Ehresmann, Kozul, and Cartan connections

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Contents

1 Introduction

These are typset notes based on a small graduate lecture course given by Professor José Figueroa-O'Farrill at the University of Edinburgh in Autumn 2019.

2 Fibre bundles

Definition 2.1 (Fibre Bundle)**.** *A **fibre bundle** consists of a smooth surjection $\pi : E \to M$ between manifolds E (the **total space**) and M (the **base space**) and such that $\forall a \in M$ there exists a neighbourhood $U \ni a$ and a diffeomorphism $\varphi : \pi^{-1} \to U \times F$ (a **local trivialisation**) for some manifold F (the **typical fibre**) such that the following triangle commutes*

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ \downarrow \pi & \swarrow pr_2 & \\ U & & \end{array}$$

We often write $F \to E \xrightarrow{\pi} M$. If we can take $U = M$ we say that E is a **trivial bundle**. Now suppose that $(U, \varphi), (V, \psi)$ are local trivialisations with $U \cap V \neq \emptyset$. Then we have two ways to view $\pi^{-1}(U \cap V)$ as a product.

$$\begin{array}{ccccc} (U \cap V) \times F & \xleftarrow{\psi} & \pi^{-1}(U \cap V) & \xrightarrow{\varphi} & (U \cap V) \times F \\ & \searrow pr_2 & \downarrow \pi & \swarrow pr_2 & \\ & & U \cap V & & \end{array}$$

and hence

$$\psi \circ \varphi^{-1} : (U \cap V) \times F \to (U \cap V) \times F$$
$$(a, p) \mapsto (a, \Phi(a, p))$$

where $\Phi(a, \cdot) : F \to F$ is a diffeomorphism, and hence it defines a **transition function** $g : U \cap V \to Diff(F)$.

Definition 2.2. *Let $F \to E \xrightarrow{\pi} M$. A collection $\{(U_\alpha, \varphi_\alpha)\}$ of local trivialisations where $M = \cup_\alpha U_\alpha$ is called a* ***trivialising atlas*** *for $E \xrightarrow{\pi} M$.*

Let us introduce the notation $U_{\alpha\beta} = U_\alpha \cap U_\beta$, etc. and $g_{\alpha\beta}$ the transition function defined by $\varphi_\alpha \circ \varphi_\beta^{-1}$.

Fact 2.3. *The transition functions satisfy the* ***cocycle conditions***

- $\forall a \in U_\alpha$, $g_{\alpha\alpha}(a) = \mathrm{id}_F$
- $\forall a \in U_{\alpha\beta}$, $g_{\alpha\beta}(a) g_{\beta\alpha}(a) = \mathrm{id}_F$
- $\forall a \in U_{\alpha\beta\gamma}$, $g_{\alpha\beta}(a) g_{\beta\gamma}(a) = g_{\alpha\gamma}(a)$.

Definition 2.4. *Let $E \xrightarrow{\pi} M$, $E' \xrightarrow{\pi'} N$ be fibre bundles. A bundle map is a pair (Φ, ϕ) of smooth maps $\Phi : E \to E'$, $\phi : M \to N$ such that the following commutes*

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & E' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow{\phi} & N \end{array}$$

Since π is surjective, ϕ is uniquely determined by Φ, which is said to ***cover*** *ϕ. Notice that Φ is* ***fibre preserving***.

Definition 2.5. *Let $f : M \to N$ be smooth and $E \xrightarrow{\pi_E} N$ a fibre bundle. Then we can define the* ***pullback bundle*** *$f^*E \to M$ as the categorical pullback, i.e.*

$$f^*E \equiv \{(a, e) \in M \times E \,|\, \pi_E(e) = f(a)\}$$

Restricting the canonical projections from $M \times E$ we get maps $\pi : f^*E \to M$, $\Phi : f^*E \to E$ making the following commute

$$\begin{array}{ccc} f^*E & \xrightarrow{\Phi} & E \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_E} \\ M & \xrightarrow{f} & N \end{array}$$

Taking $a \in M$, and (V, ψ) a local trivialisation for $E \to N$ with $f(a) \in V$, then $(f^{-1}(V), \varphi)$ with $\varphi : \pi^{-1}(f^{-1}(V)) \to f^{-1}(V) \times F$ defined by $\varphi(b, e) = (b, pr_2(\psi(e))$ is a local trivialisation for $f^*E \to M$. This shows that $f^*E \to M$ is a fibre bundle, and it has fibres $(f^*E)_a = E_{f(a)}$.

Definition 2.6. *A* ***section*** *of a fibre bundle $F \to E \xrightarrow{\pi} M$ is a smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$.*

Sections *may not exist*, but if the fibre bundle is trivial, then any smooth map $\sigma : M \to F$ defines a sections by $s(a) = (a, \sigma(a))$. Since fibres are locally trivial, they admit local sections $s_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$ via local smooth maps $\sigma_\alpha : U_\alpha \to F$. A section $s : N \to E$ can be pulled back via $f : M \to N$ to give a section $f^*s : M \to f^*E$ via $(f^*s)(a) = (a, s(f(a)))$.

Definition 2.7. *Consider $F \to E \xrightarrow{\pi} M$. Then the fibres $E_a = \pi^{-1}(a) \subset E$ are submanifolds of E. The tangent space at $e \in E_a$ is $\vartheta_e = \ker((\pi_*)_e : T_eE \to T_eM)$ and is called the* ***vertical subspace*** *of T_eE*

In the absence of any additional structure, there is no preferred complementary subspace of T_eE.

Definition 2.8. *A* ***connection*** *on $E \to M$ is a smooth choice of complementary subspace $\mathscr{H}_e \subset T_eE$ i.e. $T_eE = \vartheta_e \oplus \mathscr{H}_e$. That is, a connection is a distribution $\mathscr{H} \subset TE$*

Note $(\pi_*)_e|_{\mathscr{H}_e} : \mathscr{H}_e \overset{\cong}{\to} T_{\pi(e)}M$, so $\mathscr{H}$ gives a choice of how to lift tangent vectors, and so curves, from M to E.
Given a distribution one can ask whether it is integrable (in the sense of Frobenius), i.e. is E foliated by submanifolds whose tangent spaces are $\mathscr{H}$. We shall see that the obstruction to the integrability of $\mathscr{H}$ can be interpreted as the 'curvature' of the connection.

3 Principal fibre bundles

We now specialise to principal fibre bundles, so called because the typical fibre is a principally homogeneous space for a lie group.

Definition 3.1. *A* ***Lie group*** *consists of a manifold G which is also a group such that group multiplication $G \times G \to G$, $(g,h) \mapsto gh$, and group inversion $G \to G$, $g \mapsto g^{-1}$, are smooth maps*

For $g \in G$ a Lie group, we define diffeomorphisms $L_g : G \to G$, $L_g(h) = gh$, and $R_g : G \to G$, $R_g(h) = hg$, call **left** & **right** multiplication.

Definition 3.2. *Recall that given a diffeomorphism $F : M \to N$ we define the* ***pushforward*** *$F_* : \mathfrak{X}(M) \to \mathfrak{X}(N)$ by, for $\xi \in \mathfrak{X}(M)$, $f \in C^\infty(N)$, $(F_*\xi)(f) = \xi(f \circ F)$.*

Remark. *Note that given any smooth map of manifolds $F : M \to N$, the derivative $dF : TM \to TN$ gives a map $\forall a \in M$, $dF_a : T_aM \to T_{F(a)}N$ which for $\xi \in T_aM$, $f \in C^\infty(N)$ acts as $(dF_a(\xi))(f) = \xi(f \circ F)$. This is often written as F_*, but the two concepts are subtly different.*

Definition 3.3. *A vector field $\xi \in \mathfrak{X}(G)$ is* ***left invariant*** *if $\forall g \in G$, $(L_g)_*\xi = \xi$. Similarly we define right invariant.*

Lemma 3.4. *If ξ is a LIVF, $\xi_g = (L_g)_*\xi_e$, where $e \in G$ is the identity.*

Proof. Let $f \in C^\infty(G)$. Then

$$(L_g)_*\xi = \xi \Rightarrow \xi(f \circ L_g) = \xi(f)$$

Now evaluating at $g \in G$, $\xi_g \in T_gG$ so $\xi_g(f \circ L_g) = ((L_g)_*\xi_e)(f)$. Result follows. □

It can be shown that the lie bracket of two left invariant vector fields is also left invariant.

Definition 3.5. *The vector space of left invariant vector fields is the* ***Lie algebra*** *$\mathfrak{g}$ of G.*

Since a LIVF is uniquely determined by its value at the identity, we have that $\mathfrak{g} \cong T_eG$ as a vector space, but we can also transport the Lie bracket from $\mathfrak{g}$ to T_eG so they are isomorphic as algebras.

Definition 3.6. *The maps $(L_{g^{-1}})_* : T_gG \to T_eG \cong \mathfrak{g}$ define a $\mathfrak{g}$-valued one form θ called the* ***left invariant Maurer-Cartan one-form****. If ξ is a LIVF, $\theta(\xi) = \xi_e$.*

By definition, θ is left invariant.

Theorem 3.7. *The MC one form satisfies the* ***structure equation***

$$d\theta = -\frac{1}{2}[\theta, \theta]$$

i.e. for $\xi, \eta \in \mathfrak{X}(G)$*,* $d\theta(\xi, \eta) = -[\theta(\xi), \theta(\eta)]$

Proof. We will need the following result:

Claim: For $\theta \in \Omega^1(M)$, $X, Y \in \mathfrak{X}(M)$

$$d\theta(X, Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y])$$

To show this take coordinates such that $\theta = \theta_a dx^a, X = X^a \partial_a, Y = Y^a \partial_a$. Then

$$\begin{aligned} d\theta(X, Y) &= (\partial_b \theta_a X^c Y^d)(dx^b \wedge dx^a)(\partial_c, \partial_d) \\ &= \partial_b \theta_a (X^b Y^a - X^a Y^b) \\ &= X^b \partial_b (\theta_a Y^a) - Y^b \partial_b (\theta_a X^a) - \theta_a (X^b \partial_b Y^a - Y^b \partial_b X^a) \\ &= X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y]) \end{aligned}$$

Now if X, Y are LIVFs, $\theta(X), \theta(Y)$ are constant, so on these

$$d\theta(X, Y) + \theta([X, Y]) = 0$$

Moreover for LIVFs $\theta([X, Y]) = [\theta(X), \theta(Y)]$. Now LIVFs span the space of vector fields, and all the operations are linear, so we are done. □

Proposition 3.8. *If G is a matrix Lie group,* $\theta_g = g^{-1} dg$*.*

Proof. In a matrix group, we have the correspondence $X \in \mathfrak{g} \Leftrightarrow \exp(tX) \in G$. Take a basis $\{T_a\}$ of $T_e G$ and give $g \in G$ coordinates x^a if $g = \exp(\sum_a x^a T_a)$. Then let g be constant and take a curve through g, $\gamma : \mathbb{R} \to G$, $\gamma(t) = \exp\left[\sum_a (x^a + t\xi^a) T_a\right]$ with tangent vector $g\left(\sum_a \xi^a T_a\right) \in T_g G$. Under $L_{g^{-1}}$, this is a curve through e with tangent vector $\left(\sum_a \xi^a T_a\right) \in T_e G$. Hence if we write $\xi = \sum_a \xi^a \frac{\partial}{\partial x^a}$ for the the vector generating γ we get

$$\theta_g = \sum_a T_a dx^a = g^{-1} dg$$

□

Every $g \in G$ defines a diffeomorphism $L_g R_{g^{-1}} : G \to G$, $h \mapsto ghg^{-1}$. Since $e = geg^{-1}$ its derivative belongs to $GL(T_e G) = GL(\mathfrak{g})$.

Definition 3.9. *The* ***adjoint representation*** *of G on $\mathfrak{g}$ is given by* $\mathrm{Ad}_g = (L_g)_* (R_g^{-1})_*$

Lemma 3.10. $R_g^* \theta = \mathrm{Ad}_{g^{-1}} \theta$

Proof.

$$\begin{aligned} R_g^* \theta_{hg} &= \theta_{hg} (R_g)_* \\ &= (L_{(hg)^{-1}})_* (R_g)_* \\ &= (L_{g^{-1}})_* (L_{h^{-1}})_* (R_g)_* \\ &= (L_{g^{-1}})_* (R_g)_* (L_{h^{-1}})_* \\ &= \mathrm{Ad}_{g^{-1}} \theta_h \end{aligned}$$

□

Definition 3.11. *The **left action** of a Lie group G on a manifold M is a smooth map $G \times M \to M$, $(g, a) \mapsto ga$ satisfying the axioms $\forall g, h \in G$, $\forall a \in M$*

- $g(ha) = (gh)a$
- $ea = a$

Right action is defined equivalently.

Left and right actions are equivalent if we take $ga = ag^{-1}$.

Definition 3.12. *An action is **transitive** if the G-orbit of any point is M, equivalently $\forall a, binM, \exists g \in G$, $b = ga$*

Definition 3.13. *An action is **free** if the only element which fixes any point is the identity.*

Definition 3.14. *A **G-torsor** (or principally homogeneous G-space) is a manifold M on which G acts freely and transitively*

Given a G-torsor M, any point in M defines a diffeomorphism $g \cong M$, and as such G-torsors are said to be like a Lie group where we have 'forgotten' the identity.

Definition 3.15. *A **principal G-bundle** is a fibre bundle $P \xrightarrow{\pi} M$ together with a smooth rights G-action $(p, g) \mapsto r_g(p)$ which preserves fibres ($\pi \circ r_g = \pi$) and acts freely and transitively.*

It follows that fibres are G-orbits and hence $M = {}^{P}\!/_{G}$. The condition of local triviality now says that the local trivialisation $\pi^{-1}(U) \xrightarrow{\varphi} U \times G$ are G-equivariant, i.e. where $\varphi(p) = (\pi(p), \gamma(p))$, $\gamma : \pi^{-1}(U) \to G$ a G-equivariant ($\gamma \circ r_g = R_g \circ \gamma$) fibrewise diffeomorphism

Definition 3.16. *A principal G-bundle is **trivial** is $\exists$ a G-equivariant diffeomorphism $P \xrightarrow{\psi} M \times G$.*

Proposition 3.17. *A principal G-bundle $P \xrightarrow{\pi} M$ admits a section iff it is trivial*

Proof. If $P \xrightarrow{\pi} M$ is trivial, $\psi : P \to M \times G$ defines a section $s : M \to P$ by $s(a) = \psi^{-1}(a, e)$. Conversely, is s is a section, define ψ by $\psi(p) = (\pi(p), \chi(p))$ where $\chi(p)$ is uniquely defined by $p = s(\pi(p))\chi(p)$. Notice that since $pg = s(\pi(p))\chi(p)g = s(\pi(pg))\chi(p)g$ so $\chi(pg) = \chi(p)g$. □

Example 3.18. *Let G be a Lie group and $H \leq G$ a closed subgroup. Then $G \xrightarrow{\pi} {}^{G}\!/_{H}$ is a principal H-bundle. Therefore homogeneous spaces are examples of principal bundles.*

Since principal fibre bundles are locally trivial, they admit local sections. Let $\{(U_\alpha, \varphi_\alpha)\}$ be a trivialising atlas for $G \to P \xrightarrow{\pi} M$. The canonical local sections $s_\alpha : U_\alpha \to \pi^{-1}(U_\alpha)$ are given by $s_\alpha(a) = \varphi_\alpha^{-1}(a, e)$. On $U_{\alpha\beta}$ we have sections s_α, s_β. Writing $\varphi_\alpha(p) = (\pi(p), g_\alpha(p))$ for $g_\alpha : U_\alpha \to G$ equivariant we have that for $p \in \pi^{-1}(U_{\alpha\beta})$.

$$\begin{aligned}(\pi(p), g_\alpha(p)) = \varphi_\alpha(p) &= (\varphi_\alpha \circ \varphi_\beta^{-1} \circ \varphi_\beta)(p) = (\varphi_\alpha \circ \varphi_\beta^{-1})(\pi(p), g_\beta(p)) \\ \Rightarrow (\pi(p), \underbrace{g_\alpha(p)g_\beta^{-1}(p)}_{\equiv \hat{g}_{\alpha\beta}(p)} g_\beta(p)) &= (\varphi_\alpha \circ \varphi_\beta^{-1})(\pi(p), g_\beta(p))\end{aligned}$$

Note that $\hat{g}_{\alpha\beta}(pg) = g_\alpha(pg)g_\beta^{-1}(pg) = g_\alpha(p)gg^{-1}g_\beta(p) = \hat{g}_{\alpha\beta}(p)$ and so is constant along the fibres. Hence $\exists g_{\alpha\beta} : U_{\alpha\beta} \to G$ s.t. $\hat{g}_{\alpha\beta} = \pi^* g_{\alpha\beta}$ and $(\varphi_\alpha \circ \varphi_\beta^{-1})(a, g) = (a, g_{\alpha\beta}(a)g)$. It follows that the $g_{\alpha\beta}$ obey the cocycle

conditions.
Now note $g_\alpha \circ s_\alpha : U_\alpha \to G$ is a constant map taking value e, and so letting $p = s_\beta(a)$

$$\begin{aligned} g_\alpha(p) = \hat{g}_{\alpha\beta}(p) g_\beta(p) \Rightarrow g_\alpha(s_\beta(a)) &= g_{\alpha\beta}(a)(g_\beta \circ s_\beta)(a) \\ &= (g_\alpha \circ s_\alpha)(a) g_{\alpha\beta}(a) \\ &= g_\alpha(s_\alpha(a) g_{\alpha\beta}(a)) \\ \Rightarrow s_\beta(a) &= s_\alpha(a) g_{\alpha\beta}(a) \quad \text{as } g_\alpha \text{ a diffeomorphism} \end{aligned}$$

4 Ehresmann Connections

Let $P \stackrel{\pi}{\to} M$ be a principal G-bundle. Taking $p \in P$, the derivative $(\pi_*)_p : T_pP \to T_{\pi(p)}M$ is a surjective map.

