Lecture VI (EKC) Cartan connections (II)

In the last lecture we defined a Cartan geometry of type G/H (called a Klein geometry) on M to be an equivalence class of atlases $\{(U_\alpha,\theta_\alpha)\}_{\alpha\in A}$, where $\theta_\alpha \in \Omega^1(U_\alpha;\mathfrak{g})$ with $\mathrm{pr}\circ\theta_\alpha : T_aM \longrightarrow \mathfrak{g}/\mathfrak{h}$ a vector space isomorphism for all $a\in U_\alpha$, and where on $U_{\alpha\beta}$,

$$\theta_\beta = \mathrm{Ad}(h_{\alpha\beta}^{-1})\circ\theta_\alpha + h_{\alpha\beta}^*\vartheta_H \qquad \exists\, h_{\alpha\beta} : U_{\alpha\beta}\longrightarrow H.$$

The curvature of such a Cartan geometry is a collection $\{\Omega_\alpha\in\Omega^2(U_\alpha;\mathfrak{g})\}$ where $\Omega_\alpha = d\theta_\alpha + \frac{1}{2}[\theta_\alpha,\theta_\alpha]$. Examples of flat ($\Omega_\alpha=0$) Cartan geometries are locally isomorphic to G/H with atlas $\{(U_\alpha,\sigma_\alpha^*\vartheta_G)\}_{\alpha\in A}$.

In this lecture we give two other characterisations of a Cartan geometry and we define finally the notion of a Cartan connection and show that it is a special case of an Ehresmann connection.

A Klein geometry G/H has kernel K : the largest subgroup of H which is normal in G. If $K=1$, we say that G/H is effective. If $K\neq 1$, $(G/K)/(H/K)$ is effective. It is often convenient to consider locally effective Klein geometries, where K is a discrete subgroup.

It follows from a somewhat technical (but not hard) result that if G/H is effective, then if $\theta = \mathrm{Ad}(k^{-1})\cdot\theta + k^*\vartheta_H$ $\exists\, k : U\to H$, then $k=1$. (It follows that $k : U\to K$, but for effective G/H, $K=1$.)

This means that given a Cartan atlas $\{(U_\alpha,\theta_\alpha)\}_{\alpha\in A}$ modelled on an effective G/H, then in overlaps $U_{\alpha\beta}$, $\theta_\beta = \mathrm{Ad}(h_{\alpha\beta}^{-1})\circ\theta_\alpha + h_{\alpha\beta}^*\vartheta_H$, for a <u>unique</u> $h_{\alpha\beta}: U_{\alpha\beta}\longrightarrow H$. Indeed, if $\theta_\beta = \mathrm{Ad}(\tilde h_{\alpha\beta}^{-1})\circ\theta_\alpha + \tilde h_{\alpha\beta}^*\vartheta_H$, then letting $k = \tilde h_{\alpha\beta}^{-1} h_{\alpha\beta}$, we would have $\theta_\alpha = \mathrm{Ad}(\tilde h_{\beta\alpha}^{-1})\circ\theta_\beta + \tilde h_{\beta\alpha}^*\vartheta_H$ so that

$$\begin{aligned}\theta_\beta &= \mathrm{Ad}(h_{\alpha\beta}^{-1})\circ\left(\mathrm{Ad}(\tilde h_{\alpha\beta})\circ\theta_\beta + \tilde h_{\beta\alpha}^*\vartheta_H\right) + h_{\alpha\beta}^*\vartheta_H \\ &= \mathrm{Ad}(k^{-1})\circ\theta_\beta + \underbrace{\mathrm{Ad}(h_{\alpha\beta}^{-1})\circ\tilde h_{\beta\alpha}^*\vartheta_H + h_{\alpha\beta}^*\vartheta_H}_{k^*\vartheta_H}\end{aligned}$$

(Exercise!)

It also follows from uniqueness that $\{h_{\alpha\beta}: U_{\alpha\beta}\longrightarrow H\}$ define a (Čech) cocycle. Therefore they are the transition functions of a principal H-bundle $P\xrightarrow{\pi} M$, where $P = \bigsqcup_{\alpha\in A}(\{\alpha\}\times U_\alpha\times H)/\sim$ where $(\alpha,a,h)\sim(\beta,b,\tilde h) \Leftrightarrow a=b,\ \tilde h = h_{\alpha\beta}^{-1}(a)h$ $\forall\,\alpha,\beta\in A$, $a\in U_{\alpha\beta}$ and $h,\tilde h\in H$, and $\pi(\alpha,a,h)=a$. The right action of H on P is defined by $r_h[(\alpha,a,\tilde h)] = [(\alpha,a,\tilde h h)]$ which is well-defined since the identifications use left-multiplication.

Let $X\in\mathfrak{h}$. Then $X^L\in\mathfrak{X}(H)$ is the corresponding left-invariant vector field. We extend it to $U\times H$ as $(0,X^L) =: \xi_X\in\mathfrak{X}(U\times H)$. Since X^L is left-invariant and the identifications involve left-multiplication, the vector fields ξ_X glue to give a well-defined vector field $\xi_X\in\mathfrak{X}(P)$. Analogously to Lemma 33, we have:

<u>**43. Lemma**</u> Let $r_h: P\longrightarrow P$ denote the right action of $h\in H$ on P. Then $\forall\, X\in\mathfrak{h}$, $(r_h)_*\xi_X = \xi_{\mathrm{Ad}(h)^{-1}X}$.

<u>*Proof*</u> It is enough to check this locally on $U\times H$. Here, $r_h = \mathrm{id}\times R_h$ where $R_h: H\longrightarrow H$ is right-multiplication by h. Let $L_h: H\longrightarrow H$ denote left multiplication by h. Then on $U\times H$,

$$\begin{aligned}(r_h)_*\xi_X &= (\mathrm{id}\times R_h)_*(0,X^L)\\ &= (0,(R_h)_*X^L)\\ &= (0,(R_h)_*(L_{h^{-1}})_*X^L) \qquad \text{(since } X^L \text{ is left-invariant)}\\ &= (0,(\mathrm{Ad}(h^{-1})\cdot X)^L)\\ &= \xi_{\mathrm{Ad}(h)^{-1}X}\ . \blacksquare\end{aligned}$$

The Cartan atlas $\{(U_\alpha, \theta_\alpha)\}_{\alpha\in A}$ does not just give $P \xrightarrow{\pi} M$, but also a one-form $\omega \in \Omega^1(P;\mathfrak{g})$ with values in $\mathfrak{g}$, defined locally by

$$\omega : T_{(a,h)}(U_\alpha \times H) \longrightarrow T_a U_\alpha \times \mathfrak{h} \longrightarrow \mathfrak{g}$$

$$(v,y) \longmapsto (v, \vartheta_H(y)) \longmapsto \mathrm{Ad}(h^{-1})\,\theta_\alpha(v) + \vartheta_H(y) =: \omega_\alpha(v,y)$$

On $U_{\alpha\beta}$, we also have $\omega_\beta(v,y) = \mathrm{Ad}(h^{-1})\,\theta_\beta(v) + \vartheta_H(y)$. The transition function is $U_{\alpha\beta} \times H \xrightarrow{f_{\alpha\beta}} U_{\alpha\beta} \times H$ sending $(a,h) \longmapsto (a, h_{\alpha\beta}(a)^{-1}h)$.
We claim that the $\{\omega_\alpha\}$ glue to define $\omega \in \Omega^1(P;\mathfrak{g})$.

