i}|_o + \sum_i f_i(o)\,\bar{\nabla}_{\phi_h\cdot\xi_X}\phi_h\cdot\xi_i|_o\end{aligned}$$

$$
\begin{aligned}
\overline{\nabla}_{\phi_h\cdot\xi}\,\phi_h\cdot\eta\big|_o &= \overline{\nabla}_{\xi_{Ad(h)X}} \sum_i (\phi_h\cdot f_i)(\phi_h\cdot\xi_i)\big|_o \\
&= \sum_i \xi_{Ad(h)X}(\phi_h\cdot f_i)\ (\phi_h\cdot\xi_i)\big|_o + \sum_i (\phi_h\cdot f_i)(o)\overline{\nabla}_{\xi_{Ad(h)X}}\,\phi_h\cdot\xi_i\big|_o \\
&= \sum_i \xi_{Ad(h)X}(\phi_h\cdot f_i)\ \xi_{Ad(h)X_i}\big|_o + \sum_i f_i(o)\ \overline{\nabla}_{\phi_h\cdot\xi_X}\,\phi_h\cdot\xi_i\big|_o
\end{aligned}
$$

since $\phi_h\cdot o = o$

Both expressions agree, proving the claim. ■

Now we let $\xi,\eta\in\mathfrak{X}(M)$ and $a\in M$. Let $g\in G$ be such that $\phi_g\cdot o = a$.
We define
$$\nabla_\xi\eta\big|_a := \phi_g\cdot\left(\overline{\nabla}_{\phi_{g^{-1}}\cdot\xi}\,\phi_{g^{-1}}\cdot\eta\big|_o\right)$$

The $Ad(H)$-invariance of $\overline{\nabla}_\xi\eta\big|_o$ $\forall\,\xi,\eta\in\mathfrak{X}(M)$ shows that this definition is well-defined; ie: independent of which $g\in G$ we choose: if $\phi_{g'}\cdot o = a$ then $g' = gh$ $\exists h\in H$, and $\phi_h\cdot\overline{\nabla}_{\phi_h^{-1}\cdot\xi}\,\phi_h^{-1}\cdot\eta\big|_o = \overline{\nabla}_\xi\eta\big|_o$ shows that
$$
\begin{aligned}
\phi_{g'}\cdot\overline{\nabla}_{\phi_{g'}^{-1}\cdot\xi}\,\phi_{g'}^{-1}\cdot\eta\big|_o &= \phi_g\cdot\phi_h\cdot\overline{\nabla}_{\phi_h^{-1}\cdot\phi_g^{-1}\cdot\xi}\,\phi_h^{-1}\cdot\phi_g^{-1}\cdot\eta\big|_o \\
&= \phi_g\,\overline{\nabla}_{\phi_g^{-1}\cdot\xi}\,\phi_g^{-1}\cdot\eta\big|_o .
\end{aligned}
$$

It remains to show that

(i) $\nabla_\xi\eta$ so defined $\in\mathfrak{X}(M)$
(ii) $\eta\mapsto\nabla\eta$ defines a connection

} <u>Exercise</u>: routine checks.

(iii) ∇ is G-invariant $\Leftarrow$ by construction
(iv) $\nabla_{\xi_X}\xi_Y\big|_o = \xi_{\alpha(X,Y)}\big|_o$ $\Leftarrow$ since $\nabla_{\xi_X}\xi_Y\big|_o = \overline{\nabla}_{\xi_X}\xi_Y\big|_o = \xi_{\alpha(X,Y)}\big|_o$. ■

The set of Nomizu maps is a vector space. The invariant affine connection corresponding to $\alpha = 0$ is called the canonical invariant connection of the reductive homogeneous space G/H.
The torsion and curvature tensors of an invariant connection are themselves invariant tensors and define tensors $T:\Lambda^2\mathfrak{m}\to\mathfrak{m}$ and $R:\Lambda^2\mathfrak{m}\to\mathfrak{h}<\mathfrak{gl}(\mathfrak{m})$ by
$$T(X,Y) = \alpha(X,Y) - \alpha(Y,X) - [X,Y]_\mathfrak{m}$$
$$R(X,Y)Z = \alpha(X,\alpha(Y,Z)) - \alpha(Y,\alpha(X,Z)) - \alpha([X,Y]_\mathfrak{m},Z) - [[X,Y]_\mathfrak{h},Z]$$

In particular, for the canonical invariant connection,

$$T(X,Y) = -[X,Y]_{\mathfrak{m}} \quad \text{and} \quad R(X,Y) = -[X,Y]_{\mathfrak{h}}$$

If (and only if) the canonical connection is torsion-free, the homogeneous space is symmetric. The Lie algebra $\mathfrak{g}$ admits an involution $\tau : \mathfrak{g} \to \mathfrak{g}$, $\tau^2 = id_{\mathfrak{g}}$, with $\tau(X) = \begin{cases} X & \text{if } X \in \mathfrak{h} \\ -X & \text{if } X \in \mathfrak{m} \end{cases}$ which preserves the Lie bracket: $\tau([X,Y]) = [\tau(X), \tau(Y)]$.