Definition 4.1. *The kernel V_p is called the* ***vertical subspace****. A vector field $\xi \in \mathfrak{X}(P)$ is called* ***vertical*** *if $\forall p \in P$, $\xi_p \in V_p$.*

Lemma 4.2. *The Lie bracket of two vertical vector fields is vertical*

Lemma 4.3. *The vertical subspaces span a G-invariant integrable distribution*

Proof. Note $\pi \circ r_g = \pi \Rightarrow \pi_*(r_g)_* = \pi_* \Rightarrow (r_h)_* V_p = V_{pg}$ so G-invariant. Integrable by the previous lemma. □

Definition 4.4. *An* ***Ehresmann connection*** *on P is a smooth choice of horizontal subspaces $H_p \subset T_pP$ s.t. $T_pP = V_p \oplus H_p$ and $(r_g)_* H_p = H_{pg}$. Equivalently an Ehresmann connection is a G-invariant distribution $H \subset TP$ complementary to V.*

Example 4.5. *A G-invariant Riemannian metric on P defines an Ehresmann connection by $H_p = V_p^\perp$.*

The G action on P defines a smooth map $\mathfrak{g} \to \mathfrak{X}(P)$ assigning to every $X \in \mathfrak{g}$ the **fundamental vector field** ξ_X defined at $p \in P$ by

$$(\xi_X)_p = \frac{d}{dt}\left(pe^{tX}\right)\bigg|_{t=0}$$

Lemma 4.6. *ξ_X is vertical*

Proof.

$$\pi_* \left.\xi_X\right|_p = \frac{d}{dt}\pi\left(pe^{tX}\right)\bigg|_{t=0} = \frac{d}{dt}\pi\left(p\right)\bigg|_{t=0} = 0$$

□

As the G action is free, $\forall p \in P$ the map $X \mapsto (\xi_X)_p$ is an isomorphism $\mathfrak{g} \stackrel{\cong}{\to} V_p$.

Lemma 4.7. $(r_g)_* \xi_X = \xi_{\mathrm{Ad}_{g^{-1}}(X)}$

Proof.

$$(r_g)_*(\xi_X)_p = \frac{d}{dt} r_g \left(pe^{tX}\right)\Big|_{t=0} = \frac{d}{dt}\left(pe^{tX}g\right)\Big|_{t=0} = \frac{d}{dt}\left(pgg^{-1}e^{tX}g\right)\Big|_{t=0} = \left(\xi_{\mathrm{Ad}_{g^{-1}}(X)}\right)_{pg}$$

□

Definition 4.8. *The **connection one form** of a connection $H \subset TP$ is the $\mathfrak{g}$-valued one form $\omega \in \Omega^1(P;\mathfrak{g})$ defined by*

$$\omega(\xi) = \begin{cases} X & \xi = \xi_X \\ 0 & \xi \in H \end{cases}$$

Proposition 4.9. *The connection one form obeys $r_g^*\omega = \mathrm{Ad}_{g^{-1}} \circ \omega$*

Proof. Let ξ be horizontal. Then $(r_g)_*\xi$ is also horizontal as H G-invariant. Then $(r_g^*\omega)(\xi) = \omega((r_g)_*\xi) = 0$. Note in this case $(\mathrm{Ad}_{g^{-1}} \circ \omega)(\xi) = 0$ too.
Now if $\xi = \xi_X$, $(\mathrm{Ad}_{g^{-1}} \circ \omega)(\xi) = \mathrm{Ad}_{g^{-1}}(X) = \omega(\xi_{\mathrm{Ad}_{g^{-1}}(X)}) = \omega((r_g)_*\xi_X) = (r_g^*\omega)(\xi)$ □

It turn out we also have a converse:

Proposition 4.10. *If $\omega \in \Omega^1(P;\mathfrak{g})$ satisfies $r_g^*\omega = \mathrm{Ad}_{g^{-1}} \circ \omega$ and $\omega(\xi_X) = X$, then $H \equiv \ker \omega$ is a connection on P.*

Now define the pullback of ω along local sections to be $A_\alpha \equiv s_\alpha^*\omega \in \Omega^1(U_\alpha;\mathfrak{g})$.

Proposition 4.11. *Let $\omega_\alpha \equiv \mathrm{Ad}_{g_\alpha^{-1}} \circ \pi^* A_\alpha + g_\alpha^*\theta$ where θ is the LI Maurer-Cartan one form on G. Then $\omega_\alpha = \omega|_{\pi^{-1}U_\alpha}$*

Proof. The proof will have two steps:

Claim: ω and ω_α agree on the image of s_α

Since $\pi \circ s_\alpha = \mathrm{id}|_{U_\alpha}$, $T_pP = \mathrm{Im}(s_\alpha \circ \pi)_* \oplus V_p$ for $p = s_\alpha(a)$. Hence $\forall \xi \in T_pP$, $\exists! \xi^v \in V_p$ s.t. $\xi = (s_\alpha)_*\pi_*\xi + \xi^v$. Then using $g_\alpha(p) = (g_\alpha \circ s_\alpha)(a) = e$

$$\begin{aligned}
\omega_\alpha(\xi) &= (\pi^* s_\alpha^*\omega)(\xi) + (g_\alpha^*\theta_e)(\xi) \text{ (at } p, \mathrm{Ad}_{g_\alpha^{-1}} = \mathrm{id}) \\
&= \omega((s_\alpha)_*\pi_*\xi) + \theta_e((g_\alpha)_*\xi) \\
&= \omega((s_\alpha)_*\pi_*\xi) + \theta_e((g_\alpha)_*\xi^v) \text{ as } (g_\alpha)_*(s_\alpha)_* = (g_\alpha \circ s_\alpha)_* = 0 \\
&= \omega((s_\alpha)_*\pi_*\xi) + \omega(\xi^v) \\
&= \omega(\xi)
\end{aligned}$$

Claim: ω and ω_α transform in the same way under the right G action.

$$\begin{aligned}
r_g^*(\omega_\alpha)_{pg} &= \mathrm{Ad}_{g_\alpha(pg)^{-1}} \circ r_g^*\pi^* s_\alpha^*\omega + r_g^* g_\alpha^*\theta \\
&= \mathrm{Ad}_{(g_\alpha(p)g)^{-1}} \circ r_g^*\pi^* s_\alpha^*\omega + g_\alpha^* R_g^*\theta \\
&= \mathrm{Ad}_{g^{-1}g_\alpha(p)^{-1}} \circ \pi^* s_\alpha^*\omega + g_\alpha^*(\mathrm{Ad}_{g^{-1}} \circ \theta) \\
&= \mathrm{Ad}_{g^{-1}}\left(\mathrm{Ad}_{g_\alpha(p)^{-1}} \circ \pi^* s_\alpha^*\omega + g_\alpha^*\theta\right) \\
&= \mathrm{Ad}_{g^{-1}} \circ (\omega_\alpha)_p
\end{aligned}$$

Hence we are done. □

Now as ω is a global one form, ω_α and ω_β must agree on $U_{\alpha\beta}$, allowing us to relate A_α and A_β, namely on $U_{\alpha\beta}$

$$\begin{aligned}
A_\alpha = s_\alpha^*\omega_\alpha = s_\alpha^*\omega_\beta &= s_\alpha^*\left(\mathrm{Ad}_{g_\beta(s_\alpha)^{-1}} \circ \pi^* A_\beta + g_\beta^*\theta\right) \\
&= \mathrm{Ad}_{g_{\alpha\beta}} \circ A_\beta + g_{\beta\alpha}^*\theta
\end{aligned}$$

Example 4.12. *For matrix Lie groups,* $g_{\beta\alpha}^*\theta = g_{\beta\alpha^{-1}}dg_{\alpha\beta} = -dg_{\alpha\beta}g_{\alpha\beta}^{-1}$*, so*

$$A_\alpha = g_{\alpha\beta}A_\beta g_{\alpha\beta}^{-1} - dg_{\alpha\beta}g_{\alpha\beta}^{-1}$$

Similarly, one can ask how $\{A_\alpha\}$ depends on the choice of local section.

Fact 4.13. *If s'_α is another local section for U_α, $\exists h_\alpha : U_\alpha \to G$ s.t. $s'_\alpha(a) = s_\alpha(a)h_\alpha(a)$ and then*

$$A'_\alpha = \mathrm{Ad}_{h_\alpha^{-1}} \circ A_\alpha + h_\alpha^*\theta$$

Idea. *We now have three different ways to understand connections on a principal G-bundle $P \xrightarrow{\pi} M$, namely;*

1. *a G-invariant horizontal distribution $H \subset TP$*
2. *a one form $\omega \in \Omega^1(P;\mathfrak{g})$ satisfying $\omega(\xi_X) = X$ and $r_g^*\omega = \mathrm{Ad}_{g^{-1}} \circ \omega$*
3. *a family of one forms $\left\{A_\alpha \in \Omega^1(U_\alpha;\mathfrak{g}\right\}$ satisfying $A_\alpha = \mathrm{Ad}_{g_{\alpha\beta}} \circ A_\beta + g_{\beta\alpha}^*\theta$ on $U_{\alpha\beta} \neq \emptyset$*

If $P \xrightarrow{\pi} M$ is a principal G-bundle, G-equivariant bundle diffeomorphisms are called **gauge transformations** and one can ask how an Ehresmann connection transforms. Let $H \subset TP$ be a G-invariant horizontal distribution. Then let $H^\Phi \equiv \Phi_* H$ be the gauge-transformed distribution.

Lemma 4.14. *$H^\Phi \subset TP$ is an Ehresmann connection*

Proof.

$$(r_g)_* H^\Phi_{\Phi(p)} = (r_g)_*\Phi_* H_p = \Phi_*(r_g)_* H_p = \Phi_* H_{pg} = H^\Phi_{(\Phi(pg)} = H^\Phi_{\Phi(p)g}$$

and H^Φ is complementary to V because $\Phi_* T_pP \stackrel{\cong}{\to} T_{\Phi(p)}P$ and Φ_* preserves $V = \ker \pi_*$ because $\pi \circ \Phi = \pi$ □

Exercise 4.15. *Let Φ be a gauge transformation in a principal G-bundle $P \xrightarrow{\pi} M$. Let ξ_X denote a fundamental vector fields for the G-action on P. Show that ξ_X is gauge invariant, i.e. $\Phi_*\xi_X = \xi_X$. Further, show that if ω is the connection one form for an Ehresmann connection H then $(\Phi^{-1})^*\omega$ is the connection one form for H^Φ.*

Let $\{A_\alpha\}$, $\left\{A_\alpha^\Phi\right\}$ be the gauge fields corresponding to the Ehresmann connections H, H^Φ. Since Φ preserves fibres it makes sense to restrict to $\pi^{-1}U_\alpha$. Applying the trivialisation $\varphi_\alpha(\Phi(p)) = (\pi(p), g_\alpha(\Phi(p)))$ which defines $\bar\phi_\alpha : \pi^{-1}U_\alpha \to G$ by $\bar\phi_\alpha(p) = g_\alpha(\Phi(p))g_\alpha(p)^{-1}$.

Lemma 4.16. *$\bar\phi_\alpha$ is constant on the fibres*

Proof.

$$\begin{aligned}
\bar\phi_\alpha(pg) &= g_\alpha(\Phi(pg))g_\alpha(pg)^{-1} \\
&= g_\alpha(\Phi(p)g)g_\alpha(pg)^{-1} \\
&= g_\alpha(\Phi(p))g(g_\alpha(p)g)^{-1} \\
&= g_\alpha(\Phi(p))g_\alpha(p)^{-1} \\
&= \bar\phi_\alpha(p)
\end{aligned}$$

□

Hence $\bar{\phi}_\alpha$ defines a smooth map $\phi_\alpha : U_\alpha \to G$. On overlaps $U_{\alpha\beta} \neq \phi$ we have that $\forall a \in U_{\alpha\beta}$, $p \in \pi^{-1}(a)$, hence

$$\begin{aligned}\phi_\alpha(a) &= g_\alpha(\Phi(p))g_\alpha(p)^{-1} \\ &= g_\alpha(\Phi(p)) \cdot \underbrace{g_\beta(\Phi(p))^{-1}g_\beta(\Phi(p))}_{e}\underbrace{g_\beta(p)^{-1}g_\beta(p)}_{e}g_\alpha(p)^{-1} \\ &= g_{\alpha\beta}(a)\phi_\beta(a)g_{\alpha\beta}(a)^{-1} \text{ since } \pi(p) = \pi(\Phi(p)) = a\end{aligned}$$

Remark. *We will see later that $\{\phi_\alpha\}$ defines a section of a fibre bundle $\mathrm{Ad}\, P$ on M associated to the principal bundle P.*

Exercise 4.17. *Show that on U_α, $A_\alpha^\Phi = \mathrm{Ad}_{\phi_\alpha} \circ (A_\alpha - \phi_\alpha^* \theta) = \phi_\alpha A_\alpha \phi_\alpha^{-1} - d\phi_\alpha \phi_\alpha^{-1}$, which is a gauge transform*

5 Kozul Connections

Definition 5.1. *A real, rank k, **vector bundle** $E \xrightarrow{\pi} M$ is a fibre bundle whose fibres are k-dimensional real vector spaces and whose local tirivialisations $\psi : \pi^{-1}U \to U \times \mathbb{R}^k$ restrict fibrewise to isomorphisms $\psi : E_a \to \{a\} \times \mathbb{R}^k$ of real vector spaces.*

Let $P \xrightarrow{\pi} M$ be a principal G-bundle and let $\rho : G \to GL(V)$ be a Lie group homomorphism (i.e. a representation of G), where V is a f.d. vector space. Since G acts freely on P, it also acts freely on $P \times V$ via the right action

$$(p, v)g = (pg, \rho(g^{-1})v)$$

We let $E \equiv P \times_G V$ denote the quotient ${}^{(P \times V)}\!/_{G}$ via the above action. It is the total space of a vector bundle $E \xrightarrow{\varpi} M$ where

$$\begin{aligned}\varpi : P \times_G V &\to M \\ [(p, v)] &\mapsto \pi(p)\end{aligned}$$