44. Proposition The following triangle commutes on $U_{\alpha\beta} \times H$:

$$\begin{tikzcd} T_a U_{\alpha\beta} \times T_h H \arrow[rr, "(f_{\alpha\beta})_*"] \arrow[dr, "\omega_\alpha"'] & & T_a U_{\alpha\beta} \times T_{h_{\alpha\beta}(a)^{-1}h} \arrow[dl, "\omega_\beta"] \\ & \mathfrak{g} & \end{tikzcd}$$

To prove this proposition we need some preparation.

45. Lemma Let $\mu : H \times H \longrightarrow H$ and $\imath : H \longrightarrow H$ denote the group multiplication and inversion maps. Let $\vartheta_H \in \Omega^1(H;\mathfrak{h})$ be the left-invariant Maurer–Cartan one-form. Then

$$(\mu^* \vartheta_H)(v) = \mathrm{Ad}(h_2^{-1})\,\vartheta_H((pr_1)_* v) + \vartheta_H((pr_2)_* v) \quad \text{for } v \in T_{(h_1,h_2)}(H\times H),$$

and

$$(\imath^* \vartheta_H)(v) = -\mathrm{Ad}(h)\,\vartheta_H(v) \quad \text{for } v \in T_h H.$$

Proof of the lemma

It is simpler notationally for matrix groups, for which $\vartheta_H|_h = h^{-1}dh$.
Hence, $\imath^*\vartheta_H|_{h^{-1}} = h\, dh^{-1} = -h\, h^{-1} dh\, h^{-1} = -\mathrm{Ad}(h)\cdot \vartheta_H|_h$, proving the second identity, and

$$\mu^*\vartheta_H\big|_{(h_1,h_2)} = (h_1h_2)^{-1} d(h_1h_2) = h_2^{-1} h_1^{-1} dh_1\, h_2 + h_2^{-1} dh_2$$
$$= \mathrm{Ad}(h_2^{-1})\cdot \vartheta_H|_{h_1} + \vartheta_H|_{h_2}. \quad \blacksquare$$

It is a good exercise to prove it for general Lie groups.

We will now use this lemma to prove the proposition.

Proof of the proposition

We notice that $f_{\alpha\beta}(a,h) = (a, h_{\alpha\beta}(a)^{-1}h) = (\mathrm{id}\circ pr_1, \mu\circ(\iota\circ h_{\alpha\beta}\circ pr_1 \times pr_2))(a,h)$

so that if $(v,y)\in T_a U_{\alpha\beta}\times T_h H$, $(f_{\alpha\beta})_*(v,y) = (v, \mu_*(\iota_*\circ(h_{\alpha\beta})_* v, y)) \in T_a U_{\alpha\beta}\times T_{h_{\alpha\beta}(a)^{-1}h}H$

and hence

$$(\omega_\beta\circ (f_{\alpha\beta})_*)(v,y) = \omega_\beta(v, \mu_*(\iota_*\circ(h_{\alpha\beta})_* v, y))$$
$$= \mathrm{Ad}(h_{\alpha\beta}(a)^{-1}h)^{-1}\,\theta_\beta(v) + \vartheta_H(\mu_*(\iota_*\circ(h_{\alpha\beta})_* v, y))$$

From the lemma,

$$\vartheta_H(\mu_*(\iota_*\circ(h_{\alpha\beta})_* v, y)) = (\mu^*\vartheta_H)(\iota_*(h_{\alpha\beta})_* v, y)$$
$$= \mathrm{Ad}(h^{-1})\,\vartheta_H(\iota_*(h_{\alpha\beta})_* v) + \vartheta_H(y)$$

and also from the lemma,

$$\vartheta_H(\iota_*(h_{\alpha\beta})_* v) = (\iota^*\vartheta_H)(h_{\alpha\beta}^* v)$$
$$= -\mathrm{Ad}(h_{\alpha\beta}(a))\,(h_{\alpha\beta}^*\vartheta_H)(v)$$

Hence,

$$(\omega_\beta\circ(f_{\alpha\beta})_*)(v,y) = \mathrm{Ad}(h)^{-1}\,\mathrm{Ad}(h_{\alpha\beta}(a))\,\theta_\beta(v) - \mathrm{Ad}(h)^{-1}\,\mathrm{Ad}(h_{\alpha\beta}(a))\,(h_{\alpha\beta}^*\vartheta_H)(v) + \vartheta_H(y)$$
$$= \mathrm{Ad}(h)^{-1}\,\mathrm{Ad}(h_{\alpha\beta}(a))\,\big(\underbrace{\theta_\beta(v) - (h_{\alpha\beta}^*\vartheta_H)(v)}_{\mathrm{Ad}(h_{\alpha\beta}(a)^{-1})\circ\theta_\alpha(v)}\big) + \vartheta_H(y)$$
$$= \mathrm{Ad}(h)^{-1}\circ\theta_\alpha(v) + \vartheta_H(y)$$
$$= \omega_\alpha(v,y). \quad \blacksquare$$

46. Definition The one-form $\omega\in\Omega^1(P;\mathfrak{g})$ is called a Cartan connection.

47. Proposition The Cartan connection $\omega\in\Omega^1(P;\mathfrak{g})$ obeys the following:

(i) for each $p\in P$, $\omega_p : T_pP \to \mathfrak{g}$ is a vector space isomorphism

(ii) $r_h^*\omega = \mathrm{Ad}(h^{-1})\circ\omega \qquad \forall h\in H$

(iii) $\omega(\xi_X) = X \qquad \forall X\in\mathfrak{h}$

Remark Properties (ii) & (iii) are reminiscent of an Ehresmann connection except that ω takes values in $\mathfrak{g}$ and not $\mathfrak{h}$. Condition (i) has no analogue for an Ehresmann connection.

Proof (i) $\dim P = \dim H + \dim M = \dim \mathfrak{h} + \dim \mathfrak{g}/\mathfrak{h} = \dim \mathfrak{g}$, so it suffices to show that $\omega_p : T_pP \to \mathfrak{g}$ is injective for all p. Let $a = \pi(p)$ and (U,θ) a Cartan gauge with $a \in U$. Then if $(v,y) \in T_aU \times T_hH$ is such that $\omega(v,y) = Ad(h^{-1})\cdot\theta(v) + \vartheta_H(y) = 0$, we need to show that $(v,y) = 0$.