We now set the stage for the discussion of Cartan connections. Let $G \xrightarrow{\pi} G/H =: M$ be a reductive homogeneous space, and let ϑ_G denote the LI MC one-form on G. If $\sigma_1 : U \to G$ is a local section, we may pull back $\sigma_1^* \vartheta_G \in \Omega^1(U; \mathfrak{g})$. If $\sigma_2 : U \to G$ is another local section, then $\exists\, h : U \to H$ such that $\forall a \in U$, $\sigma_2(a) = \sigma_1(a) h(a)$.

38. Lemma $\quad \sigma_2^* \vartheta_G = \mathrm{Ad}(h)^{-1} \circ \sigma_1^* \vartheta_G + h^* \vartheta_H$

Proof It is easiest to prove this in the language of matrix groups, so let's do that. $\sigma^* \vartheta_G = \sigma^{-1} d\sigma$ in this case. Then,

$$\begin{aligned} \sigma_2^* \vartheta_G = \sigma_2^{-1} d\sigma_2 &= (\sigma_1 h)^{-1} d(\sigma_1 h) \\ &= h^{-1} \sigma_1^{-1} (d\sigma_1\, h + \sigma_1 dh) \\ &= h^{-1} \sigma_1^{-1} d\sigma_1\, h + h^{-1} dh \\ &= \mathrm{Ad}(h^{-1}) \circ \sigma_1^* \vartheta_G + h^* \vartheta_H \, . \; \blacksquare \end{aligned}$$

We write $\sigma_1^* \vartheta_G = \theta_1 + \omega_1$ where $\theta_1 \in \Omega^1(U; \mathfrak{m})$ and $\omega_1 \in \Omega^1(U; \mathfrak{h})$, and similarly for $\sigma_2^* \vartheta_G$. Then from the Lemma,

$$\theta_2 + \omega_2 = \mathrm{Ad}(h^{-1}) \circ (\theta_1 + \omega_1) + h^* \vartheta_H$$

and hence comparing $\mathfrak{m}$ and $\mathfrak{h}$ components:

$$\theta_2 = \mathrm{Ad}(h^{-1})\cdot\theta_1 \qquad \text{and} \qquad \omega_2 = \mathrm{Ad}(h^{-1})\cdot\omega_1 + h^*\vartheta_H$$

So that the $\mathfrak{m}$-component transforms tensorially, but the $\mathfrak{h}$-component transforms like a gauge field.

The MC structure equation says that ϑ_G is flat:

$$d\vartheta_G + \tfrac{1}{2}[\vartheta_G,\vartheta_G] = 0 .$$

We can pull this 2-form back via a local section $\sigma : U \to G$, and decomposing into $\mathfrak{m}$ and $\mathfrak{h}$ components:

$$d(\theta+\omega) + \tfrac{1}{2}[\theta+\omega,\theta+\omega] = 0$$

$$\therefore \quad d\theta + [\omega,\theta] + \tfrac{1}{2}[\theta,\theta]_{\mathfrak{m}} = 0 \Rightarrow \Theta = -\tfrac{1}{2}[\theta,\theta]_{\mathfrak{m}}$$

$$d\omega + \tfrac{1}{2}[\omega,\omega] + \tfrac{1}{2}[\theta,\theta]_{\mathfrak{h}} = 0 \Rightarrow \Omega = -\tfrac{1}{2}[\theta,\theta]_{\mathfrak{h}}$$

or equivalently

$$T(X,Y) = \Theta(\xi_X,\xi_Y) = -[X,Y]_{\mathfrak{m}}$$

$$R(X,Y) = \Omega(\xi_X,\xi_Y) = -[X,Y]_{\mathfrak{h}} \in \mathrm{End}(\mathfrak{m}) .$$

So ω is the canonical invariant connection on G/H.

Cartan geometries are modelled on $G \to G/H$ and Cartan connections will generalise $\vartheta_G \in \Omega^1(G;\mathfrak{g})$. The curvature of a Cartan connection measures how far the Cartan geometry is to G/H.

The tangent space is the best 'linear approximation' to a manifold. The Cartan viewpoint is to view the tangent space not as a linear representation of $GL(n,\mathbb{R})$ but as a (reductive) homogeneous space of the affine group $\mathrm{Aff}(n,\mathbb{R}) \cong GL(n,\mathbb{R}) \ltimes \mathbb{R}^n$, so that $T_a M \cong \mathrm{Aff}(n,\mathbb{R})/GL(n,\mathbb{R})$. This motivates why we may wish to approximate a manifold by a homogeneous space.

Cartan connections

We saw three descriptions of an Ehresmann connection on a ppal H-bundle $P \xrightarrow{\pi} M$: ① $\mathcal{H} \subset TP$ ② $\omega \in \Omega^1(P;\mathfrak{h})$ and ③ $\{A_\alpha \in \Omega^1(U_\alpha;\mathfrak{h})\}$. Descriptions ① & ② pertain to P, whereas ③ pertains to M. Similarly, we will consider two descriptions of Cartan connections: a bundle description and a 'basic' description. We start with the latter.

the nomenclature is due to Sharpe.