Definition 5.2. *$E \xrightarrow{\varpi} M$ is called an **associated vector bundle** to the PFB $P \to M$, associated via the representation ρ.*

Let $\{(U_\alpha, \varphi_\alpha)\}$ be a trivialising atlas for P with transition function $\{g_{\alpha\beta} : U_{\alpha\beta} \to G\}$ obeying the cocycle conditions. We may then trivialise $P \times_G V$ on each U_α, and the transition functions are $\{\rho \circ g_{\alpha\beta} : U_{\alpha\beta} \to GL(V)\}$. More concretely we define $P \times_G V \equiv {}^{\sqcup_\alpha U_\alpha \times V}\!/_{\sim}$ where $(a, v) \sim (a, \rho(g_{\alpha\beta}(a))v)$
Let $P \xrightarrow{\pi} M$ be a G-PFB and $E \equiv P \times_G V \xrightarrow{\varpi} M$ an associated VB with $\rho : G \to GL(V)$. Let $\Gamma(E) = \{s : M \to E \,|\, \varpi \circ s = \mathrm{id}_M\}$ denote the $C^\infty(M)$-module of sections of E, and $C_G^\infty(P, V) = \{\zeta : P \to V \,|\, \forall g \in G, r_g^* \zeta = \rho(g)^{-1} \circ$ the G-equivariant functions $P \to V$. We can give $C_G^\infty(P, V)$ the structure of a $C^\infty(M)$-module by declaring that for $f \in C^\infty(M)$, $f\zeta = \pi^* f \zeta$

Proposition 5.3. *There is a $C^\infty(M)$-module isomorphism*

$$\Gamma(E) \cong C_G^\infty(P, V)$$

Proof. Let $\sigma \in \Gamma(E)$. Let $\psi_\alpha : \varpi^{-1}U_\alpha \to U_\alpha \times V$ be a local trivialisation and define $\sigma_\alpha : U_\alpha \to V$, $(\psi_\alpha \circ \sigma)(a) = (a, \sigma_\alpha(a))$. On overlaps the local functions $\sigma_\alpha, \sigma_\beta$, are related by $\sigma_\alpha(a) = \rho(g_{\alpha\beta}(a))\sigma_\beta(a)$, where $g_{\alpha\beta}$ are the transition functions of $P \to M$. We now define $\zeta_\alpha : \pi^{-1}U_\alpha \to V$ by $\zeta_\alpha((\pi^* s_\alpha)(p)) = \sigma_\alpha(\pi(p))$ and extend by $\zeta_\alpha((\pi^* s_\alpha)(p)g) = \rho(g)^{-1}\sigma_\alpha(\pi(p))$.

Let $\pi(p) = a \in U_{\alpha\beta}$. Then

$$\begin{aligned}\zeta_\beta(p) = \zeta(s_\alpha(a)g_\alpha(p)) &= \zeta(s_\beta(a)g_{\beta\alpha}(a)g_\alpha(p)) \\ &= \rho(g_{\beta\alpha}(a)g_\alpha(p))^{-1} \circ \sigma_\beta(a) \\ &= \rho(g_\alpha(p))^{-1} \circ \rho(g_{\alpha\beta}(a)) \circ \sigma_\beta(a) \\ &= \rho(g_\alpha(p))^{-1} \circ \sigma_\alpha(a) \\ &= \rho(g_\alpha(p))^{-1}\zeta_\alpha(s_\alpha(a)) \\ &= \zeta_\alpha(s_\alpha(a)g_\alpha(p)) = \zeta_\alpha(p)\end{aligned}$$

The $\{\zeta_\alpha\}$ are constructed to define a function $\zeta : P \to V$ such that $r_g^*\zeta = \rho(g)^{-1} \circ \zeta$. If $f \in C^\infty(M)$, then $f\sigma \in \Gamma(E)$ and $(f\sigma)_\alpha = f\sigma_\alpha$ since ψ_α is fibrewise linear. Then by definition

$$\begin{aligned}\rho(g_\alpha(p))^{-1} \circ \pi^*(f\sigma_\alpha) &= \rho(g_\alpha(p))^{-1} \circ (\pi^* f)(\pi^*\sigma_\alpha) \\ &= (\pi^* f)\rho(g_\alpha(p))^{-1} \circ (\pi^*\sigma_\alpha) \\ &= (\pi^* f)\zeta_\alpha(p)\end{aligned}$$

so the map $\Gamma(E) \to C^\infty_G(P, V)$, thus defined, is $C^\infty(M)$-linear.
Conversely, given a G-equivariant $\zeta : P \to V$, we define $\sigma \in \Gamma(E)$ as follows: let $s_\alpha : U_\alpha \to P$ be the canonical local sections. Then let $\sigma_\alpha = s_\alpha^*\zeta$. For $a \in U_{\alpha\beta}$,

$$\sigma_\beta(a) = \zeta(s_\beta(a)) = \zeta(s_\alpha(a)g_{\alpha\beta}(a)) = \rho(g_{\alpha\beta}(a))^{-1}\zeta(s_\alpha(a)) = \rho(g_{\beta\alpha}(a))\sigma_\alpha(a)$$

□

Example 5.4. *Let ω, ω' be connection one forms for Ehresmann connections $\mathscr{H}, \mathscr{H}'$ on $P \to M$. Then $r_g^*\omega = \mathrm{Ad}_{g^{-1}} \circ \omega$ and similarly for ω'. Now if ξ is vertical, $\omega(\xi) = \omega'(\xi)$, and hence $\tau \equiv \omega - \omega' \in \Omega^1(P; \mathfrak{g})$ is* ***horizontal*** *(i.e. $\tau(\xi) = 0$ if ξ vertical).*
Now let $\tau_\alpha = s_\alpha^*\tau \in \Omega^1(U_\alpha; \mathfrak{g})$. Then $\tau_\alpha = s_\alpha^*\omega - s_\alpha^*\omega' = A_\alpha - A'_\alpha$. On $U_{\alpha\beta}$, $A_\alpha = \mathrm{Ad}_{g_{\alpha\beta}} \circ A_\beta + g_{\beta\alpha}^*\theta$, and likewise for A'_α, $\Rightarrow \tau_\alpha = \mathrm{Ad}_{g_{\alpha\beta}} \circ \tau_\beta$. Hence $\{\tau_\alpha\}$ defines $\tau \in \Omega^1(M; \mathrm{ad}\, P)$ where $\mathrm{ad}\, P \equiv P \times_G \mathfrak{g}$.

Example 5.5. *Take $H \leq G$ closed and $M = {G}/{H}$. Then $G \xrightarrow{\pi} M$ is a principal H-bundle. Let $\rho : H \to GL(V)$ be a representation. Then $E \equiv G \times_H V \to M$ is a* ***homogeneous vector bundle***. *Then $\Gamma(E) \cong \left\{f : G \to V \mid f(ph) = \rho(h)^{-1}f(p)\right\}$ as $C^\infty(M)$-modules. On $\Gamma(E)$ we have a rep of G given by $(g \cdot f)(g_1) = g(g^{-1}g_1)$*

There is a sort of converse to the associated VB construction. If $E \xrightarrow{\pi} M$ is a real rank k vector bundle, we may associate with it a principal $GL(k, \mathbb{R})$-bundle in one of two ways as follows:

1. Let $\{(U_\alpha, \psi_\alpha)\}$ be a trivialising atlas for E, with $\psi_\alpha : \pi^{-1}U_\alpha \to U_\alpha \times \mathbb{R}^k$ and transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to GL(k, \mathbb{R})$. We can then glue $U_\alpha \times GL(k, \mathbb{R})$ and $U_\beta \times GL(k, \mathbb{R})$ along $U_{\alpha\beta}$ by

$$(a, A) \sim (a, g_{\alpha\beta}(a)A)$$

which is equivariant under right multiplication by $GL(k, \mathbb{R})$. The resulting principal $GL(k, \mathbb{R})$-bundle is denoted $GL(E) \xrightarrow{\varpi} M$ and it follows that $E \to M$ is the vector bundle associated to $GL(E)$ view the identity rep

2. The PFB $GL(E) \xrightarrow{\varpi} M$ can understood as the **bundle of frames** of $E \xrightarrow{\pi} M$. Let $GL(E)_a = \{\text{ordered bases for } E_a\}$. Let $u = (u_1, \ldots, u_n)$ be a frame for E_a. Then $\varpi(u) = a$ defines $\varpi : GL(E) \to M$. If $A \in GL(k, \mathbb{R})$, uA defined by $(uA)_i = \sum_j u_j A_{ji}$ is another frame for E_a. Given frames u, u' for E_a, $\exists!\, A \in GL(k, \mathbb{R})$ s.t. $u' = uA$. Let (U, ψ) be a local trivialisation for E. We define a reference frame $\bar{u}(a)$ for each $a \in U$ by $\psi(\bar{u}_i(a)) = (a, e_i)$, where $\{e_i\}$ is the standard bases for $\mathbb{R}^k$. This defines a trivialisation $\Psi : \varpi^{-1}U \to U \times GL(k, \mathbb{R})$ by $\Psi(u) = (a, A(u))$ where u is a frame for E_a and $A(u) \in GL(k, \mathbb{R})$ is the unique element sending u to $\bar{u}(a)$. Now for $B \in GL(k, \mathbb{R})$, we have

$$\bar{u}(a)A(uB) = uB = (\bar{u}(a)A(u))B \Rightarrow A(uB) = A(u)B$$

Hence Ψ is $GL(k,\mathbb{R})$-equivariant. Let $\{(U_\alpha, \Psi_\alpha)\}$ denote the reslting trivialising atlas. Then if $a \in U_{\alpha\beta}$ and u is a frame for E_a, then $\Psi_\alpha(u) = (a, A_\alpha(u))$ where $\bar{u}_\alpha(u)A_\alpha(u) = u$. Now note

$$\begin{aligned}
\bar{u}_\beta(a)_i &= \psi^{-1}(a, e_i) \\
&= \psi_\alpha^{-1} \circ \psi_\alpha \circ \psi_\beta^{-1}(a, e_i) \\
&= \psi_\alpha^{-1}(a, g_{\alpha\beta}(a)e_i) \\
&= \psi_\alpha^{-1}(a, \sum_j e_j (g_{\alpha\beta}(a))_{ji}) \\
&= \sum_j \psi_\alpha^{-1}(a, e_j) g_{\alpha\beta}(a)_{ji} \\
&= \sum_j \bar{u}_\alpha(a)_j g_{\alpha\beta}(a)_{ji} \\
\Rightarrow \bar{u}_\beta(a) &= \bar{u}_\alpha(a) g_{\alpha\beta}(a) \\
\Rightarrow A_\alpha(u) &= g_{\alpha\beta}(a) A_\beta(u)
\end{aligned}$$

Definition 5.6. *Let $E \xrightarrow{\pi} M$ be a vector bundle. A* ***Kozul connection*** *on E is an $\mathbb{R}$-bilinear map*

$$\begin{aligned}
\nabla : \mathfrak{X}(M) \times \Gamma(E) &\to \Gamma(E) \\
(X, s) &\mapsto \nabla_X s
\end{aligned}$$

satisfying that, $\forall f \in C^\infty(M), X \in \mathfrak{X}(M), s \in \Gamma(E)$

1. $\nabla_{fX} s = f \nabla_X s$
2. $\nabla_X(fs) = X(f)s + f\nabla_X s$

Suppose that $E = P \times_G V$ for some G-PFB $P \xrightarrow{\pi} M$ Then an Ehresmann connection on P induces a Kozul connection on E. For this it is convenient to use the $C^\infty(M)$-module isomorphism $\Gamma(E) \cong C^\infty_G(P, V)$ and we will define ∇ on $C^\infty_G(P, V)$:
Let $\mathscr{H} \subset TP$ be an Ehresmann connection. We define $h : T_pP \to T_pP$ to be the projector onto $\mathscr{H}$ along $\ker(\pi_*)$. If we write $\xi \in T_pP$ as $\xi^h + \xi^v$ where $\xi^h \in \mathscr{H}_p$ and $\pi_*(\xi^v) = 0$, then $h(\xi) = \xi^h$. Let $h^* : T_p^*P \to T_p^*P$ be the dual (i.e $(h^*\alpha)(\xi) = \alpha(h(\xi))$). Let $X \in \mathfrak{X}(M)$. Then given $p \in P_a$ let $\xi \in T_pP$ be s.t. $\pi_*\xi = X(a)$. We define $\nabla_X \psi|_p = (d\psi)_p(h\xi)$, i.e. $d^\nabla \psi = h^* d\psi$. This is well defined because if $\pi_*\xi = \pi_*\xi'$, $h\xi = h\xi'$. Further, $\nabla_X \psi \in C^\infty_G(P, V)$ because the split $TP = \mathcal{V} \oplus \mathscr{H}$ is G-invariant, and hence $r_g^* h^* = h^* r_g^*$. Hence

$$\begin{aligned}
r_g^* d^\nabla \psi &= r_g^* h^* d\psi \\
&= h^* r_g^* d\psi \\
&= h^* d(\rho(g)^{-1} \circ \psi) \\
&= \rho(g)^{-1} \circ h^* d\psi = \rho(g)^{-1} d^\nabla \psi
\end{aligned}$$

Proposition 5.7. *∇ defines a Kozul connection on E*

Proof.

$$\begin{aligned}
\nabla_{fX}\psi &= d\psi(h(f\xi)) \\
&= d\psi(h[(\pi^* f)\xi]) \\
&= \pi^* f d\psi(h\xi) \\
&= f\nabla_X\psi \\
\nabla_X(f\psi) &= \nabla_X[(\pi^* f)\psi] \\
&= d\left[(\pi^* f)\psi\right](h\xi) \\
&= (\pi^* df)(h\xi) + (\pi^* f)\nabla_X\psi \\
&= \pi^*(df(\pi_* h\xi))\psi + f\nabla_X\psi \\
&= \pi^*(df(\pi_*\xi))\psi + f\nabla_X\psi \\
&= \pi^*(Xf)\psi + f\nabla_X\psi \\
&= X(f)\psi + f\nabla_X\psi
\end{aligned}$$

□

We will now define a more calculationally useful formula for the Kozul connection of $P \times_G V$ induced by the Ehresmann connection on P. Let $\psi \in C^\infty_G(P, V)$ and let $\xi \in \mathfrak{X}(P)$. We decompose $\xi = h\xi + \xi^v$ where $\pi_*\xi^v = 0$. Then

$$d\psi(h\xi) = d\psi(\xi - \xi^v) = d\psi(\xi) - d\psi(xi^v)$$

The derivative $\xi^v\psi$ only depends on the value of ξ^v at a point, so we can take ξ^v to be the fundamental vector field $\xi_{\omega(\xi^v)} = \xi_{\omega(\xi)}$ corresponding to the G-action. Therefore

$$\begin{aligned}
\xi^v\psi = \xi_{\omega(\xi)}\psi = \; & \frac{d}{dt}\psi \circ r_{\exp(t\omega(\xi))}\bigg|_{t=0} \\
&= \frac{d}{dt}\rho(\exp(-t\omega(\xi))) \circ \psi\bigg|_{t=0} \\
&= -\rho(\omega(\xi)) \circ \psi
\end{aligned}$$

Therefore $d\psi(h\xi) = d\psi(\xi) + \rho(\omega(\xi)) \circ \psi$, or abstracting ξ,

$$d^\nabla\psi = d\psi + \rho(\omega) \cdot \psi$$

Finally, we give a formula for $\nabla_X\sigma$, where $\sigma \in \Gamma(P \times_G V)$, now viewed as a family $\{\sigma_\alpha : U_\alpha \to V\}$ of functions transforming in overlaps as $\sigma_\alpha(A) = \rho(g_{\alpha\beta}(a))\sigma_\beta(a)$;

$$\begin{aligned}
d^\nabla\sigma_\alpha &= d^\nabla s^*_\alpha\psi = d^\nabla(\psi \circ s_\alpha) = d(\psi \circ s_\alpha) \circ h \\
&= d(s^*_\alpha\psi) \circ h = s^*_\alpha(d\psi) \circ h \\
&= s^*_\alpha d^\nabla\psi = s^*_\alpha(d\psi + \rho(\omega) \circ \psi) \\
&= ds^*_\alpha\psi + \rho(s^*_\alpha\omega) \circ s^*_\alpha\psi \\
&= d\sigma_\alpha + \rho(A_\alpha) \circ \sigma_\alpha
\end{aligned}$$

Hence, if $X \in \mathfrak{X}(M)$,

$$\nabla_X\sigma_\alpha \equiv X(\sigma_\alpha) + \rho(A_\alpha(X)) \cdot \sigma_\alpha$$

Exercise 5.8. *Show that $\nabla_X\sigma_\alpha$ transforms like σ_α on overlaps, that is*

$$\nabla_X\sigma_\alpha = \rho(g_{\alpha\beta}) \circ \nabla_X\sigma_\beta$$

*Note this justifies the name **covariant derivative**.*

In summary, given a G-PFB, $P \to M$, and a f.d. rep $\rho : G \to GL(V)$, we construct a VB $P \times_G V \to M$. Every VB is obtained in this way from its frame bundle. We then introduced the notion of a Kozul connection on a VB and showed that an Ehresmann connection on P induces a Kozul connection on $P \times_G V$. The converse is also true: a Kozul connection on E induces an Ehresmann connection on $GL(E)$.