Let $\omega(v,y) = 0$, so $Ad(h^{-1})\cdot\theta(v) = -\vartheta_H(y) \in \mathfrak{h}$ and hence $\theta(v) \in Ad(h)\mathfrak{h} = \mathfrak{h}$ and hence $pr_{\mathfrak{g}/\mathfrak{h}}\,\theta(v) = 0 \Rightarrow v = 0$ by the regularity property of θ.
Therefore $\vartheta_H(y) = 0$, but ϑ_H is injective, hence $y = 0$ as well.

(ii) Enough to check this in a Cartan gauge (U,θ). Let $(v,y) \in T_aU \times T_hH$. Then for $k \in H$, $(r_k^*\omega)(v,y) = \omega(v,(R_k)_*y) = Ad(hk)^{-1}\cdot\theta(v) + \vartheta_H((R_k)_*y)$

but $R_k^*\vartheta_H = Ad(k^{-1})\circ\vartheta_H$, so that

$$\begin{aligned}(r_k^*\omega)(v,y) &= Ad(k^{-1})\,Ad(h)^{-1}\,\theta(v) + Ad(k^{-1})\,\vartheta_H(y)\\ &= Ad(k^{-1})\left[Ad(h)^{-1}\,\theta(v) + \vartheta_H(y)\right]\\ &= Ad(k^{-1})\;\omega(v,y).\end{aligned}$$

(iii) In a Cartan chart, $\xi_X = (0, X^L) \in \mathfrak{X}(U\times H)$, hence

$$\omega(\xi_X) = Ad(h)^{-1}\,\theta(0) + \vartheta_H(X^L) = 0 + X = X. \;\blacksquare$$

Remark ω parallelises P, just like ϑ_G parallelises G in the Klein model.
Given $X \in \mathfrak{g}$, we get a vector field $\xi_X \in \mathfrak{X}(P)$ defined by $\xi_X(p) = \omega_p^{-1}(X)$, but unlike the case of (G,ϑ_G), this is not a Lie algebra morphism; although if $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$, $[\xi_X,\xi_Y] = \xi_{[X,Y]}$. The curvature of ω is the obstruction to $X \mapsto \xi_X$ defining a LA morphism $\mathfrak{g} \to \mathfrak{X}(P)$.
(See Additional Remarks at end.)

Notice that if $\{(U_\alpha,\theta_\alpha)\}_{\alpha\in A}$ is a Cartan atlas trivialising P, then if $s_\alpha : U_\alpha \to P|_{U_\alpha}$ are the canonical sections $s_\alpha(a) = [(a,e)]$, identity in H
$(s_\alpha^*\omega)(v) = \omega(v,0) = \theta_\alpha(v)$. So θ_α are the 'gauge fields' of the Cartan connection. Let $\Omega = d\omega + \frac{1}{2}[\omega,\omega] \in \Omega^2(P;\mathfrak{g})$ denote the curvature of the Cartan connection. Then $s_\alpha^*\Omega = d\theta_\alpha + \frac{1}{2}[\theta_\alpha,\theta_\alpha]$.

Bundle automorphisms of P (covering the identity) are the gauge symmetries of the Cartan geometry.

We can now give the standard definition of a Cartan geometry modelled on a Klein geometry.

48. Definition A Cartan geometry (P, ω) on M modelled on G/H consists of the following:

(a) a principal H-bundle $P \to M$

(b) $\omega \in \Omega^1(P; \mathfrak{g})$ satisfying

(i) for each $p \in P$, $\omega_p : T_p P \to \mathfrak{g}$ is a vector space isomorphism

(ii) $r_h^* \omega = \mathrm{Ad}(h^{-1}) \circ \omega$ for all $h \in H$

(iii) $\omega(\xi_X) = X$ for all $X \in \mathfrak{h}$.

Let $\Omega = d\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2(P; \mathfrak{g})$ be the curvature of ω. The projection $\mathrm{pr}_{\mathfrak{g}/\mathfrak{h}} \circ \Omega \in \Omega^2(P; \mathfrak{g}/\mathfrak{h})$ is the torsion of ω. The Cartan geometry is torsion-free if $\Omega \in \Omega^2(P; \mathfrak{h})$. A Cartan geometry is effective/reductive if so is the model geometry.

49. Lemma Let (P, ω) be a Cartan geometry on M modelled on G/H. Let $\psi : P \to H$ be a smooth map and $f : P \to P$ be such that $f(p) = r_{\psi(p)}(p)$. Then $f^* \omega = \mathrm{Ad}(\psi^{-1}) \omega + \psi^* \vartheta_H$ and $f^* \Omega = \mathrm{Ad}(\psi) \circ \Omega$.

Proof The expression for $f^* \Omega$ follows from that of $f^* \omega$ as in Lemma 41. To calculate $f^* \omega$ we work relative to a Cartan gauge (U, θ) and on $U \times H$. Then $f : U \times H \to U \times H$, defined by $f(a, h) = (a, h\psi(a,h))$, may be written as $f = (\mathrm{id} \circ \mathrm{pr}_1, \mu \circ (\mathrm{pr}_2 \times \psi))$. Hence if $(v, y) \in T_a U \times T_h H$,

$$f_* (v, y) = (v, \mu_* (y, \psi_*(v,y))) \in T_a U \times T_{h\psi(a,h)} H .$$

Therefore,

$$\begin{aligned}
(f^* \omega)(v, y) &= \omega(v, \mu_*(y, \psi_*(v,y))) \\
&= \mathrm{Ad}(h\psi(a,h))^{-1} \circ \theta(v) + \vartheta_H(\mu_*(y, \psi_*(v,y))) \\
&= \mathrm{Ad}(\psi^{-1}) \circ \mathrm{Ad}(h^{-1}) \cdot \theta(v) + (\mu^* \vartheta_H)(y, \psi_*(v,y)) \\
\text{(by Lemma 45)} \quad &= \mathrm{Ad}(\psi^{-1}) \circ \mathrm{Ad}(h^{-1}) \circ \theta(v) + \mathrm{Ad}(\psi^{-1}) \cdot \vartheta_H(y) + \vartheta_H(\psi_*(v,y)) \\
&= \mathrm{Ad}(\psi^{-1}) \circ \left(\mathrm{Ad}(h^{-1}) \cdot \theta(v) + \vartheta_H(y) \right) + (\psi^* \vartheta_H)(v, y) \\
&= \left(\mathrm{Ad}(\psi^{-1}) \circ \omega + \psi^* \vartheta_H \right)(v, y). \quad \blacksquare
\end{aligned}$$

50. Corollary Ω is horizontal; ie: if either u,v are tangent to the fibre, $\Omega(u,v)=0$.

Proof Let $u,v \in T_p P$ and v tangent to the fibre. Let $\Psi : P \to H$ be any smooth map sending $p \mapsto e$ and such that $(\Psi_*)_p v = -\omega_p(v) \in \mathfrak{h}$ and define $f : P \to P$ by $f(q) = q\Psi(q)$. From the previous lemma, at $p \in P$ we have

$$f^*\omega = \mathrm{Ad}(\Psi^{-1})\,\omega + \Psi^*\vartheta_H = \omega + \Psi^*\vartheta_H$$

and $f^*\Omega = \Omega$

Therefore,

$$\omega_p(f_* v) = \omega_p(v) + \vartheta_H(\Psi_* v) = \omega_p(v) - \omega_p(v) = 0$$

and hence $f_* v = 0$. Therefore,

$$\Omega(u,v) = \Omega(f_* u, f_* v) = \Omega(f_* u, 0) = 0 \;. \blacksquare$$

It follows that Ω defines a 2-form on $TP/_{\ker \pi_*} \cong \pi^* TM$.