39. Definition A Cartan gauge with model G/H on a manifold M is a pair (U,θ) where $U \subset M$ is open and $\theta \in \Omega^1(U;\mathfrak{g})$ satisfying the regularity condition: $T_a U \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h}$ is a vector space isomorphism for every $a \in U$. A Cartan atlas with model G/H on M is a collection $\{(U_\alpha,\theta_\alpha)\}_{\alpha\in A}$ of Cartan gauges (with model G/H on M) such that ① $\bigcup_{\alpha\in A} U_\alpha = M$ and ② On $U_{\alpha\beta}$, $\theta_\beta = \mathrm{Ad}(h_{\alpha\beta}^{-1})\circ\theta_\alpha + h_{\alpha\beta}^*\vartheta_H$ $\exists\, h_{\alpha\beta}: U_{\alpha\beta} \longrightarrow H$.

(cf. with gauge fields for canonical invariant connection on G/H.)

As for manifold atlases, we say that two Cartan atlases are equivalent if their union is also a Cartan atlas and there is a unique maximal atlas equivalent to a given one.

40. Definition A Cartan structure on M is an equivalence class of Cartan atlases on M. A Cartan geometry is a manifold M together with a Cartan structure.

Given a Cartan gauge (U,θ), we define the curvature $\Omega \in \Omega^2(U;\mathfrak{g})$ by $\Omega = d\theta + \frac{1}{2}[\theta,\theta]$.

41. Lemma Let $\{(U_\alpha,\Theta_\alpha)\}_{\alpha\in A}$ be a Cartan atlas and $\{\Omega_\alpha \in \Omega^2(U_\alpha;\mathfrak{g})\}$ the curvatures. Then on $U_{\alpha\beta}$, $\Omega_\beta = \mathrm{Ad}(h_{\alpha\beta}^{-1})\circ\Omega_\alpha$.

Proof Differentiate $\Theta_\beta = \mathrm{Ad}(h_{\alpha\beta}^{-1})\circ\Theta_\alpha + h_{\alpha\beta}^*\vartheta_H$.

$dh_{\alpha\beta}^* = h_{\alpha\beta}^* d$ & MC structure eqn.

$$
\begin{aligned}
d\Theta_\beta &= d\left(\mathrm{Ad}(h_{\alpha\beta}^{-1})\circ\Theta_\alpha\right) + d\,h_{\alpha\beta}^*\vartheta_H \\
&= \mathrm{Ad}(h_{\alpha\beta}^{-1})\circ d\Theta_\alpha - [\mathrm{Ad}(h_{\alpha\beta}^{-1})\circ\Theta_\alpha, h_{\alpha\beta}^*\vartheta_H] - \tfrac{1}{2}h_{\alpha\beta}^*[\vartheta_H,\vartheta_H]
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ \Omega_\beta &= d\Theta_\beta + \tfrac{1}{2}[\Theta_\beta,\Theta_\beta] \\
&= \mathrm{Ad}(h_{\alpha\beta}^{-1})\circ d\Theta_\alpha - [\mathrm{Ad}(h_{\alpha\beta}^{-1})\circ\Theta_\alpha, h_{\alpha\beta}^*\vartheta_H] - \tfrac{1}{2}h_{\alpha\beta}^*[\vartheta_H,\vartheta_H] \\
&\quad + \tfrac{1}{2}[\mathrm{Ad}(h_{\alpha\beta}^{-1})\circ\Theta_\alpha + h_{\alpha\beta}^*\vartheta_H, \mathrm{Ad}(h_{\alpha\beta}^{-1})\circ\Theta_\alpha + h_{\alpha\beta}^*\vartheta_H] \\
&= \mathrm{Ad}(h_{\alpha\beta}^{-1})\circ d\Theta_\alpha + \tfrac{1}{2}[\mathrm{Ad}(h_{\alpha\beta}^{-1})\circ\Theta_\alpha, \mathrm{Ad}(h_{\alpha\beta}^{-1})\circ\Theta_\alpha] \\
&= \mathrm{Ad}(h_{\alpha\beta}^{-1})\circ d\Theta_\alpha + \tfrac{1}{2}\mathrm{Ad}(h_{\alpha\beta}^{-1})\circ[\Theta_\alpha,\Theta_\alpha] \\
&= \mathrm{Ad}(h_{\alpha\beta}^{-1})\circ\Omega_\alpha . \quad \blacksquare
\end{aligned}
$$

We see that $\Omega_\alpha = 0$ is an intrinsic condition, independent of the atlas.

42. Definition A Cartan geometry whose curvature vanishes everywhere is said to be *flat*.

Examples

1. G/H with $\Theta_\alpha = \sigma_\alpha^*\vartheta_G$
2. open subsets of G/H
3. $\Gamma\backslash G/H$ (locally G/H), $\Gamma\subset G$ acts on G/H via deck transformations.

Of course not all Cartan geometries are flat.

Next time: the bundle description and comparison with Ehresmann connections.