6 Curvature

Let $P \xrightarrow{\pi} M$ be a principal G-bundle and $\rho : G \to GL(V)$ a Lie group homomorphism. Let $E \equiv P \times_G V \xrightarrow{\varpi} M$ be the associated VB. We saw in the last lecture that we have a $C^\infty(M)$-module isomorphism

$$\{s : M \to E \,|\, \varpi \circ s = \mathrm{id}_M\} = \Gamma(E) \cong C^\infty_G(P, V) = \left\{\zeta : P \to V \,|\, r_g^* \zeta = \rho(g^{-1}) \circ \zeta\right\}$$

with module actions $f \cdot \zeta = (\pi^* f)\zeta$.
We wish to generalise this from functions to forms. We define $\Omega^k(P, V)$ to be the k-forms on P with values in V. If $p \in P, \omega \in \Omega^k(P, V)$, then $\omega_p : \Lambda^k T_p P \to V$ is linear. Let $\Omega^k_G(P, V) \subset \Omega^k(P, V)$ denote those V-valued k-forms ω which are both

- **horizontal**: $\forall \xi$ vertical, $i_\xi \omega = 0$
- **invariant**: $\forall g \in G$, $r_g^* \omega = \rho(g^{-1}) \circ \omega$.

Forms $\omega \in \Omega^k(P, V)$ are said to be basic since they come from bundle valued forms on the base. Indeed, we have

Proposition 6.1. *There is an isomorphism of $C^\infty(M)$-modules*

$$\Omega^K_G(P, V) \cong \Omega^k(M, P \times_G V)$$

where for $\omega \in \Omega^k_G(P, V)$, $f \cdot \omega = (\pi^* f)\omega$

Proof. Similar to $k = 0$ case. Define $\sigma \in \Omega^k(M, P \times_G V)$ locally by $\left\{\sigma_\alpha \in \Omega^k(U_\alpha, V)\right\}$ obeying $\sigma_\alpha(a) = \rho(g_{\alpha\beta}(a))\sigma_\beta(a)$. Then $\zeta_\alpha(p) = \rho(g_\alpha(p))^{-1} \circ \pi^* \sigma_\alpha$ is clearly horizontal. It can be shown to be invariant and that $\forall p \in \pi^{-1} U_{\alpha\beta}$, $\zeta_\alpha(p) = \zeta_\beta(p)$. Conversely, if $\zeta \in \Omega^k_G(P, V)$, we define $\sigma_\alpha = s_\alpha^* \zeta$ and one can show that $\forall a \in U_{\alpha\beta}$, $\sigma_\alpha(a) = \rho(g_{\alpha\beta}(a))\sigma_\beta(a)$ □

If $\sigma \in \Gamma(P \times_G V)$, $d^\nabla \sigma_\alpha = \rho(g_{\alpha\beta}) d^\nabla \sigma_\beta$, and hence $d^\nabla \sigma \in \Omega^1(M, P \times_G V)$.

Lemma 6.2. *Let $\alpha \in \Omega^k_G(P, V)$. Then $h^* d\alpha \in \Omega^{k+1}_G(P, V)$.*

Proof. $h^* d\alpha$ is horizontal by construction, so we check invariance;

$$r_g^* h^* d\alpha = h^* r_g^* d\alpha = h^* d(r_g^* \alpha) = h^* d(\rho(g)^{-1} \circ \alpha) = \rho(g)^{-1} \circ h^* d\alpha$$

□

Definition 6.3. *Let $\omega \in \Omega^1(P, \mathfrak{g})$ be the connection one form of an Ehresmann connection $\mathscr{H} \subset TP$. Its **curvature** is $\Omega \equiv h^* d\omega$.*

Lemma 6.4. $\Omega \in \Omega^2_G(P, V)$.

Proof. Horizontal by construction, and by the same calculation as the lemma above it is invariant because ω is. □

Proposition 6.5. $\Omega = 0$ *iff $\mathscr{H} \subset TP$ is (Frobenius) integrable.*

Proof. we see

$$\begin{aligned}\Omega(\xi, \eta) = d\omega(h\xi, h\eta) &= h\xi \underbrace{\omega(h\eta)}_{=0} - h\eta \underbrace{\omega(h\xi)}_{=0} - \omega([h\xi, h\eta]) \\ &= \omega([h\xi, h\eta])\end{aligned}$$

Hence

$$\begin{aligned}\Omega = 0 &\Leftrightarrow \forall \xi, \eta\ [h\xi, h\eta] \text{ is horizontal}\\ &\Leftrightarrow [\mathscr{H}, \mathscr{H}] \subset \mathscr{H}\\ &\Leftrightarrow \mathscr{H} \subset TP \text{ is integrable.}\end{aligned}$$

□

Proposition 6.6 (Structure equation)**.** $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$

Proof. We need to show $\Omega(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$.
Let ξ, η be horizontal. Then $h\xi = \xi$ and $h\eta = \eta$, . hence $\Omega(\xi, \eta) = d\omega(\xi, \eta)$ and $\omega(\xi) = 0 = \omega(\eta)$.
Let η be horizontal and $\xi = \xi_X$ be vertical. Then $h\xi = 0$, $h\eta = \eta$, and $\omega(\eta)$. Hence we need

$$0 = d\omega(\xi_X, \eta) = -\eta\omega(\xi_X) - \omega([\xi_X, \eta]) = -\underbrace{\eta X}_{=0} - \omega([\xi_X, \eta])$$

i.e that $[\xi_X, \mathscr{H}] \subset \mathscr{H}$. This is the case as $\mathscr{H}$ is invariant.
Let $\xi = \xi_X, \eta = \xi_Y$ vertical. Then $h\xi_X = 0 = h\xi_Y$ and $\omega(\xi_X), \omega(\xi_Y) = Y$. So we must show that

$$\begin{aligned}0 &= d\omega(\xi_X, \xi_Y) + [\omega(\xi_X), \omega(\xi_Y)]\\ &= \xi_X Y - \xi_Y X - \omega([\xi_X, \xi_Y]) + [X, Y]\\ &= -\omega(\xi_{[X,Y]}) + [X, Y]\end{aligned}$$

so done. □

Corollary 6.7 (Bianchi Identity)**.** $h^* d\Omega = 0$

Proof.

$$h^* d\Omega = h^* d(d\omega + \frac{1}{2}[\omega, \omega]) + h^*[d\omega, \omega] = [h^* d\omega, h^*\omega] = 0$$

since $h^*\omega = 0$ □

Let's define $d^\nabla : \Omega^k_G(P, V) \to \Omega^{k+1}_G(P, V)$ by $d^\nabla = h^* d$. Then, unlike d, d^∇ need not be a differential, and the obstruction is the curvature:

Proposition 6.8. $\forall \alpha \in \Omega^k_G(P, V)$, $d^\nabla(d^\nabla \alpha) = \rho(\Omega) \wedge \alpha$

Proof.

$$\begin{aligned}d^\nabla \alpha &= d\alpha + \rho(\omega) \wedge \alpha\\ \Rightarrow d^\nabla(d^\nabla \alpha) &= d(d\alpha + \rho(\omega) \wedge \alpha) + \rho(\omega) \wedge (d\alpha + \rho(\omega) \wedge \alpha)\\ &= \rho(d\omega) \wedge \alpha - \rho(\omega) \wedge d\alpha + \rho(\omega) \wedge d\alpha + \rho(\omega) \wedge \rho(\omega) \wedge \alpha\\ &= \rho(d\omega) \wedge \alpha + \frac{1}{2}[\rho(\omega), \rho(\omega)] \wedge \alpha\\ &= \rho(d\omega + \frac{1}{2}[\omega, \omega]) \wedge \alpha\\ &= \rho(\Omega) \wedge \alpha\end{aligned}$$

□

Exercise 6.9. *Write $F_\alpha = s_\alpha^*\Omega$. Express F_α in terms of $A_\alpha = s_\alpha^*\omega$ and relate F_α, F_β on $U_{\alpha\beta} \neq \emptyset$*

7 Homogeneous spaces and Invariant Connections I

Let G be a Lie group acting transitively on a manifold M, Pick $a \in M$ and let $H \subset G$ be the stabiliser subgroup. It is a closed subgroup, and then $M \cong {}^{G}\!/_{H}$, where the diffeomorphism is G-equivariant and $G \circlearrowright {}^{G}\!/_{H}$ is induced by left multiplication in G. If $g \in G$, we let $\phi_g : M \to M$ denote the corresponding diffeomorphism. If $X \in \mathfrak{g}$, we define a vector field $\xi_X \in \mathfrak{X}(M)$ by

$$(\xi_X f)(m) = \frac{d}{dt} f\left(\phi_{\exp(-tX)}(m)\right)\Big|_{t=0}$$

Then $[\xi_X, \xi_Y] = \xi_{[X,Y]}$.
Since H stabilises $a \in M$, $\forall h \in H$, $(\phi_h)_* : T_aM \to T_aM$, and we get a Lie group homomorphism $\lambda : H \to GL(T_aM)$ called the **linear isotropy representation**. We will use the same notation for the induced Lie algebra rep $\lambda : \mathfrak{h} \to \mathfrak{gl}(T_aM)$. Evaluating at $a \in M$, we get a surjective linear map $\mathfrak{g} \to T_aM$, $X \mapsto \xi_X|_a$ whose kernel is $\mathfrak{h}$.

Definition 7.1. *We say that ${}^{G}\!/_{H}$ is* ***reductive*** *if the short exact sequence*

$$0 \to \mathfrak{h} \to \mathfrak{g} \to T_aM \to 0$$

splits as H-modules. In other words if $\exists \mathfrak{m} \subset \mathfrak{g}$ such that $\mathfrak{g} \oplus \mathfrak{m}$ and $\forall h \in H$, $\mathrm{Ad}_h : \mathfrak{m} \to \mathfrak{m}$. In that case $T_aM \cong \mathfrak{m}$ as H-modules.

If $g \in G$ and $\phi_g \in \mathrm{Diff}(M)$, we define $\phi_g \cdot f = f \circ \phi_{g^{-1}}$ and $\phi_g \cdot \xi = (\phi_g)_*\xi$ where

$$((\phi_g)_*\xi)_a = ((\phi_g)_*)_{\phi_g^{-1}(a)} \xi_{\phi_g^{-1}(a)}$$

It follows that

$$\phi_g \cdot (Xf) = (\phi_g \cdot X)(\phi_g \cdot f)$$
$$\phi_g \cdot (fX) = (\phi_g \cdot f)(\phi_g \cdot X)$$

Now let ∇ be an affine connection, (i.e. $\nabla_{fX}Y = f\nabla_X Y$, $\nabla_X(fY) = X(f)Y + f\nabla_X Y$) . Let $\phi \in \mathrm{Diff}(M)$. Define ∇^ϕ by

$$\nabla^\phi_X Y = \phi \cdot \nabla_{\phi^{-1}\cdot X}(\phi^{-1} \cdot Y)$$

Lemma 7.2. ∇^ϕ *is an affine connection*

Proof.

$$\begin{aligned}
\nabla^\phi_{fX} Y &= \phi \cdot \nabla_{\phi^{-1}\cdot(fX)}(\phi^{-1}Y) \\
&= \phi \cdot \nabla_{(\phi^{-1}\cdot f)(\phi^{-1}\cdot X)}(\phi^{-1}Y) \\
&= \phi \cdot \left(\phi^{-1} \cdot f \nabla_{\phi^{-1}\cdot X}(\phi^{-1} \cdot Y\right) \\
&= (\phi \cdot \phi^{-1} f)(\phi \cdot \nabla_{\phi^{-1}X}(\phi^{-1} \cdot Y) \\
&= f\nabla^\phi_X Y \\
\nabla + X^\phi(fY) &= \phi \cdot \left(\nabla_{\phi^{-1}\cdot X}\phi^{-1}(fY)\right) \\
&= \phi \cdot \left(\nabla_{\phi^{-1}\cdot X}(\phi^{-1}f)(\phi^{-1}Y)\right) \\
&= \phi \cdot \left((\phi^{-1}X)(\phi^{-1}f)(\phi^{-1}Y) + (\phi^{-1}f)\nabla_{\phi^{-1}X}(\phi^{-1}Y)\right) \\
&= (\phi \cdot \phi^{-1} \cdot X(f))(\phi \cdot \phi^{-1} \cdot Y) + (\phi \cdot \phi^{-1} \cdot f)\nabla_{\phi^{-1}X}(\phi^{-1}Y) \\
&= X(f)Y + f\nabla^\phi_X Y
\end{aligned}$$

□

Definition 7.3. *An affine connection* ∇ *on a reductive homogeneous* $M = {}^{G}\!/_{H}$ *is said to be* ***G-invariant*** *if* $\forall g \in G$, $\nabla^{\phi_g} = \nabla$. *i.e*

$$\phi_g \cdot \nabla_X Y = \nabla_{\phi_g X}(\phi_g Y)$$

If $H = \{e\}$, $M = G$, *then* ∇ *is* ***left invariant*** *if*

$$L_g \cdot \nabla_X Y = \nabla_{L_g \cdot X}(L_g \cdot Y)$$

Suppose that X, Y are left invariant , so that $L_g \cdot X = X, L_g \cdot Y = Y$. In that case, the left invariance of ∇ implies that $\nabla_X Y$ is also left invariant. Now, on a Lie group we may trivialise the tangent bundle via left translations. That means that we have a global fram $(X_1, \ldots, X_n)$ consisting of left invariant vector fields. The connection is therefor uniquely determined by n^3 numbers Γ_{ij}^k defined by

$$\nabla_{X_i} X_j = \sum_k \Gamma_{ij}^k X_k$$

These are the components relative to the basis $\{X_i\}$ of a linear map $\Lambda : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. The torsion and curvature tensors are also left-invariant and are given in terms of Λ by

$$T(X,Y) = \Lambda_X Y - \Lambda_Y X - [X,Y]$$
$$R(X,Y)Z = [\Lambda_X, \Lambda_Y] Z - \Lambda_{[X,Y]} Z$$

for LI $X, Y, Z \in \mathfrak{X}(G)$. We see that curvature measure the failure of Λ to be a Lie algebra homomorphism. In particular, taking $\Lambda = 0$, we see tat there exists a flat connection with torsion given by $T(X,Y) = -[X,Y]$ relative to which LI vf on G are **parallel** (i.e. $\nabla X = 0$). Of course, there exists another flat connection annihilating the right-invariant vector fields.