Notice that each fibre F of P is identified with H up to left multiplication by some element of H. Since ϑ_H is left-invariant, it defines a "Maurer-Cartan" form ϑ_F on the fibre. And the fact that $\vartheta_F(\xi_X) = X$ for $X = \mathfrak{h}$ shows that $\vartheta_F = \omega|_F$. It follows from this result that Ω vanishes when restricted to any fibre.

So a Cartan geometry (P,ω) deforms (G,ϑ_G) by changing $G \rightsquigarrow P$ and $\vartheta_G \rightsquigarrow \omega$, but in such a way that fibrewise we still have (H,ϑ_H).

The tangent bundle of G/H is a vector bundle associated to $G \to G/H$ via the linear isotropy representation $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}} : H \to GL(\mathfrak{g}/\mathfrak{h})$, so that $T(G/H) \cong G \times_H \mathfrak{g}/\mathfrak{h}$.

In a similar way, the tangent bundle of a Cartan geometry (P,ω) modelled on G/H is isomorphic to an associated vector bundle $P\times_H \mathfrak{g}/\mathfrak{h}$.

51. Proposition (P,ω) a Cartan geometry on M modelled on G/H. There is a canonical bundle isomorphism $\varphi : TM \xrightarrow{\cong} P\times_H \mathfrak{g}/\mathfrak{h}$, such that for all $p\in P$ with $\pi(p)=x$, there is an H-equivariant vector space isomorphism:

$$\varphi_p : T_xM \longrightarrow \mathfrak{g}/\mathfrak{h} \quad \text{such that} \quad \varphi_{ph} = \mathrm{Ad}(h^{-1})\circ \varphi_p \quad \forall h\in H.$$

Proof

$$\begin{tikzcd}
0 \arrow[r] & T_p(F_x) \arrow[r] \arrow[d, "\vartheta_H"', "\cong"] & T_pP \arrow[r, "(\pi_*)_p"] \arrow[d, "\omega"', "\cong"] & T_xM \arrow[r] \arrow[d, dashed, "\exists!\,\varphi_p\ \cong"] & 0 \\
0 \arrow[r] & \mathfrak{h} \arrow[r] & \mathfrak{g} \arrow[r, "\rho"] & \mathfrak{g}/\mathfrak{h} \arrow[r] & 0
\end{tikzcd}$$

If $v\in T_xM$, we may write $v = (\pi_*)_p(u) = (\pi_*)_{ph}((r_h)_*u)$ $\exists u\in T_pP$. Thus,

$$\begin{aligned}
\varphi_{ph}(v) &= \varphi_{ph}\left((\pi_*)_{ph}(r_h)_* u\right) \\
\text{(commutativity of square)} \quad &= \rho\left(\omega_{ph}((r_h)_*u)\right) \\
&= \rho\left(\mathrm{Ad}(h)^{-1}\circ \omega_p(u)\right) \\
&= \mathrm{Ad}(h)^{-1}\left(\varphi_p(\pi_{*p}u)\right) \\
&= \mathrm{Ad}(h)^{-1}\,\varphi_p(v)
\end{aligned}$$

This allows us to define a bundle map

$$\begin{aligned}
q : P\times \mathfrak{g} &\longrightarrow TM \\
(p, X) &\longmapsto \left(\pi(p), \varphi_p^{-1}(\rho(X))\right)
\end{aligned}$$

Then
$$\begin{aligned}
q(ph, \mathrm{Ad}(h)^{-1}X) &= \left(\pi(ph), \varphi_{ph}^{-1}\left(\rho(\mathrm{Ad}(h)^{-1}X)\right)\right) \\
&= \left(\pi(p), (\mathrm{Ad}(h)\varphi_{ph})^{-1}\rho(X)\right) \\
&= \left(\pi(p), \varphi_p^{-1}(\rho(X))\right) \\
&= q(p, X)
\end{aligned}$$

$\Rightarrow$ q induces $\bar{q} : P\times_H \mathfrak{g}/\mathfrak{h} \longrightarrow TM$ which covers the identity & is a linear iso on the fibres. ∎

52. Corollary (P,ω) a Cartan geometry on M modelled on G/H. Then vector fields $\xi \in \mathfrak{X}(M)$ are in bijective correspondence with functions $\bar{\xi} : P \to \mathfrak{g}/\mathfrak{h}$ such that $\bar{\xi}(ph) = \mathrm{Ad}(h^{-1}) \circ \bar{\xi}(p)$ $\forall p \in P, h \in H$:

$$\xi \longmapsto \bar{\xi} = \{ p \in P \longmapsto \varphi_p(\xi_{\pi(p)}) \in \mathfrak{g}/\mathfrak{h} \}$$

53. Definition The curvature function $K : P \to \mathrm{Hom}(\Lambda^2 \mathfrak{g}/\mathfrak{h}, \mathfrak{g})$ of a Cartan connection ω is defined by

$$K(p)(X,Y) := \Omega_p(\omega_p^{-1}(X), \omega_p^{-1}(Y)) \qquad \forall p \in P,\ X,Y \in \mathfrak{g}$$

54. Lemma The curvature function is well-defined and is H-equivariant:

$$K(ph)(X,Y) = \mathrm{Ad}(h^{-1})\ K(p)(\mathrm{Ad}(h)X, \mathrm{Ad}(h)Y)$$

Proof Fix $p \in P$ and let $\tilde{X} = X + W$, $\tilde{Y} = Y + Z$ $\exists W, Z \in \mathfrak{h}$. Then $\Omega_p(\omega_p^{-1}\tilde{X}, \omega_p^{-1}\tilde{Y}) = \Omega_p(\omega_p^{-1}X, \omega_p^{-1}Y)$ since $\omega_p^{-1}(Z)$, $\omega_p^{-1}(W)$ are tangent to the fibres & Ω is horizontal. Therefore $K(p) \in \mathrm{Hom}(\Lambda^2(\mathfrak{g}/\mathfrak{h}), \mathfrak{g})$. The equivariance follows from the equivariance of ω & Ω. ■

It follows that the curvature of a Cartan connection defines a curvature section of the bundle $P \times_H \mathrm{Hom}(\Lambda^2 \mathfrak{g}/\mathfrak{h}, \mathfrak{g})$

A Cartan connection is torsion-free iff the curvature function takes values in $\mathrm{Hom}(\Lambda^2 \mathfrak{g}/\mathfrak{h}, \mathfrak{h}) \subset \mathrm{Hom}(\Lambda^2 \mathfrak{g}/\mathfrak{h}, \mathfrak{g})$.