8 Invariant Connections

What did we do last time? We were looking at Homogeneous spaces $M \equiv {}^{G}\!/_{H}$, $H \leq G$ a closed subgroup. We had fibre

$$H \to G \xrightarrow{\pi} M$$

and for $g \in G$ we have $\phi_g : M \to M$ acting by multiplication, i.e. $\phi_g(a) = g \cdot a$. As a result of the quotient have $eH = o \in M$ s.t

$$\forall h \in H \ \phi_h(o) = o$$

Then

$$(\phi_h)_* : T_o M \to T_o M$$

Hence we may make the following def:

Definition 8.1. *The* ***linear isotropy representation***

$$\lambda : H \to GL(T_o M)$$

is given by $\lambda_h = (\phi_h)_*$.

We also have the map

$$\xi : \mathfrak{g} \to \mathfrak{X}(M)$$
$$X \mapsto \xi_X$$

s.t. $[\xi_X, \xi_Y] = \xi_{[X,Y]}$. Composing with evaluation yields

$$ev_o \circ \xi : \mathfrak{g} \to T_o M$$

where $\ker(ev_o \circ \xi) = \mathfrak{h} \subset \mathfrak{g}$. This is bijective so in fact

$$T_oM \cong \mathfrak{g}/\mathfrak{h}$$

and we get commuting diagram

$$\begin{array}{ccc} T_oM & \xrightarrow{\lambda_h} & T_oM \\ \cong\downarrow & & \downarrow\cong \\ \mathfrak{g}/\mathfrak{h} & \xrightarrow{\mathrm{Ad}(h)} & \mathfrak{g}/\mathfrak{h} \end{array}$$

Definition 8.2. *An affine connection ∇ on TM is* ***G-invariant*** *if*

$$\forall g \in G,\ \nabla^{\phi_g} = \nabla$$
$$\phi_g \nabla_\xi \eta = \nabla_{\phi_g \xi}(\phi_g \eta)$$

If $H = \{e\}, M = G$, then ∇ is left invariant if for all left invariant vector fields ξ_X, ξ_Y $\nabla_{\xi_X}\xi_Y$ is also LI. This is then uniquely determined by its value at e. Hence ∇ defines a bilinear map

$$\mathfrak{g} \times \mathfrak{g} \xrightarrow{\alpha} \mathfrak{g}$$
$$(X, Y) \mapsto \nabla_{\xi_X}\xi_Y\big|_e$$

We can then **curry** a map as given $\alpha : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ we can get

$$\Lambda : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$$
$$X \mapsto \Lambda_X$$

where $\Lambda_X(Y) = \alpha(X, Y)$.

Exercise 8.3. *Show that the torsion T and curvature R of Λ are left invariant and given by*

$$T(X, Y) = \Lambda_X Y - \Lambda_Y X - [X, Y]$$
$$R(X, Y)Z = [\Lambda_X, \Lambda_Y]\, Z - \Lambda_{[X,Y]} Z$$

Note R is the obstruction to Λ being a Lie algebra homomorphism.

Claim: $\exists$ a LI connection ∇ corresponding to $\Lambda = 0$.

Such Λ is flat, but has torsion $T(X, Y) = -[X, Y]$. As such ∇ is characterised by $\forall$ LI ξ, $\nabla\xi = 0$. Now let $H \neq \{e\}$ be closed and reductive: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ where $\mathrm{Ad}_H(\mathfrak{m}) \subset \mathfrak{m}$. Note $\mathfrak{m} \cong \mathfrak{g}/\mathfrak{h}$, so in the previous case $\mathfrak{m} \cong T_oM$

Aside. *There is a "holonomy principle" that*

$$\left\{G\text{-invariant tensor fields on } G/H\right\} \overset{ev_o}{\leftrightarrow} \{\mathrm{Ad}(H)\text{-invariant tensor on } \mathfrak{m}\}$$

This comes about, as if we take a tensor T at o, we can define a tensor field on G/H by

$$\mathcal{T}(a) = \phi_g T$$

for and $g \in G$ s.t. $\phi_g o = a$. Then if we have another representative g' then

$$g^{-1}g' \in H \Leftrightarrow \phi_{g^{-1}g'} o = o$$

so

$$T = \phi_{g^{-1}g'} T$$

Claim: An invariant connection ∇ is determined by a bilinear map

$$\alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$$

which is H invariant, the **Nomizu map**.

We can take natural coordinates for M in the neighbourhood $V \subset \mathfrak{m}$ of o by exponentiating $\mathfrak{m}$. The projection π is a local diffeo on $U = \exp(V)$.
In a basis for $\mathfrak{m}$, $\{e_i\}$,

$$\begin{aligned} V &\to U \\ \sum x^i e_i &\mapsto \exp\left(\sum x^i e_i\right) \end{aligned}$$

Now $\forall g \in U$, $\pi(g) = \phi_g \cdot o$, Let $\overline{V} = \{\phi_g \cdot o | g \in U\}$. For $X \in \mathfrak{m}$ define $\xi_X \in \mathfrak{X}(\overline{V})$ by

$$\begin{aligned} (\xi_X)_{\phi_g o} &\equiv ((\phi_g)_*)_o(\pi_*)_e X \\ &= ((\phi_g \circ \pi)_*)_e X \\ &= ((\pi \circ L_g)_*)_e X = (\pi_*)_g X^L_g \end{aligned}$$

where X^L is the LIVF defined by $X^L\big|_e = X$. Hence ξ_X is π-related to X^L. Then $[\xi_X, \xi_Y]$ is π-related to $\left[X^L, Y^L\right] = [X, Y]^L$.
Now let $W \subset V$ s.t. $\forall h \in H$, $\mathrm{Ad}_h W \subset V$, and def $\overline{W}$ accordingly. Then for $h \in H$, $\phi_h : \overline{W} \to \overline{V}$. As such

$$\begin{aligned} \phi_h \phi_g \cdot o &= \phi_h \phi_g \phi_{h^{-1}} \phi_h \cdot o \\ &= \phi_{hgh^{-1}} \cdot o \in \overline{V}. \end{aligned}$$

We will now need the following lemma

Lemma 8.4. *$\forall g \in \exp(W)$, $h \in H$,*

$$(\phi_h)_* \xi_X = \xi_{\mathrm{Ad}_h X}$$

at $\phi_g o$, i.e. at all point in $\overline{V}$.

Proof.

$$\begin{aligned} [(\phi_h)_* \xi_X]_{\phi_h \phi_g o} &= (\phi_h)_* (\xi_X)_{\phi_g o} \\ &= (\phi_h)_* (\phi_g)_* \pi_* X \\ &= (\phi_{hg})_* \pi_* X \\ &= (\phi_{hgh^{-1}})_* (\phi_h)_* \pi_* X \\ &= (\xi_{\mathrm{Ad}(h) X})_{\phi_{hgh^{-1}} o} = (\xi_{\mathrm{Ad}_h X})_{\phi_h \phi_g o} \end{aligned}$$

recalling the commuting diagram

$$\begin{array}{ccc} T_o M & \xrightarrow{(\phi_h)_*} & T_o M \\ {\scriptstyle \pi_*} \big\uparrow & & \big\uparrow {\scriptstyle \pi_*} \\ \mathfrak{m} & \xrightarrow{\mathrm{Ad}(h)} & \mathfrak{m} \end{array}$$

□

Lemma 8.5. *Let $X, Y \in \mathfrak{m}$, and $\xi_X, \xi_Y \in \mathfrak{X}(\overline{V})$. Then $[\xi_X, \xi_Y]\big|_o = \pi_* [X, Y]_{\mathfrak{m}}$*

Proof. We saw above that $[\xi_X, \xi_Y]$ is π-related to $\left[X^L, Y^L\right] = [X, Y]^L$. Hence $[\xi_X, \xi_Y] = \xi_{[X,Y]}$ and evaluating at $o \in M$ gives

$$[\xi_X, \xi_Y]\big|_o = \xi_{[X,Y]_{\mathfrak{h}}}\Big|_o + \xi_{[X,Y]_{\mathfrak{m}}}\Big|_o = \xi_{[X,Y]_{\mathfrak{m}}}\Big|_o = \pi_* [X, Y]_{\mathfrak{m}}$$

□

Theorem 8.6 (Nomizu)**.** *There is a bijective correspondence*

$$\{G\text{-invariant affine connections on } M\} \leftrightarrow \{\mathrm{Ad}(h)\text{-invariant bilinear maps } \alpha : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}\}$$

given by $\alpha(X,Y) = \nabla_{\xi_X}\xi_Y|_o$

Note $\exists!$ G-invariant connection ∇ with $\alpha = 0$, and this is called the **canonical connection**. If you curry this map again you can show

$$\begin{aligned} T(X,Y) &= \alpha(X,Y) = \alpha(Y,X) - [X,Y]_{\mathfrak{m}} \\ R(X,Y)Z &= \alpha(X,\alpha(Y,Z)) - \alpha(Y,\alpha(X,Z)) - \alpha([X,Y]_{\mathfrak{m}}, Z) - [X,Y]_{\mathfrak{h}} Z \end{aligned}$$

If $\alpha = 0$ we get

$$\begin{aligned} T(X,Y) &= -[X,Y]_{\mathfrak{m}} \\ R(X,Y) &= -[X,Y]_{\mathfrak{h}} \end{aligned}$$

If $T = 0$, M is said to be **symmetric**.

9 Cartan Connections

Again consider homogeneous reductive spaces

$$\begin{array}{ccc} H & \longrightarrow & G \\ & & \downarrow \pi \\ & & M = G/H \end{array}$$

With a local section $\sigma : U \to G$ we can pull back the LI MC 1-form $\vartheta_G \in \Omega^1(G;\mathfrak{g})$

$$\sigma^*\vartheta_G \in \Omega^1(U;\mathfrak{g})$$

Recall the MC 1-form satisfies structure equation s

$$d\vartheta_G + \frac{1}{2}[\vartheta_G, \vartheta_G] = 0$$

Then given two such sections σ_i we have

$$\forall a \in U,\ \sigma_2(a) = \sigma_1(a)h(a)$$

for some $h : U \to H$, a uniquely defined function.

Lemma 9.1.

$$\sigma_2^*\vartheta_G = \mathrm{Ad}(h^{-1}) \cdot \sigma_1^*\vartheta_G + h^*\vartheta_H$$

Proof. We will notationally use the idea of matrix groups but in general the proof works. Then

$$\sigma^*\vartheta_g = \sigma^{-1}d\sigma\,.$$

Then

$$\begin{aligned} \sigma_2^*\vartheta_G &= \sigma_2^{-1}d\sigma_2 \\ &= (\sigma_1 h)^{-1} d(\sigma_1 h) \\ &= h^{-1}\sigma_1^{-1}(d\sigma_1 h + \sigma_1 dh) \\ &= h^{-1}(\sigma_1^{-1}d\sigma_1)h + h^{-1}dh \end{aligned}$$

so done. □

As we are in the reductive case, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and we can decompose. Write $\sigma_1^* \vartheta_G = \theta_1 + \omega_1$ for $\theta_1 \in \Omega^1(U, \mathfrak{m})$, $\omega_1 \in \Omega^1(U, \mathfrak{h})$. Then

$$\theta_2 + \omega_2 = \mathrm{Ad}(h)^{-1}(\theta_1 + \omega_1) + h^* \vartheta_H$$

so

$$\begin{aligned} \theta_2 &= \mathrm{Ad}(h)^{-1}\theta_1 \\ \omega_2 &= \mathrm{Ad}(h)^{-1}\omega_1 + h^*\vartheta_H \end{aligned}$$

decomposing. Hence θ_2 transforms as a tensor, ω_2 as a gauge field. Now if we let $\sigma = \sigma_1$ and the structure equation becomes

$$\begin{aligned} d(\theta + \omega) + \frac{1}{2}[\theta + \omega, \theta + \omega] &= 0 \\ d\theta + d\omega + \frac{1}{2}[\theta, \theta] + \frac{1}{2}[\omega, \omega] + [\omega, \theta] &= 0 \end{aligned}$$

As such decomposing

$$\begin{aligned} d\theta + \frac{1}{2}[\theta, \theta]_{\mathfrak{m}} + [\omega, \theta] &= 0 & \Rightarrow & & \Theta \equiv d\theta + [\omega, \theta] &= -\frac{1}{2}[\theta, \theta]_{\mathfrak{m}} \\ d\omega + \frac{1}{2}[\theta, \theta]_{\mathfrak{h}} + \frac{1}{2}[\omega, \omega] &= 0 & \Rightarrow & & \Omega \equiv d\omega + \frac{1}{2}[\omega, \omega] &= -\frac{1}{2}[\theta, \theta]_{\mathfrak{h}} \end{aligned}$$

As such

$$\begin{aligned} \Theta(\xi_X, \xi_Y) &= -[X, Y]_{\mathfrak{m}} \\ \Omega(\xi_X, \xi_Y) &= -[X, Y]_{\mathfrak{h}} \end{aligned}$$

Gauge fields for the canonical invariant connection on G/H are $\sigma^* \vartheta_G$.
With this motivation with us, the Cartan connections are going to be generalisation of these where in the gauge field descriptions these are local 1-forms on the base. The Cartan viewpoint is to view TM not as a linear rep of $GL(n, \mathbb{R})$, but as a homogeneous space of the affine group $\mathbb{A}(n, \mathbb{R}) \cong GL(n, \mathbb{R}) \ltimes \mathbb{R}^n$ such that $T_a M \cong \mathbb{A}(n, \mathbb{R}) / GL(n, \mathbb{R})$.

Definition 9.2. *A* ***Cartan gauge*** *(def from Sharpe, Jose doesn't like) with model G/H on M is a pair (U, θ) where $U \subset M$ open and $\theta \in \Omega^1(U, \mathfrak{g})$ satisfying* ***regularity***

$$T_a M \xrightarrow{\theta_a} \mathfrak{g} \xrightarrow{pr} \mathfrak{g}/\mathfrak{h}$$

is an isomorphism $\forall a \in U$.

This is the analogue of a chart

Definition 9.3. *A* ***Cartan atlas*** *is a collection of Cartan gauges $\{(U_\alpha, \theta_\alpha)\}$ s.t*

- $\bigcup_\alpha U_\alpha = M$
- *on $U_{\alpha\beta}$*

$$\theta_\beta = \mathrm{Ad}(h_{\alpha\beta}^{-1})\theta_\alpha + h_{\alpha\beta}^* \vartheta_H$$

for some $h_{\alpha\beta} : U_{\alpha\beta} \to H$.

This is very analogous to atlases.

Definition 9.4. *Two atlases are* ***equivalent*** *if their union is an atlas.*

Definition 9.5. *A **Cartan structure** on M is an equivalence class (equivalently maximal atlas) of Cartan atlases. A **Cartan geometry** is a manifold M together with a Cartan structure.*

Definition 9.6. *The **curvature** of a Cartan gauge (U, θ) is $\Omega \in \Omega^2(U, \mathfrak{g})$ given by*

$$\Omega = d\theta + \frac{1}{2}[\theta, \theta]$$

If I have a Cartan atlas, I can ask how respective curvatures Ω_α change on overlaps.

Lemma 9.7. *On $U_{\alpha\beta}$*

$$\Omega_\beta = \mathrm{Ad}(h_{\alpha\beta}^{-1})\Omega_\alpha$$

Proof.

$$\begin{aligned}
&\theta_\beta = \mathrm{Ad}(h_{\alpha\beta}^{-1})\theta_\alpha + h_{\alpha\beta}^*\vartheta_H \\
\Rightarrow d\theta_\beta + \frac{1}{2}[\theta_\beta, \theta_\beta] &= d\left(\underbrace{\mathrm{Ad}(h_{\alpha\beta}^{-1})\theta_\alpha}_{h_{\alpha\beta}^{-1}\theta_\alpha h_{\alpha\beta}} + h_{\alpha\beta}^*\vartheta_H\right) + \frac{1}{2}\left[\mathrm{Ad}(h_{\alpha\beta}^{-1})\theta_\alpha + h_{\alpha\beta}^*\vartheta_H, \mathrm{Ad}(h_{\alpha\beta}^{-1})\theta_\alpha + h_{\alpha\beta}^*\vartheta_H\right] \\
&= \mathrm{Ad}(h_{\alpha\beta}^{-1})d\theta_\alpha - \left[\mathrm{Ad}(h_{\alpha\beta}^{-1})\theta_\alpha, h_{\alpha\beta}^*\vartheta_H\right] - \frac{1}{2}h_{\alpha\beta}^*[\vartheta_H, \vartheta_H] + \frac{1}{2}\mathrm{Ad}(h_{\alpha\beta}^{-1})[\theta_\alpha, \theta_\alpha] \\
&+ \frac{1}{2}\left[h_{\alpha\beta}^*\vartheta_H, h_{\alpha\beta}^*\vartheta_H\right] + \left[\mathrm{Ad}(h_{\alpha\beta}^{-1})\theta_\alpha, h_{\alpha\beta}^*\vartheta_H\right] \\
&= \mathrm{Ad}(h_{\alpha\beta}^{-1})\left(d\theta_\alpha + \frac{1}{2}[\theta_\alpha, \theta_\alpha]\right)
\end{aligned}$$

□

Hence setting $\Omega_\alpha = 0$ is an *extrinsic* statement of an atlas.