Exercise Show that $K(p)(X,Y) = [X,Y] - \omega_p[\omega_p^{-1}X, \omega_p^{-1}Y]$

55. Lemma (Bianchi identity) $d\Omega = [\Omega, \omega]$.

Proof Mutatis mutandis as for Ehresmann connections. ■

Let V be a vector space and $f: P \to V$ a function.
A Cartan connection $\omega \in \Omega^1(P;\mathfrak{g})$ defines a universal covariant derivative as follows: if $X \in \mathfrak{g}$ and if $\xi_X = \omega^{-1}(X)$, then $\tilde{D}_X f := \xi_X f$. Since this is linear in $X \in \mathfrak{g}$, we get

$$\begin{aligned} \tilde{D}: \Omega^0(P;V) &\longrightarrow \Omega^0(P; V\otimes \mathfrak{g}^*) \\ f &\longmapsto \tilde{D}f \end{aligned}$$

where $\tilde{D}f$ is defined by $(\iota_X)_* \tilde{D}f = \tilde{D}_X f$,
where

$$\begin{aligned} \iota_X : V\otimes \mathfrak{g}^* &\to V \\ v\otimes\eta &\mapsto \eta(X)v \end{aligned}$$

56. Definition Let $\rho: H \to GL(V)$ be a representation.
We define

$$\Omega^k(P;\rho) := \{ \alpha \in \Omega^k(P;V) \mid r_h^* \alpha = \rho(h^{-1})\circ\alpha \ \ \forall h \in H \}$$

the k-forms on P transforming according to ρ.

57. Proposition $\tilde{D}: \Omega^0(P;\rho) \longrightarrow \Omega^1(P;\rho) \cong \Omega^0(P;\rho\otimes \mathrm{Ad}^*)$

Proof Let $p \in P$, $X \in \mathfrak{g}$, $f \in \Omega^0(P;\rho)$. Then

$$\begin{aligned} (\iota_X)_*(r_h^*(\tilde{D}f))(p) &= (\iota_X)_*(\tilde{D}f(ph)) \\ &= (\tilde{D}_X f)(ph) \\ &= \omega_{ph}^{-1}(X) f \end{aligned}$$

but $r_h^*\omega = \mathrm{Ad}(h^{-1})\circ\omega$ says that $\omega_{ph}\circ(r_h)_* = \mathrm{Ad}(h^{-1})\circ\omega_p$
so inverting, $(r_{h^{-1}})_* \circ \omega_{ph}^{-1} = \omega_p^{-1}\circ \mathrm{Ad}(h)$ or $\omega_{ph}^{-1} = (r_h)_*\omega_p^{-1}\mathrm{Ad}(h)$,
so

$$(\iota_X)_*(r_h^* \tilde{D}f)(p) = ((r_h)_* \omega_p^{-1}(\mathrm{Ad}(h)X))f$$

For any $Y \in \mathfrak{X}(P)$,

$$((r_h)_* Y)f = Y(r_h^* f) = Y(\rho(h^{-1})\circ f) = \rho(h^{-1})Yf$$

so taking $Y = \omega_p^{-1}(\mathrm{Ad}(h)X)$, we get

$$((r_h)_* \omega_p^{-1}(\mathrm{Ad}(h)X))f = \rho(h^{-1})\, \omega_p^{-1}(\mathrm{Ad}(h)X) f = \rho(h^{-1}) \tilde{D}_{\mathrm{Ad}(h)X} f$$

so that

$$(\zeta_X)_*(r_h^* \tilde{D} f)(p) = \rho(h^{-1}) \tilde{D}_{\mathrm{Ad}(h)X} f \; . \blacksquare$$

Even if (V,ρ) is irreducible, $(V \otimes \mathfrak{g}^*, \rho \otimes \mathrm{Ad}^*)$ need not be. Decomposing $V \otimes \mathfrak{g}^*$ into irreducibles, decomposes $\tilde{D}$ and in this way we get many "famous" differential operators: $\partial, \bar{\partial}$, div, curl, ...

58. Lemma Let $X \in \mathfrak{h}$ and $f \in \Omega^0(P;\rho)$. Then $(\zeta_X)_* \tilde{D} f = -\rho_*(X) f$, where $\rho_* : \mathfrak{h} \to \mathrm{End}(V)$ is the LA hom. induced by $\rho : H \to GL(V)$.

Proof

$$(\zeta_X)_*(\tilde{D} f)(p) = \omega_p^{-1}(X) f = \frac{d}{dt} f(p e^{tX})\Big|_{t=0} = \frac{d}{dt} \rho(e^{-tX}) f(p)\Big|_{t=0}$$

$$= -\rho_*(X) f(p) \; . \blacksquare$$

Reductive Cartan geometries

Now assume that (P,ω) is reductive, so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $\mathrm{Ad}(H)\mathfrak{m} \subseteq \mathfrak{m}$, then the Cartan connection decomposes as $\omega = \omega_{\mathfrak{h}} + \omega_{\mathfrak{m}}$ and so does any Cartan gauge $\Theta = \Theta_{\mathfrak{h}} + \Theta_{\mathfrak{m}}$ and so does $\tilde{D} = \tilde{D}_{\mathfrak{h}} + \tilde{D}_{\mathfrak{m}}$. If $X \in \mathfrak{h}$, $\tilde{D}_X f = -\rho(X) f$, so $\tilde{D}_{\mathfrak{h}} = -\rho$. But as we will see below, $\tilde{D}_{\mathfrak{m}}$ defines a Koszul connection on any associated vector bundle $P \times_H V$.

It follows from the defining properties of a Cartan connection, that $\omega_{\mathfrak{h}} \in \Omega^1(P;\mathfrak{h})$ is the connection one-form for an Ehresmann connection on the principal H-bundle $P \to M$. In contrast, the component $\omega_{\mathfrak{m}} \in \Omega^1(P;\mathfrak{m})$ satisfies: (1) it is horizontal, since $\omega_{\mathfrak{m}}(\xi_X) = 0$ for $X \in \mathfrak{h}$ (and those span the tangent spaces to the fibres and $r_h^* \omega_{\mathfrak{m}} = \mathrm{Ad}(h^{-1}) \circ \omega_{\mathfrak{m}}$. This means that $\omega_{\mathfrak{m}}$ induces a one-form in M with values in the associated vector bundle $P \times_H \mathfrak{m}$, which is isomorphic to TM. In other words, $\omega_{\mathfrak{m}}$ is a soldering form on P.

In summary, a reductive Cartan geometry (P, ω) on M is equivalent to an Ehresmann connection together with a soldering form on P.

Let (P, ω) be a reductive Cartan geometry on M modelled after G/H. Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be a reductive split. We saw that $\omega_{\mathfrak{h}} \in \Omega^1(P; \mathfrak{h})$ is the connection one-form of an Ehresmann connection, whereas $\omega_{\mathfrak{m}} \in \Omega^1(P; \mathfrak{m})$ is a soldering form. The curvature also splits as $\Omega = \Omega_{\mathfrak{h}} + \Omega_{\mathfrak{m}}$, where from $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$, we get that

$$\Omega_{\mathfrak{h}} = d\omega_{\mathfrak{h}} + \tfrac{1}{2}[\omega_{\mathfrak{h}}, \omega_{\mathfrak{h}}] + \tfrac{1}{2}[\omega_{\mathfrak{m}}, \omega_{\mathfrak{m}}]_{\mathfrak{h}}$$

$$\Omega_{\mathfrak{m}} = d\omega_{\mathfrak{m}} + [\omega_{\mathfrak{h}}, \omega_{\mathfrak{m}}] + \tfrac{1}{2}[\omega_{\mathfrak{m}}, \omega_{\mathfrak{m}}]_{\mathfrak{m}}$$

Therefore the $\mathfrak{h}$-component of the curvature of the Cartan connection is not necessarily the curvature of the Ehresmann connection, but receives a correction from the soldering form:

$$\Omega_{\mathfrak{h}}^{\text{Cartan}} = \Omega^{\text{Ehresmann}} + \tfrac{1}{2}[\omega_{\mathfrak{m}}, \omega_{\mathfrak{m}}]_{\mathfrak{h}}$$

whereas the torsion of the Cartan connection $(\Omega_{\mathfrak{m}})$ is not necessarily the torsion of the "affine connection" defined by $\omega_{\mathfrak{h}}$:

$$\Theta^{\text{Cartan}} = \Omega_{\mathfrak{m}}^{\text{Cartan}} = \Theta + \tfrac{1}{2}[\omega_{\mathfrak{m}}, \omega_{\mathfrak{m}}]_{\mathfrak{m}}.$$

Let's now consider the universal covariant derivative $\widetilde{D} = \widetilde{D}_{\mathfrak{h}} + \widetilde{D}_{\mathfrak{m}}$. The $\mathfrak{m}$-component $\widetilde{D}_{\mathfrak{m}}$ defines a Koszul connection on any associated vector bundle $E := P \times_H V$ for (V, ρ) a representation of H. Indeed, let $\psi : \Gamma(E) \longrightarrow \Omega^0(P;\rho)$ be the $C^\infty(M)$-module isomorphism. We define $\nabla_\xi : \Gamma(E) \longrightarrow \Gamma(E)$ by the commutativity of the following square:

$$\begin{tikzcd} \Gamma(E) \arrow[r, "\nabla_\xi"] \arrow[d, "\psi"', "\cong"] & \Gamma(E) \arrow[d, "\psi", "\cong"'] \\ \Omega^0(P;\rho) \arrow[r, "\tilde{\xi}"] & \Omega^0(P;\rho) \end{tikzcd}$$

ie: $\psi(\nabla_\xi s) = \tilde{\xi}\, \psi(s)$

where $\tilde{\xi}$ is the horizontal lift of ξ: the unique vector field on P such that $(\pi_*)_p \tilde{\xi} = \xi_{\pi(p)}$ and $\omega_{\mathfrak{h}}(\tilde{\xi}) = 0$.

59. Proposition ∇ defines a Koszul connection on E.

Proof ∇_ξ is $\mathbb{R}$-linear and if $f \in C^\infty(M)$, $\pi^* f\, \tilde{\xi}$ is the horizontal lift of $f\xi$, so that $\nabla_{f\xi} s = f \nabla_\xi s$:

$$\psi(\nabla_{f\xi} s) = \widetilde{f\xi}\, \psi(s) = \pi^* f\, \tilde{\xi}\, \psi(s) = \pi^* f\, \psi(\nabla_\xi s) = \psi(f \nabla_\xi s).$$

Finally, the derivation property:

$$\begin{aligned} \psi(\nabla_\xi(fs)) &= \tilde{\xi}\, \psi(fs) = \tilde{\xi}\left(\pi^* f\, \psi(s)\right) = \tilde{\xi}(\pi^* f)\, \psi(s) + \pi^* f\, \tilde{\xi}\, \psi(s) \\ &= \pi^*(\xi f)\, \psi(s) + \pi^* f\, \psi(\nabla_\xi s) = \psi((\xi f) s) + \psi(f \nabla_\xi s) \\ &= \psi((\xi f) s + f \nabla_\xi s). \blacksquare \end{aligned}$$

60. Proposition Let (U, θ) be a gauge for a reductive Cartan geometry, $\sigma : U \to P|_U$ the section such that $\theta = \sigma^* \omega$, $\xi \in \mathfrak{X}(U)$ and $\phi = \sigma^* \Phi$ where $\Phi \in \Omega^0(P;\rho)$. Then $\nabla_\xi \phi := \xi(\phi) - \rho_*(\theta_{\mathfrak{h}}(\xi))\phi$ is the expression of the covariant derivative of Φ in the gauge (U, θ).

Special Geometries

We may define "special geometries" via curvature constraints.

61. Lemma Let $V \subset \mathfrak{g}$ be the vector subspace spanned by the values of the curvature form Ω. Then V is an H-submodule.

Proof Let $v = \Omega_p(\xi_p, \eta_p)$. Then

$$\begin{aligned} Ad(h^{-1})v &= Ad(h^{-1})(\Omega_p(\xi_p,\eta_p)) \\ &= (r_h^*\Omega_p)(\xi_p,\eta_p) \\ &= \Omega_{ph}((r_h)_*\xi_p,(r_h)_*\eta_p) \end{aligned}$$

which is a value of Ω. ■

In particular, if $V \subset \mathfrak{h}$, so that the Cartan geometry is torsion-free, then V is an ideal in $\mathfrak{h}$. If the geometry is torsion-free and the adjoint action of H on $\mathfrak{h}$ is irreducible, there are no special geometries arising from $\mathfrak{g}$-curvature conditions. However the H-module $\mathrm{Hom}(\Lambda^2(\mathfrak{g}/\mathfrak{h}),\mathfrak{h})$ need not be irreducible and we can define special geometries by demanding that the curvature function $K: P \to \mathrm{Hom}(\Lambda^2(\mathfrak{g}/\mathfrak{h}),\mathfrak{h})$ takes values in an H-submodule.