Definition 9.8. *A Cartan structure is **flat** if $\forall \alpha$, $\Omega_\alpha = 0$*

Example 9.9. *Flat Cartan structures:*

- *$G \to G/H$ with $(U_\alpha, \sigma_\alpha^*\vartheta_G)$*
- *an open subset $V \subset G/H$ as above.*
- *$\Gamma \subset G$ acting by covering transformations, locally like G/H.*

Definition 9.10. *A Klein geometry G/H has **kernel** K: the largest subgroup of H that is normal in G. If $K = 1$ we say that G/H is **effective**. If K is discrete we say the geometry is **locally effective**.*

Lemma 9.11. *If $K \neq 1$ then $(G/K)/(H/K)$ is effective.*

Proposition 9.12. *If G/H is effective, and $\exists k : U \to H$ s.t. $\theta = \mathrm{Ad}(k^{-1}) \cdot \theta + k^*\vartheta_H$, then $k = 1$.*

This means that, given a Cartan atlas $\{(U_\alpha, \theta_\alpha)\}$ modelled on an effective ${}^{G}\!/\!_{H}$, then in overlaps $U_{\alpha\beta}$, $\theta_\beta = \mathrm{Ad}(h_{\alpha\beta}^{-1}) \circ \theta_\alpha + h_{\alpha\beta}^* \vartheta_H$ for a unique $h_{\alpha\beta} : U_{\alpha\beta} \to H$. Indeed if $\theta_\beta = \mathrm{Ad}(\tilde{h}_{\alpha\beta}^{-1}) \circ \theta_\alpha + \tilde{h}_{\alpha\beta}^* \vartheta_H$, then letting $k = \tilde{h}_{\alpha\beta}^{-1} h_{\alpha\beta}$ we would have

$$\begin{aligned} \theta_\alpha &= \mathrm{Ad}(\tilde{h}_{\beta\alpha}^{-1}) \circ \theta_\beta + \tilde{h}_{\beta\alpha}^* \vartheta_H \\ \Rightarrow \theta_\beta &= \mathrm{Ad}(h_{\alpha\beta}^{-1}) \circ \left[\mathrm{Ad}(\tilde{h}_{\beta\alpha}^{-1}) \circ \theta_\beta + \tilde{h}_{\beta\alpha}^* \vartheta_H\right] + h_{\alpha\beta}^* \vartheta_H \\ &= \mathrm{Ad}(k^{-1}) \circ \theta_\beta + \underbrace{\mathrm{Ad}(h_{\alpha\beta}^{-1}) \circ \tilde{h}_{\beta\alpha}^* \vartheta_H + h_{\alpha\beta}^* \vartheta_H}_{k^* \vartheta_H} \end{aligned}$$

It follows from uniqueness then that $\{h_{\alpha\beta} : U_{\alpha\beta} \to H\}$ defines a (Cech) cocycle. Therefore they are the transition functions of a principle H-bundle $P \overset{\pi}{\to} M$, where $P = \sqcup_\alpha (\{\alpha\} \times U_\alpha \times H)/\sim$, $(\alpha, a, h) \sim (\beta, a, h_{\alpha\beta}^{-1}(a)h)$, and $\pi(\alpha, a, h) = a$. The right action is given by $r_h[(\alpha, a, \tilde{h})] = [(\alpha, a, \tilde{h}h)]$. This is well defined since the identification uses left multiplication.

Let $X \in \mathfrak{h}$. Then $X^L \in \mathfrak{X}(H)$ is the corresponding LIVF. We extend it to $U \times H$ as $(0, X^L) \equiv \xi_X \in \mathfrak{X}(U \times H)$. Since X^L is LI and the identifications involve left multiplication the vector fields ξ_X glue to give a well defined vector field $\xi_X \in \mathfrak{X}(P)$. We then have

Lemma 9.13. *Let $r_h : P \to P$ denote the right action of $h \in H$ on P. Then $\forall X \in \mathfrak{h}$, $(r_h)_* \xi_X = \xi_{\mathrm{Ad}(h)^{-1} X}$.*

Proof. It is sufficient to check locally on $U \times H$. Here $r_h = \mathrm{id} \times R_h$ where $R_h : H \to H$ is right multiplication by h. Let $L_h : H \to H$ be left multiplication and then on $U \times H$ we have

$$\begin{aligned} (r_h)_* \xi_X &= (\mathrm{id} \times R_h)_* (0, X^L) \\ &= (0, (R_h)_* X^L) \\ &= (0, (r_h)_* (L_{h^{-1}})_* X^L) \text{ since } X^L \text{ is LI} \\ &= (0, (\mathrm{Ad}(h^{-1}) \cdot X)^L) \\ &= \xi_{\mathrm{Ad}(h)^{-1} X} \end{aligned}$$

□

The Cartan atlas $(U_\alpha, \theta_\alpha)$ does not first just give $P \overset{\pi}{\to} M$, but also a one-form $\omega \in \Omega^1(P; \mathfrak{g})$ defined locally by

$$\begin{aligned} \omega : T_{(a,h)}(U_\alpha \times H) &\to T_a U_\alpha \times \mathfrak{h}) \to \mathfrak{g} \\ (v, y) &\mapsto (v, \vartheta_H(y)) \mapsto \mathrm{Ad}(h^{-1}) \theta_\alpha(v) + \vartheta_H(y) \equiv \omega_\alpha(v, y) \end{aligned}$$

On overlaps, we also have $\omega_\beta(v, y) = \mathrm{Ad}(h^{-1}) \theta_\beta(v) + \vartheta_H(y)$. The transition function is then $U_{\alpha\beta} \times H \overset{f_{\alpha\beta}}{\to} U_{\alpha\beta} \times H$ sending $(a, h) \mapsto (a, h_{\alpha\beta}(a)^{-1} h)$.

We will claim that the ω_α glue together properly to give a consistent ω. To prove this we will need a preparatory lemma:

Lemma 9.14. *Let $\mu : H \times H \to H$ and $i : H \to H$ denote multiplication and inversion as groups maps on H. Letting $\vartheta_H \in \Omega^1(H; \mathfrak{h})$ be the LI MC one-form we have*

$$\begin{aligned} \forall v \in T_{(h_1, h_2)}(H \times H),\ (\mu^* \vartheta_H)(v) = \mathrm{Ad}(h_2^{-1}) \vartheta_H((pr_1)_* v) + \vartheta_H((pr_2)_* v) \\ \forall v \in T_h H,\ (i^* \vartheta_H)(v) = -\mathrm{Ad}(h) \vartheta_H(v) \end{aligned}$$

Proof. It is simpler notationally for matrix groups where $\vartheta_H|_h = h^{-1} dh$. Hence

$$i^* \vartheta_H|_h = h dh^{-1} = -h h^{-1} dh h^{-1} = -\mathrm{Ad}(h)\ \vartheta_H|_h$$

Moreover we have

$$\mu^*\ \vartheta_H|_{(h_1, h_2)} = (h_1 h_2)^{-1} d(h_1 h_2) = h_2^{-1} h_1^{-2} dh_1 h_2 + + h_2^{-1} dh_2 = \mathrm{Ad}(h_2^{-1})\ \vartheta_H|_{h_1} + \vartheta_H|_{h_2}$$

□

Now we are ready to state what we want:

Proposition 9.15. *The following diagram commutes:*

$$\begin{array}{ccc} T_aU_{\alpha\beta} \times T_hH & \xrightarrow{(f_{\alpha\beta})_*} & T_aU_{\alpha\beta} \times T_{h_{\alpha\beta}(a)^{-1}h}H \\ & {\scriptstyle\omega_\alpha}\searrow \quad \swarrow{\scriptstyle\omega_\beta} & \\ & \mathfrak{g} & \end{array}$$

Proof. We notice that $f_{\alpha\beta}(a,h) = (a, h_{\alpha\beta}(a)^{-1}h) = (\mathrm{id} \circ pr_1, \mu \circ (i \circ h_{\alpha\beta} \circ pr_1 \times pr_2))(a,h)$, so that if $(v,y) \in T_aU_{\alpha\beta} \times T_hH$, $(f_{\alpha\beta})_*(v,y) = (v, \mu_*(i_* \circ (h_{\alpha\beta})_*v, y)) \in T_aU_{\alpha\beta} \times T_{h_{\alpha\beta}(a)^{-1}h}H$. Hence

$$\begin{aligned} (\omega_\beta \circ (f_{\alpha\beta})_*)(v,y) &= \omega_\beta(v, \mu_*(i_* \circ (h_{\alpha\beta})_*v, y)) \\ &= \mathrm{Ad}(h_{\alpha\beta}(a)^{-1}h)^{-1}\theta_\beta(v) + \vartheta_H(\mu_*(i_* \circ (h_{\alpha\beta})_*v, y)) \end{aligned}$$

Using the lemma we have that

$$\begin{aligned} \vartheta_H(\mu_*(i_* \circ (h_{\alpha\beta})_*v, y)) &= (\mu^*\vartheta_H)(i_*(h_{\alpha\beta})_*v, y) \\ &= \mathrm{Ad}(h^{-1})\vartheta_H(i_*(h_{\alpha\beta})_*v) + \vartheta_H(y) \\ \vartheta_H(i_*(h_{\alpha\beta})_*v) &= (i^*\vartheta_H)(h^*_{\alpha\beta}v) \\ &= -\mathrm{Ad}(h_{\alpha\beta}(a))(h^*_{\alpha\beta}\vartheta_H)(v) \end{aligned}$$

Hence

$$\begin{aligned} (\omega_\beta \circ (f_{\alpha\beta})_*)(v,y) &= \mathrm{Ad}(h)^{-1}\mathrm{Ad}(h_{\alpha\beta}(a))\theta_\beta(v) - \mathrm{Ad}(h)^{-1}\mathrm{Ad}(h_{\alpha\beta}(a))(h^*_{\alpha\beta}\vartheta_H)(v) + \vartheta_H(y) \\ &= \mathrm{Ad}(h)^{-1}\mathrm{Ad}(h_{\alpha\beta}(a))\left[\theta_\beta(v) - (h^*_{\alpha\beta}\vartheta_H)(v)\right] + \vartheta_H(y) \\ &= \mathrm{Ad}(h)^{-1} \circ \theta_\alpha(v) + \vartheta_H(y) \\ &= \omega_\alpha(v,y) \end{aligned}$$

□

Definition 9.16. *The one-form $\omega \in \Omega^1(P;\mathfrak{g})$ is called a **Cartan connection***

Proposition 9.17. *The Cartan connection $\omega \in \Omega^1(P;\mathfrak{g})$ obeys the following:*

1. *$\forall p \in P$, $\omega_p : T_pP \to \mathfrak{g}$ is a vector space isomorphism*
2. *$\forall h \in H$, $r^*_h\omega = \mathrm{Ad}(h^{-1}) \circ \omega$*
3. *$\forall X \in \mathfrak{h}$, $\omega(\xi_X) = X$*

Proof. We may separate the proof:

1. $\dim P = \dim H + \dim M = \dim \mathfrak{h} + \dim \mathfrak{g}/\mathfrak{h} = \dim \mathfrak{g}$, so it suffices to show that ω_p is injective. Now if $(v,y) \in T_aU \times T_hH$ is such that $\omega(v,y) = \mathrm{Ad}(h^{-1})\theta(v) + \vartheta_H(y) = 0$, we have $\mathrm{Ad}(h^{-1})\theta(v) = -\vartheta_H(y) \in \mathfrak{h}$ and hence $\theta(v) \in \mathrm{Ad}(h)\mathfrak{h} = \mathfrak{h} \Rightarrow pr_{\mathfrak{g}/\mathfrak{h}}\theta(v) = 0$. By the regularity property of θ, $v = 0$. Hence $\vartheta_H(y) = 0$, but as ϑ_H is injective, we have $y = 0$
2. It is sufficient to check in a Cartan gauge (U,θ). Let $(v,y) \in T_aU \times T_hH$. Then for $k \in H$:

$$(r^*_k\omega)(v,y) = \omega(v, (R_k)_*y) = \mathrm{Ad}(hk)^{-1} \circ \theta(v) + \vartheta_H((R_k)_*y)$$

and using $R_k^*\vartheta_H = \mathrm{Ad}(k^{-1}) \circ \vartheta_H$

$$\begin{aligned}(r_k^*\omega)(v,y) &= \mathrm{Ad}(k^{-1})\,\mathrm{Ad}(h)^{-1}\theta(v) + \mathrm{Ad}(k^{-1})\vartheta_H(y)\\ &= \mathrm{Ad}(k^{-1})\left[\mathrm{Ad}(h)^{-1}\theta(v) + \vartheta_H(y)\right]\\ &= \mathrm{Ad}(k^{-1})\omega(v,y)\end{aligned}$$

3. In a Cartan chart $\xi_X = (0, X^L) \in \mathfrak{X}(U \times H)$, hence

$$\omega(\xi_X) = \mathrm{Ad}(h)^{-1}\theta(0) + \vartheta_H(X^L) = 0 + X = X$$

□

Remark. *Properties 2 and 3 are reminiscent of an Ehresmann connection except that ω takes values in $\mathfrak{g}$ not $\mathfrak{h}$.*

Notice that if $\{(U_\alpha, \theta_\alpha)\}$ is a Cartan atlas trivialising P, then if $s_\alpha : U_\alpha \to P|_{U_\alpha}$ are the canonical sections, $s_\alpha(a) = [(a,e)]$, $(s_\alpha^*\omega)(v) = \omega(v,0) = \theta_\alpha(v)$. So θ_α are the 'gauge fields' of the Cartan connection. Let $\Omega = d\omega + \frac{1}{2}[\omega,\omega] \in \Omega^2(p;\mathfrak{g})$ denote the **curvature** of the Cartan connection. Then $s_\alpha^*\Omega = d\theta_\alpha + \frac{1}{2}[\theta_\alpha,\theta_\alpha]$. Hence bundle automorphisms of P (covering the identity) are the **gauge symmetries** of the Cartan geometry.