If H is compact, then $\mathrm{Hom}(\Lambda^2(\mathfrak{g}/\mathfrak{h}),\mathfrak{h})$ is fully reducible. For example, when $\mathfrak{g} = \underline{so}(n) \ltimes \mathbb{R}^n$ and $\mathfrak{h} = \underline{so}(n)$, $\mathrm{Hom}(\Lambda^2(\mathfrak{g}/\mathfrak{h}),\mathfrak{h}) = \mathrm{Hom}(\Lambda^2\mathbb{R}^n, \underline{so}(n))$. The subspace corresponding to those curvature functions obeying the (algebraic) Bianchi identity breaks up into three submodules: scalar, trace-free Ricci and Weyl. (And if $n=4$ Weyl splits further.)

Ehresmann generalises Cartan

Cartan connections are special types of Ehresmann connection. Let $P \to M$ and $G \to G/H$ be principal H-bundles. There is an associated fibre bundle $Q = P \times_H G$ where H acts on G by left multiplication. This is a (right) principal G-bundle on M, and we have a natural inclusion $P \subset Q$ sending $p \mapsto (p, e)$. An Ehresmann connection on Q is a $\mathfrak{g}$-valued one-form and its restriction to P gives a candidate for a Cartan connection on P. Which Ehresmann connections restrict to Cartan connections?

62. **Theorem** Let G/H be a Klein geometry and let P and Q be principal H- and G-bundles, respectively, over a manifold M. Assume that $\dim P = \dim G$ and that $\varphi : P \to Q$ is an H-bundle map. Then there is a bijection of sets:

$$\left\{ \begin{array}{l} \text{Ehresmann connections} \\ \text{on } Q \text{ whose kernels do} \\ \text{not meet } \varphi_*(TP) \end{array} \right\} \xrightarrow{\varphi^*} \left\{ \begin{array}{l} \text{Cartan} \\ \text{connections} \\ \text{on } P \end{array} \right\}.$$

Proof Let $\varpi \in \Omega^1(Q; \mathfrak{g})$ be an Ehresmann connection such that $\varphi_*(TP) \cap \ker \varpi = 0$. It follows that $\omega = \varphi^* \varpi \in \Omega^1(P; \mathfrak{g})$ with zero kernel. Since $\dim P = \dim \mathfrak{g}$, $\omega_p : T_p P \to \mathfrak{g}$ is injective $\forall p$ and hence an isomorphism.

Since $\varphi : P \to Q$ is an H-bundle map, $\forall X \in \mathfrak{h}$, the vector fields ξ_X on P and ζ_X on Q are φ-related: $(\varphi_*)_p \, \xi_X(p) = \zeta_X(\varphi(p)) \quad \forall p \in P$. Therefore $\omega(\xi_X) = (\varphi^* \varpi)(\xi_X) = \varpi(\varphi_* \xi_X) = \varpi(\zeta_X) = X, \quad \forall X \in \mathfrak{g}$.

Also, $r_h^* \omega = r_h^* \varphi^* \varpi = \varphi^* r_h^* \varpi = \varphi^*(\mathrm{Ad}(h^{-1}) \cdot \varpi) = \mathrm{Ad}(h^{-1}) \cdot \varphi^* \varpi = \mathrm{Ad}(h^{-1}) \circ \omega$, so that $\omega = \varphi^* \varpi$ is a Cartan connection.

Next we define a correspondence

$$\left\{\begin{matrix}\text{Cartan}\\ \text{connections}\\ \text{on } P\end{matrix}\right\} \xrightarrow{j} \left\{\begin{matrix}\text{Ehresmann connections}\\ \text{on } Q \text{ whose kernels do}\\ \text{not meet } \varphi_*(TP)\end{matrix}\right\}.$$

Given a Cartan connection ω on P, we extend it to a form $\varpi = j(\omega)$ on $P\times G$ by

$$\varpi_{(p,g)} = Ad(g^{-1})\circ \pi_P^*\omega_p + \pi_G^*\vartheta_G|_g$$

where $\pi_P : P\times G \to P$ and $\pi_G : P\times G \to G$ are the canonical projections. We notice that $\varpi(0\times X^L) = X \quad \forall X\in\mathfrak{g}$. Also if $i : P \to P\times G$ sends $p \mapsto (p,e)$, then $i^*\varpi = \omega$. In particular, ϖ does not vanish on $T(P\times\{e\})$. Let $\gamma\in G$ and consider $id\times R_\gamma$ on $P\times G$. Then

$$\begin{aligned}
(id\times R_\gamma)^*\varpi_{(p,g\gamma)} &= \varpi_{(p,g\gamma)}\circ(id\times R_\gamma)_* \\
&= \left(Ad(g\gamma)^{-1}\circ\pi_P^*\omega + \pi_G^*\vartheta_G\right)\circ(id\times R_\gamma)_* \\
&= Ad(g\gamma)^{-1}\circ\omega\circ(\pi_P)_*\circ(id\times R_\gamma)_* + \vartheta_G\circ(\pi_G)_*\circ(id\times R_\gamma)_* \\
&= Ad(g\gamma)^{-1}\circ\omega\circ(\pi_P)_* + \vartheta_G\circ(R_\gamma)_*\circ(\pi_G)_* \\
&= Ad(\gamma)^{-1}\left(Ad(g)^{-1}\circ\pi_P^*\omega + \pi_G^*\vartheta_G\right) \\
&= Ad(\gamma)^{-1}\circ\varpi_{(p,g)}
\end{aligned}$$

We now check that ϖ is basic for $P\times G \to P\times_H G$ which means that it is both horizontal and 'invariant'. The latter condition says $\alpha_h^*\varpi = \varpi$, where $\alpha_h : P\times G \to P\times G$ sends $(p,g)\mapsto(ph,h^{-1}g)$. We calculate

$$\begin{aligned}
(\alpha_h^*\varpi)_{(p,g)} &= \varpi_{(ph,h^{-1}g)}\circ(\alpha_h)_* \\
&= Ad(h^{-1}g)^{-1}\pi_P^*\omega\circ(\alpha_h)_* + \pi_G^*\vartheta_G\circ(\alpha_h)_* \\
&= Ad(h^{-1}g)^{-1}\,\omega\circ(\pi_P)_*\circ(\alpha_h)_* + \vartheta_G\circ(\pi_G)_*\circ(\alpha_h)_* \\
&= Ad\,g^{-1}\circ Ad\,h\circ\omega\circ(R_h)_*\circ(\pi_P)_* + \vartheta_G\circ(L_{h^{-1}})_*\circ(\pi_G)_* \\
\left(R_h^*\omega = Ad(h)^{-1}\circ\omega \text{ \& } \vartheta_G \text{ is LI} \Rightarrow\right) \quad &= Ad\,g^{-1}\circ\pi_P^*\omega + \pi_G^*\vartheta_G \\
&= \varpi_{(p,g)}
\end{aligned}$$

To show ϖ is horizontal, let $X \in \mathfrak{h} \subset \mathfrak{g}$ and $\xi_X \in \mathfrak{X}(P\times G)$ corresponding to the right H action on $P \times G$:

$$\begin{aligned} P\times G\times H &\longrightarrow P\times G \\ (p,g,h) &\longmapsto (ph, h^{-1}g) = \big((\mu_P\times\mu_G)\circ(\mathrm{id}\times\mathrm{id}\times\iota\times\mathrm{id})\circ(\mathrm{id}\times\Delta\times\mathrm{id})\circ\rho\big)(p,g,h) \end{aligned}$$

where $\rho : \begin{array}{l} P\times G\times H \longrightarrow P\times H\times G \\ (p,g,h) \longmapsto (p,h,g) \end{array}$, $\mathrm{id}\times\Delta\times\mathrm{id} : \begin{array}{l} P\times H\times G \longrightarrow P\times H\times H\times G \\ (p,h,g) \longmapsto (p,h,h,g) \end{array}$

$\mathrm{id}\times\mathrm{id}\times\iota\times\mathrm{id} : \begin{array}{l} P\times H\times H\times G \longrightarrow P\times H\times H\times G \\ (p,h,h,g) \longmapsto (p,h,h^{-1},g) \end{array}$ & $\mu_P\times\mu_G : \begin{array}{l} P\times H\times H\times G \longmapsto P\times G \\ (p,h,h^{-1},g) \longmapsto (ph, h^{-1}g) \end{array}$

Then
$$\begin{aligned} (\xi_X)_{(p,g)} &= \big((\mu_P\times\mu_G)\circ(\mathrm{id}\times\mathrm{id}\times\iota\times\mathrm{id})\circ(\mathrm{id}\times\Delta\times\mathrm{id})\circ\rho\big)_{*(p,g,e)}(0,0,X) \\ &= (\mu_P\times\mu_G)_*\circ(\mathrm{id}\times\mathrm{id}\times\iota\times\mathrm{id})_*\circ(\mathrm{id}\times\Delta\times\mathrm{id})_{*(p,e,g)}(0,X,0) \\ &= (\mu_P\times\mu_G)_*(\mathrm{id}\times\mathrm{id}\times\iota\times\mathrm{id})_{*(p,e,e,g)}(0,X,X,0) \\ &= (\mu_P\times\mu_G)_{*(p,e,e,g)}(0,X,-X,0) \\ &= \big((\mu_P)_{*(p,e)}(0,X), (\mu_G)_{*(e,g)}(-X,0)\big) \\ &= \big(\omega_p^{-1}(X), -\vartheta_{G_g}^{-1}(\mathrm{Ad}(g^{-1})X)\big) \end{aligned}$$

$\therefore$

$$\begin{aligned} \varpi_{(p,g)}(\xi_X) &= \varpi_{(p,g)}\big(\omega_p^{-1}(X), -\vartheta_G^{-1}(\mathrm{Ad}(g^{-1})X)\big) \\ &= \big(\mathrm{Ad}(g^{-1})\circ(\pi_P^*\circ\omega) + \pi_G^*\vartheta_G\big)\big(\omega_p^{-1}(X), -\vartheta_G^{-1}(\mathrm{Ad}(g^{-1})X)\big) \\ &= \mathrm{Ad}(g^{-1})X - \mathrm{Ad}(g^{-1})X = 0 . \end{aligned}$$

Therefore ϖ descends to $\varpi \in \Omega^1(P\times_H G, \mathfrak{g})$ and satisfies the properties of an Ehresmann connection which, in addition, obeys $\ker\varpi \cap \varphi_*(TP) = 0$.

Finally we show that φ^* and j are mutual inverses:

$$\begin{aligned} \varphi^*(j(\omega_p)) = \varphi^*\varpi_{(p,e)} &= \mathrm{Ad}(e)^{-1}\circ\varphi^*\pi_P^*\omega_p + \varphi^*\pi_G^*\vartheta_{G_e} \\ &= (\pi_P\circ\varphi)^*\omega_p + 0 \qquad \text{(since } \pi_G\circ\varphi \text{ is constant)} \\ &= \omega_p , \qquad \text{(since } \pi_P\circ\varphi = \mathrm{id}) \end{aligned}$$

so that $\varphi^*\circ j = \mathrm{id}$. To show that $j\circ\varphi^* = \mathrm{id}$, it suffices to show that φ^* is injective. Indeed, if φ^* is injective, then $(j\circ\varphi^*)\varpi = \varpi$ by applying φ^* to both sides & using injectivity.

Now then, φ^* is injective because if $\varphi^*\varpi_1 = \varphi^*\varpi_2$, then ϖ_1 and ϖ_2 agree on the image $\varphi_*(TP)$ and hence on all the right translates $(r_g)_*\varphi_*(TP)$. But they also agree on ξ_X for $X \in \mathfrak{g}$ and those two kinds of vectors span TQ. ■

Additional remarks

Let (P,ω) be a Cartan geometry on M modelled on G/H. Then as remarked earlier, ω defines a parallelism: $\omega^{-1}: \mathfrak{g} \to \mathfrak{X}(P)$, sending $X \in \mathfrak{g} \longmapsto \xi_X \in \mathfrak{X}(P)$. Let's investigate when ω^{-1} is a Lie algebra morphism. Let $X, Y \in \mathfrak{g}$. Then let's calculate $\xi_{[X,Y]} - [\xi_X, \xi_Y]$. We apply ω to both sides to obtain

$$
\begin{aligned}
\omega(\xi_{[X,Y]}) - \omega([\xi_X,\xi_Y]) &= [X,Y] + \big(d\omega(\xi_X,\xi_Y) - \xi_X\,\omega(\xi_Y) + \xi_Y\,\omega(\xi_X)\big) \\
&= [X,Y] + \big(\Omega(\xi_X,\xi_Y) - [\omega(\xi_X),\omega(\xi_Y)]\big) + \cancelto{0}{\xi_X Y} - \cancelto{0}{\xi_Y X} \\
&= [X,Y] + \Omega(\xi_X,\xi_Y) - [X,Y] \\
&= \Omega(\xi_X,\xi_Y).
\end{aligned}
$$

This exhibits the curvature of a Cartan connection as the obstruction to $\mathfrak{X}(P) \xrightarrow{\omega} \mathfrak{g}$ being a LA homomorphism. Notice that from Corollary 50 it follows that if either $X \in \mathfrak{h}$ or $Y \in \mathfrak{h}$, then

$$[\xi_X, \xi_Y] = \xi_{[X,Y]}$$

Indeed, an alternative definition of a Cartan connection on P (of type $\mathfrak{g}/\mathfrak{h}$) as $\omega \in \Omega^1(P;\mathfrak{g})$ s.t. $\omega_p : T_pP \to \mathfrak{g}$ is an isomorphism for all $p \in P$ and such that $[\xi_X,\xi_Y] = \xi_{[X,Y]}$ $\forall X \in \mathfrak{h}, Y \in \mathfrak{g}$ where $\xi : \mathfrak{g} \to \mathfrak{X}(P)$ is such that $\xi_X(p) = \omega_p^{-1}(X)$.