Remark. *ω parallelises P, just like ϑ_G parallelises G in the Klein model. Given $X \in \mathfrak{g}$ we get a vector field $\xi_X \in \mathfrak{X}(P)$ defined by $\xi_X|_p = \omega_p^{-1}(X)$, but unlike the case of (G, ϑ_G). this is not a Lie algebra morphism. This is despite that for $X \in \mathfrak{h}, Y \in \mathfrak{g}$ we do have $[\xi_X, \xi_Y] = \xi_{[X,Y]}$. The curvature ω is the obstruction to $X \mapsto \xi_X$ defining a Lie algebra morphism $\mathfrak{g} \to \mathfrak{X}(P)$. To see this, calculate*

$$\begin{aligned}\omega(\xi_{[X,Y]} - \omega([\xi_X,\xi_Y]) &= [X,Y] + (d\omega(\xi_X,\xi_Y) - \xi_X\omega(\xi_Y) + \xi_Y\omega(\xi_X))\\ &= [X,Y] + (d\Omega(\xi_X,\xi_Y) - [\omega(\xi_X),\omega(\xi_Y)]) + \xi_X Y - \xi_Y X\\ &= [X,Y] + \Omega(\xi_X,\xi_Y) - [X,Y]\\ &= \Omega(\xi_X,\xi_Y)\end{aligned}$$

We can now give the standard definition of a Cartan geometry modelled on a Klein geometry:

Definition 9.18. *A **Cartan geometry** (P,ω) on M modelled on G/H consists of the following:*

1. *a principal H-bundle $P \to M$*
2. *$\omega \in \Omega^(P;\mathfrak{g})$ satisfying*
   (a) *$\forall p \in P\, \omega_p : T_pP \to \mathfrak{g}$ is a vector space isomorphism*
   (b) *$\forall h \in H$, $r_h^*\omega = \mathrm{Ad}(h^{-1})\omega$*
   (c) *$\forall X \in \mathfrak{h}$, $\omega(\xi_X) = X$*

Definition 9.19. *Let $\Omega = d\omega + \frac{1}{2}[\omega,\omega] \in \Omega^2(P;\mathfrak{g})$ be the curvature of ω. Then projection $pr_{\mathfrak{g}/\mathfrak{h}} \circ \Omega \in \Omega^2(P;\mathfrak{g}/\mathfrak{h})$ is the **torsion** of ω. The Cartan geometry is **torsion free** if $\Omega \in \Omega^2(P;\mathfrak{h})$*

Lemma 9.20. *Let (P,ω) be a Cartan geometry on M modelled on G/H. Let $\psi : P \to H$ be a smooth and $f : P \to P$ be such that $f(p) = r_{\psi(p)}(p)$. Then $f^*\omega = \mathrm{Ad}(\psi^{-1})\omega + \psi^*\vartheta_H$ and $f^*\Omega = \mathrm{Ad}(\psi) \circ \Omega$.*

Proof. The expression for $f^*\Omega$ follows from that of $f^*\omega$. To calculate $f^*\omega$, we work relative to a Cartan gauge (U,θ) on $U \times H$. Then $f : U \times H \to U \times H$ by $f(a,h) = (a, h\psi(a,h))$ can be written as $f =$

$(\mathrm{id} \circ pr_1, \mu \circ (pr_2 \times \psi))$. Hence if $(v, y) \in T_aU \times T_hH$

$$\begin{aligned}
f_*(v,y) &= (v, \mu_*(y, \psi_*(v,y))) \in T_aU \times T_{h\psi(a,h)}H \\
\Rightarrow (f^*\omega)(v,y) &= \omega(v, \mu_*(y, \psi_*(v,y))) \\
&= \mathrm{Ad}(h\psi(a,h))^{-1} \circ \theta(v) + \vartheta_H(\mu_*(y, \psi_*(v,y))) \\
&= \mathrm{Ad}(\psi^{-1}) \circ \mathrm{Ad}(h^{-1}) \circ \theta(v) + (\mu^*\vartheta_H)(y, \psi_*(v,y)) \\
&= \mathrm{Ad}(\psi^{-1}) \circ \mathrm{Ad}(h^{-1}) \circ \theta(v) + \mathrm{Ad}(\psi^{-1}) \circ \vartheta_H(y) + \vartheta_H(\psi_*(v,y)) \\
&= \mathrm{Ad}(\psi^{-1}) \circ \left[\mathrm{Ad}(h^{-1}) \circ \theta(v) + \vartheta_H(y)\right] + (\psi^*\vartheta_H)(v,y) \\
&= \left[\mathrm{Ad}(\psi^{-1}) \circ \omega + \psi^*\vartheta_H\right](v,y)
\end{aligned}$$

□

Corollary 9.21. *Ω is horizontal, i.e. if either u, v are tangent to the fibre, $\Omega(u,v) = 0$.*

Proof. Let $u, v \in T_pP$ and v tangent to the fibre. Let $\psi : P \to H$ be any smooth map sending $p \mapsto e$ s.t. $(\psi_*)_p v = -\omega_p(v) \in \mathfrak{h}$. define $f : P \to P$ by $f(q) = q \cdot \psi(q)$. Then from the previous lemma we have that $p \in P$

$$\begin{aligned}
f^*\omega &= \mathrm{Ad}(\psi^{-1})\omega + \psi^*\vartheta_H = \omega + \psi^*\vartheta_H \\
f^*\Omega &= \Omega
\end{aligned}$$

Hence

$$\begin{aligned}
\omega_p(f_*v) &= \omega_p(v) + \vartheta_H(\psi_*v) = \omega_p(v) - \omega_p(v) = 0 \\
\Rightarrow f_*v &= 0 \\
\Rightarrow \Omega(u,v) &= \Omega(f_*u, f_*v) = \Omega(f_*u, 0) \qquad = 0
\end{aligned}$$

□

It follows that Ω defines a 2-form on ${}^{TP}\!/_{\ker \pi_*} \cong \pi^*TM$.
Note that each fibre F of P is identified with H up to left multiplication by some element of H. Since ϑ_H is left-invariant, it defines a "Maurer-Cartan" form ϑ_F on the fibre. The fact that $\forall X \in \mathfrak{h}$, $\vartheta_F(\xi_X) = X$ shows that $\vartheta_F = \omega|_F$. It then follows that Ω vanishes when restricted to any fibre. As such we can interpret a Cartan geometry (P, ω) as deforming (G, ϑ_G) in a way that fibrewise we still have (H, ϑ_H).
The tangent bundle of ${}^{G}\!/_{H}$ is a vector bundle associated to $G \to {}^{G}\!/_{H}$ via the linear isotropy representation $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}} : H \to GL(\mathfrak{g}/\mathfrak{h})$ s.t. $T({}^{G}\!/_{H}) \cong G \times_h \mathfrak{g}/\mathfrak{h}$. In a similar way, the tangent bundle of a Cartan geometry (P, ω) modelled on ${}^{G}\!/_{H}$ is isomorphic to an associated vector bundle $P \times_H \mathfrak{g}/\mathfrak{h}$.

Proposition 9.22. *Let (P, ω) be a Cartan geometry on M modelled on ${}^{G}\!/_{H}$. There is a canonical bundle isomorphism $\varphi : TM \overset{\cong}{\to} P \times_H \mathfrak{g}/\mathfrak{h}$ such that $\forall p \in \pi^{-1}(x)$, $\exists \varphi_p : T_xM \to \mathfrak{g}/\mathfrak{h}$ a H-equivariant vector space isomorphism s.t. $\forall h \in H$, $\varphi_{p \cdot h} = \mathrm{Ad}(h^{-1}) \circ \varphi_p$*

Proof. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_p(F_x) & \longrightarrow & T_pP & \xrightarrow{(\pi_*)_p} & T_xM & \longrightarrow & 0 \\
 & & \cong\big\downarrow \vartheta_H & & \cong\big\downarrow \omega & & \cong\big\downarrow \exists!\varphi_p & & \\
0 & \longrightarrow & \mathfrak{h} & \longrightarrow & \mathfrak{g} & \xrightarrow{\rho} & \mathfrak{g}/\mathfrak{h} & \longrightarrow & 0
\end{array}$$

If $v \in T_xM$, we may write $v = (\pi_*)_p(u) = (\pi_*)_{ph}((r_h)_*u)$ for some $u \in T_pP$. Thus

$$\begin{aligned}\varphi_{ph}(v) &= \varphi_{ph}((\pi_*)_{ph}((r_h)_*u))\\ &= \rho(\omega_{ph}((r_h)_*u))\\ &= \rho(\mathrm{Ad}(h)^{-1} \circ \omega_p(u))\\ &= \mathrm{Ad}(h)^{-1}(\varphi_p((\pi_*)_p u))\\ &= \mathrm{Ad}(h)^{-1}\varphi_p(v)\end{aligned}$$

This allows us to define a bundle map

$$\begin{aligned}q : P \times \mathfrak{g} &\to TM\\ (p, X) &\mapsto (\pi(p), \varphi_p^{-1}(\rho(X)))\end{aligned}$$

Then

$$\begin{aligned}q(ph, \mathrm{Ad}(h)^{-1}X) &= (\pi(ph), \varphi_{ph}^{-1}(\rho(\mathrm{Ad}(h)^{-1}X)))\\ &= (\pi(p), (\mathrm{Ad}(H)\varphi_{ph})^{-1}\rho(X))\\ &= (\pi(p), \varphi_p^{-1}(\rho(X)))\\ &= q(p, X)\end{aligned}$$

Hence q induces $\bar{q} : P \times_H \mathfrak{g}/\mathfrak{h} \to TM$, which covers the identity and is a linear iso on the fibres. □

Corollary 9.23. *Let (P, ω) be a Cartan geometry on M modelled on G/H. Then vector fields $\xi \in \mathfrak{X}(M)$ are in bijective correspondence with functions $\bar{\xi} : P \to \mathfrak{g}/\mathfrak{h}$ such that $\forall p \in P, h \in H$, $\bar{\xi}(ph) = \mathrm{Ad}(h^{-1}) \circ \bar{\xi}(p)$ by*

$$\xi \mapsto \bar{\xi} = \left\{p \in P :\mapsto \varphi_p(\xi_{\pi(p)}) \in \mathfrak{g}/\mathfrak{h}\right\}$$

Definition 9.24. *The **curvature function** $K : P \to \mathrm{Hom}(\Lambda^2 \mathfrak{g}/\mathfrak{h}, \mathfrak{g})$ of a Cartan connection ω is defined by*

$$\forall p \in P,\ \forall X, Y \in \mathfrak{g},\ K(p)(X, Y) \equiv \Omega_p(\omega_p^{-1}(X), \omega_p^{-1}(Y))$$

Lemma 9.25. *The curvature function is well defined and is H-equivariant, i.e.*

$$\forall h \in H,\ K(ph)(X, Y) = \mathrm{Ad}(h^{-1})K(p)(\mathrm{Ad}(h)X, \mathrm{Ad}(h)Y)$$

Proof. Fix $p \in P$ and let $\tilde{X} = X + W$, $\tilde{Y} = Y + Z$ for some $W, Z \in \mathfrak{h}$. Then $\Omega_p(\omega_p^{-1}(\tilde{X}), \omega_p^{-1}(\tilde{Y})) = \Omega_p(\omega_p^{-1}(X), \omega_p^{-1}(Y))$ since $\omega_p^{-1}(Z), \omega_p^{-1}(W)$ are tangent to the fibres and Ω is horizontal. Therefore $K(p) \in \mathrm{Hom}(\Lambda^2\mathfrak{g}/\mathfrak{h}, \mathfrak{g})$. The equivariance follows from the equivariance of ω, Ω. □

It follows that the curvature of a Cartan connection defines a **curvature section** of the bundle $P \times_H \mathrm{Hom}(\Lambda^2\mathfrak{g}/\mathfrak{h}, \mathfrak{g})$.

Proposition 9.26. *A Cartan connection is torsion free iff the curvature function takes values in $\mathrm{Hom}\left(\Lambda^2\mathfrak{g}/\mathfrak{h}, \mathfrak{h}\right) \subset \mathrm{Hom}\left(\Lambda^2\mathfrak{g}/\mathfrak{h}, \mathfrak{g}\right)$.*

Exercise 9.27. *Show that $K(p)(X, Y) = [X, Y] - \omega_p(\left[\omega_p^{-1}X, \omega_p^{-1}Y\right])$*

Lemma 9.28 (Bianchi identity)**.** $d\Omega = [\Omega, \omega]$

Proof. Follows *Mutatis Mutandis* as for Ehresmann connections. □

Let V be a vector space and $f : P \to V$ a function. A Cartan connection $\omega \in \Omega^1(P; \mathfrak{g})$ defines a universal covariant derivative as follows: if $X \in \mathfrak{g}$ and if $\xi_X = \omega^{-1}(X)$, then $\tilde{D}_X f \equiv \xi_X f$. Since this is linear in $X \in \mathfrak{g}$, we get

$$\begin{aligned} \tilde{D} : \Omega^0(P; V) &\to \Omega^0(P; V \otimes \mathfrak{g}^*) \\ f &\mapsto \tilde{D} f \end{aligned}$$

where we define $\tilde{D}f$ by $(i_X)_* \tilde{D} f = \tilde{D}_X f$ for

$$\begin{aligned} i_X : V \otimes \mathfrak{g}^* &\to V \\ v \otimes \eta &\mapsto \eta(X) v \end{aligned}$$

Definition 9.29. *Let $\rho : H \to GL(V)$ be a representation. We define*

$$\Omega^k(P; \rho) \equiv \left\{ \alpha \in \Omega^k(P; V) \,|\, \forall h \in H,\, r_h^* \alpha = \rho(h^{-1}) \circ \alpha \right\}$$

*the **k-forms on** P **transforming according to** ρ.*

Proposition 9.30. $\tilde{D} : \Omega^0(P; \rho) \to \Omega^1(P; \rho) \cong \Omega^0(P; \rho \otimes \mathrm{Ad}^*)$

Proof. Let $p \in P$, $X \in \mathfrak{g}$, $f \in \Omega^0(P; \mathfrak{g})$. Then

$$\begin{aligned} (i_X)_*(r_h^*(\tilde{D} f))(p) &= (i_X)_*(\tilde{D} f(ph)) \\ &= (\tilde{D}_X f)(ph) \\ &= \omega_{ph}^{-1}(X) f \end{aligned}$$

Now $r_h^* \omega = \mathrm{Ad}(h^{-1}) \circ \omega \Rightarrow \omega_{ph} \circ (r_h)_* = \mathrm{Ad}(h^{-1}) \circ \omega_p \Rightarrow (r_{h^{-1}})_* \circ \omega_{ph}^{-1} = \omega_p^{-1} \circ \mathrm{Ad}(h)$ so

$$(i_X)_*(r_h^*(\tilde{D} f))(p) = \left[(r_h)_* \omega_p^{-1}(\mathrm{Ad}(h) X) \right] f$$

If $Y \in \mathfrak{X}(P)$ we have

$$((r_h)_* Y) f = Y(r_h^* f) = Y(\rho(h^{-1}) \cdot f) = \rho(h^{-1}) Y f$$

so taking $Y = \omega_p^{-1}(\mathrm{Ad}(h) X)$ yields

$$\left[(r_h)_* \omega_p^{-1}(\mathrm{Ad}(h) X) \right] f = \rho(h^{-1}) \omega_p^{-1}(\mathrm{Ad}(h) X) f = \rho(h^{-1}) \tilde{D}_{\mathrm{Ad}(h) X} f$$

and so

$$(i_X)_*(r_h^* \tilde{D} f)(p) = \rho(h^{-1}) \tilde{D}_{\mathrm{Ad}(h) X} f$$

□

Even if (V, ρ) is irreducible, $(V \otimes \mathfrak{g}^*, \rho \otimes \mathrm{Ad}^*)$ need not be. Decomposing $V \otimes \mathfrak{g}^*$ into irreducibles decomposes $\tilde{D}$ and in this way we get 'famous' differential operators such as $\partial, \bar{\partial}, \boldsymbol{\nabla} \cdot, \boldsymbol{\nabla} \times$.

Lemma 9.31. *Let $X \in \mathfrak{h}$ and $f \in \Omega^0(P; \mathfrak{g})$. Then $(i_X)_* \tilde{D} f = -\rho_*(X) f$ where $\rho_* : \mathfrak{h} \to \mathrm{End}(V)$ is the LA hom induced by $\rho : H \to GL(V)$*

Proof.

$$
\begin{aligned}
(i_X)_*(\tilde{D}f)(p) &= \omega_p^{-1}(X)f \\
&= \frac{d}{dt} f(pe^{tX})\Big|_{t=0} \\
&= \frac{d}{dt} \rho(e^{-tX}) f(p)\Big|_{t=0} \\
&= -\rho_*(X)f(p)
\end{aligned}
$$

□

9.1 Reductive Cartan geometries

Now assume that (P,ω) is reductive, s.t. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $\mathrm{Ad}(H)\mathfrak{m} \subseteq \mathfrak{m}$. Then the Cartan connection decomposes as $\omega = \omega_\mathfrak{h} + \omega_\mathfrak{m}$, so does the Cartan gauge $\theta = \theta_\mathfrak{h} + \theta_\mathfrak{m}$, and so does $\tilde{D} = \tilde{D}_\mathfrak{h} + \tilde{D}_\mathfrak{m}$. If for $X \in \mathfrak{h}$, $\tilde{D}_X f = -\rho(X)f$, then $\tilde{D}_\mathfrak{h} = -\rho$. As we will see below, $\tilde{D}_\mathfrak{m}$ defines a Kozul connection on any associated vector bundle $P \times_H V$. It follows from the defining properties of a Cartan connection that $\omega_\mathfrak{h} \in \Omega^1(P;\mathfrak{h})$ is the connection one-form for an Ehresmann connection on the principal H-bundle $P \to M$. In contrast, the component $\omega_\mathfrak{m} \in \Omega^1(P;\mathfrak{m})$ satisfies

1. It is horizontal, i.e. $\forall X \in \mathfrak{h}$, $\omega_\mathfrak{m}(\xi_X) = 0$
2. $r_h^* \omega_\mathfrak{m} = \mathrm{Ad}(h^{-1}) \circ \omega_\mathfrak{m}$

The above two mean that $\omega_\mathfrak{m}$ induces a one-form on M with values in the associated vector bundle $P \times_H \mathfrak{m}$, which is isomorphic to TM. Thus $\omega_\mathfrak{m}$ is a **soldering form** on P.
As ω splits, so does $\Omega = \Omega_\mathfrak{h} + \Omega_\mathfrak{m}$ where the structure equation $\Omega = d\omega + \frac{1}{2}[\omega,\omega]$ gives

$$
\begin{aligned}
\Omega_\mathfrak{h} &= d\omega_\mathfrak{h} + \frac{1}{2}[\omega_\mathfrak{h},\omega_\mathfrak{h}] + \frac{1}{2}[\omega_\mathfrak{m},\omega_\mathfrak{m}]_\mathfrak{h} \\
\Omega_\mathfrak{m} &= d\omega_\mathfrak{m} + [\omega_\mathfrak{h},\omega_\mathfrak{m}] + \frac{1}{2}[\omega_\mathfrak{m},\omega_\mathfrak{m}]_\mathfrak{m}
\end{aligned}
$$

Therefore, the $\mathfrak{h}$-component of the curvature of the Cartan connection is not necessarily the curvature of the Ehresmann connection, but receives a correction from the soldering form:

$$
\Omega_\mathfrak{h}^{\text{Cartan}} = \Omega^{\text{Ehresmann}} + \frac{1}{2}[\omega_\mathfrak{m},\omega_\mathfrak{m}]_\mathfrak{h}
$$

whereas the torsion of the Cartan connection is not necessarily the torsion of the affine connection defined by $\omega_\mathfrak{h}$:

$$
\Theta^{\text{Cartan}} = \Omega_\mathfrak{m}^{\text{Cartan}} = \Theta + \frac{1}{2}[\omega_\mathfrak{m},\omega_\mathfrak{m}]_\mathfrak{m}
$$

Let's now consider the universal covariant derivative $\tilde{D} = \tilde{D}_\mathfrak{h} + \tilde{D}_\mathfrak{m}$. The $\mathfrak{m}$-component defines a Kozul connection on any associated vector bundle $E \equiv P \times_H V$ for (V,ρ) a representation of H. Indeed, let $\psi : \Gamma(E) \to \Omega^0(P;\rho)$ be the $C^\infty(M)$-modules isomorphism. We define $\nabla_\zeta : \Gamma(E) \to \Gamma(E)$ by the commutativity of the following square:

$$
\begin{array}{ccc}
\Gamma(E) & \xrightarrow{\nabla_\zeta} & \Gamma(E) \\
\psi \big\downarrow \cong & & \cong \big\downarrow \psi \\
\Omega^0(P;\rho) & \xrightarrow[\tilde{\zeta}]{} & \Omega^0(p;\rho)
\end{array}
$$

i.e. $\psi(\nabla_\zeta s) = \tilde{\zeta}\psi(s)$, where $\tilde{\zeta}$ is the **horizontal lift** of ζ, i.e the unique[1] vector field on P s.t. $(\pi_*)_p \tilde{\zeta} = \zeta_{\pi(p)}$ and $\omega_\mathfrak{h}(\tilde{\zeta}) = 0$

Proposition 9.32. *∇ defines a Kozul connection on E*

[1] Is it clear why this vector field is unique

Proof. ∇_ζ is $\mathbb{R}$-linear and if $f \in C^\infty(M)$, $(\pi^* f)\tilde{\zeta}$ is the horizontal lift of $f\zeta$, so we have $\nabla_{f\zeta} s = f\nabla_\zeta s$. Finally, to get that ∇ is a derivation, see

$$\begin{aligned}\psi(\nabla_\zeta(fs)) &= \tilde{\zeta}\psi(fs) = \tilde{\zeta}(\pi^* f\psi(s)) = \tilde{\zeta}(\pi^* f)\psi(s) + (\pi^* f)\tilde{\zeta}\psi(s) \\ &= \pi^*(\zeta f)\psi(s) + (\pi^* f)\psi(\nabla_\zeta s) = \psi((\zeta f)s) + \psi(f\nabla_\zeta s) \\ &= \psi((\zeta f)s + f\nabla_\zeta s)\end{aligned}$$

□

Proposition 9.33. *Let (U, θ) be a gauge for a reductive Cartan geometry, $\sigma : U \to P|_U$ the section such that $\theta = \sigma^*\omega$, $\zeta \in \mathfrak{X}(U)$, and $\phi = \sigma^*\Phi$ where $\Phi \in \Omega^0(P;\rho)$. Then*

$$\nabla_\zeta \phi \equiv \zeta(\phi) - \rho_*(\theta_{\mathfrak{h}}(\zeta))\phi$$

is the expression of the covariant derivative of Φ in the gauge (U, θ).

9.2 Special geometries

We may define 'special geometries' via curvature constraints

Lemma 9.34. *Let $V \subset \mathfrak{g}$ be the vector subspace spanned by the values of the curvature form Ω. Then V is a H-submodule*

Proof. Let $v = \Omega_p(\xi_p, \eta_p)$. Then

$$\begin{aligned}\mathrm{Ad}(h^{-1})v &= \mathrm{Ad}(h^{-1})(\Omega_p(\xi_p, \eta_p)) \\ &= (r_h^*\Omega_p)(\xi_p, \eta_p) \\ &= \Omega_{ph}((r_h)_*\xi_p, (r_h)_*\eta_p)\end{aligned}$$

which is a value of Ω □

In particular if $V \subset \mathfrak{h}$ is s.t. the Cartan geometry is torsion-free, then V is is an ideal. If the geometry is torsion-free and the action of H on $\mathfrak{h}$ is irreducible, there are no special geometries arising from $\mathfrak{g}$-curvature conditions. However, the H-modules $\mathrm{Hom}\left(\Lambda^2\left(\mathfrak{g}/\mathfrak{h}\right), \mathfrak{h}\right)$ need not be irreducible and we can define special geometries by damnding that the curvature function $K : P \to \mathrm{Hom}\left(\Lambda^2\left(\mathfrak{g}/\mathfrak{h}\right), \mathfrak{h}\right)$ takes values in a H-submodule. If H is compact, then $\mathrm{Hom}\left(\Lambda^2\left(\mathfrak{g}/\mathfrak{h}\right), \mathfrak{h}\right)$ is fully reducible

Example 9.35. $\mathfrak{g} = \mathfrak{so}_n \ltimes \mathbb{R}^n$ *and* $\mathfrak{h} = \mathfrak{so}_n$*. Have* $\mathrm{Hom}\left(\Lambda^2\left(\mathfrak{g}/\mathfrak{h}\right), \mathfrak{h}\right) = \mathrm{Hom}(\Lambda^2\mathbb{R}^n, \mathfrak{so}_n)$.

The subspace corresponding to those curvature functions obeying the (algebraic) Bianchi identity breaks up into three submodules: scalar, trace-free Ricci, and Weyl.
Cartan connections are special types of Ehresmann connections. Let $P \to M$, $G \to G/H$, be principal H-bundles. There is an associated fibre bundle $Q = P \times_H G$ where H acts on G by left multiplication. This is a (right) principal G-bundle and M, and we have a natural inclusion $P \subset Q$ sending $p \mapsto (p, e)$. An Ehresmann connection on Q is a $\mathfrak{g}$-valued one-form and its restriction to P gives a candidate for a Cartan connection on P.

Theorem 9.36. *Let G/H be a Klein geometry and let P, Q be principal H, G-bundles respectively over a manifold M. Assume that* $\dim P = \dim G$ *and $\varphi : P \to Q$ is a H-bundle map. Then there is a bijection of sets*

$$\{\textit{Ehresmann connections on } Q\textit{, kernels not } \varphi_*(TP)\} \xrightarrow{\varphi^*} \{\textit{Cartan connections on } P\}$$

Proof. Let $\varpi \in \Omega^1(Q;\mathfrak{g})$ be an Ehresmann connection s.t. $\varpi_*(TP) \cap \ker \varpi = 0$. It follows that $\omega = \varphi^*\varpi \in \Omega^1(p;\mathfrak{g})$ with zero kernel. Since $\dim P = \dim \mathfrak{g}$, $\omega_p : T_pP \to \mathfrak{g}$ is injective and so an isomorphism.

Since $\varphi : P \to Q$ is a H-bundle map, $\forall X \in \mathfrak{h}$ the vector fields ξ_X on P and ζ_X on Q are φ-related : i.e

$$\forall p \in P,\ (\varphi_*)_p \xi_X(p) = \zeta_X(\varphi(p))$$

Also,

$$r_h^*\omega = r_h^*\varphi^*\varpi = \varphi^* r_h^*\varpi = \varphi^*(\mathrm{Ad}(h^{-1}) \circ \varpi) = \mathrm{Ad}(h^{-1}) \circ \varphi^*\varpi = \mathrm{Ad}(h^{-1}) \circ \omega$$

so ω is a Cartan connection. Next we define a correspondence

$$\{\text{Cartan connections on } P\} \xrightarrow{j} \{\text{Ehresmann connections on } Q, \text{ kernels not } \varphi_*(TP)\}$$

Given a Cartan connection ω on P we extend it to a form $\varpi = j(\omega)$ on $P \times G$ by

$$\varpi_{(p,g)} = \mathrm{Ad}(g^{-1}) \circ \pi_P^*\omega_p + \pi_G^*\vartheta_G|_g$$

where $\pi_{P/G} : P \times G \to P/G$ are the canonical projections. We notice that $\forall X \in \mathfrak{g}$, $\varpi(0, X^L) = X$. Also, if $i : P \to P \times G$ is the injection $p \mapsto (p, e)$ then $i^*\varpi = \omega$. In particular, ϖ does not vanish on $T(P \times \{e\})$. Let $\gamma \in G$ and consider $\mathrm{id} \times R_\gamma : P \times G \to P \times G$:

$$\begin{aligned}
(\mathrm{id} \times R_\gamma)^*\varpi_{(p,g\gamma)} &= \varpi_{(p,g\gamma)} \circ (\mathrm{id} \times R_\gamma)_* \\
&= \big(\mathrm{Ad}(g\gamma)^{-1} \circ \pi_P^*\omega_p + \pi_G^*\vartheta_G\big) \circ (\mathrm{id} \times R_\gamma)_* \\
&= \mathrm{Ad}(g\gamma)^{-1} \circ \omega \circ (\pi_P)_* \circ (\mathrm{id} \times R_\gamma)_* + \vartheta_G \circ (\pi_G)_* \circ (\mathrm{id} \times R_\gamma)_* \\
&= \mathrm{Ad}(g\gamma)^{-1} \circ \omega \circ (\pi_P)_* + \vartheta_G \circ (R_\gamma)_* \circ (\pi_G)_* \\
&= \mathrm{Ad}(\gamma)^{-1} \left(\mathrm{Ad}(g)^{-1} \circ \pi_P^*\omega + \pi_G^*\vartheta_G\right) \\
&= \mathrm{Ad}(\gamma)^{-1} \circ \varpi_{(p,g)}
\end{aligned}$$

We now check that ϖ is basic for $P \times G \to P \times_H G$ which means that it is both horizontal and 'invariant'. The latter condition requires that for $\alpha_h : P \times G \to P \times G$, $(p, g) \mapsto (phh^{-1}g)$, we have $\alpha_h^*\varpi = \varpi$. We calculate

$$\begin{aligned}
(\alpha_h^*\varpi)_{(p,g)} &= \varpi_{(ph,h^{-1}g)} \circ (\alpha_h)_* \\
&= \mathrm{Ad}(h^{-1}g)^{-1}\pi_P^*\omega \circ (\alpha_h)_* + \pi_G^*\vartheta_G \circ (\alpha_h)_* \\
&= \mathrm{Ad}(h^{-1}g)^{-1}\omega \circ (\pi_P)_* \circ (\alpha_h)_* + +\vartheta_G \circ (\pi_G)_* \circ (\alpha_h)_* \\
&= \mathrm{Ad}(g^{-1}) \circ \mathrm{Ad}(h) \circ \omega \circ (R_h)_* \circ (\pi_P)_* + \vartheta_G \circ (L_{h^{-1}})_* \circ (\pi_G)_* \\
&= \mathrm{Ad}(g^{-1}) \circ \pi_P^*\omega + \pi_G^*\vartheta_G \quad (\text{as } R_h^*\omega = \mathrm{Ad}(h)^{-1}\omega \text{ and } \vartheta_G \text{ is LI}) \\
&= \varpi_{(p,g)}
\end{aligned}$$

To show ϖ is horizontal, let $X \in \mathfrak{h}$ and $\xi_X \in \mathfrak{X}(P \times G)$ corresponding to the right H-action on $P \times G$:

$$\begin{aligned}
&P \times G \times H \to P \times G \\
&\quad (p, g, h) \mapsto (ph, h^{-1}g) = ((\mu_P \times \mu_G) \circ (\mathrm{id} \times \mathrm{id} \times \imath \times \mathrm{id}) \circ (\mathrm{id} \times \Delta \times \mathrm{id}) \circ \varrho))\,(p, g, h)
\end{aligned}$$

where we have

$$\begin{aligned} \varrho : P \times G \times H &\to P \times H \times G \\ (p, g, h) &\mapsto (p, h, g) \end{aligned}$$

$$\begin{aligned} \mathrm{id} \times \Delta \times \mathrm{id} : P \times G \times H &\to P \times H \times H \times G \\ (p, h, g) &\mapsto (p, h, h, g) \end{aligned}$$

$$\begin{aligned} \mathrm{id} \times \mathrm{id} \times \imath \times \mathrm{id} : P \times H \times H \times G &\to P \times H \times H \times G \\ (p, h, h, g) &\mapsto (p, h, h^{-1}, g) \end{aligned}$$

$$\begin{aligned} \mu_P \times \mu_G : P \times H \times H \times G &\to P \times G \\ (p, h, h^{-1}, g) &\mapsto (ph, h^{-1}g) \end{aligned}$$

Then

$$\begin{aligned} (\xi_X)_{(p,g)} &= ((\mu_P \times \mu_G) \circ (\mathrm{id} \times \mathrm{id} \times \imath \times \mathrm{id}) \circ (\mathrm{id} \times \Delta \times \mathrm{id}) \circ \varrho))_{*,(p,g,e)} (0, 0, X) \\ &= (\mu_P \times \mu_G)_* \circ (\mathrm{id} \times \mathrm{id} \times \imath \times \mathrm{id})_* \circ (\mathrm{id} \times \Delta \times \mathrm{id})_{*,(p,e,g)} (0, X, 0) \\ &= (\mu_P \times \mu_G)_* \circ (\mathrm{id} \times \mathrm{id} \times \imath \times \mathrm{id})_{*,(p,e,e,g)} (0, X, X, 0) \\ &= (\mu_P \times \mu_G)_{*,(p,e,e,g)} (0, X, -X, 0) \\ &= (\mu_P)_{*,(p,e)} (0, X), (\mu_G)_{*,(e,g)} (-X, 0) \\ &= (\omega_p^{-1}(X), -(\vartheta_G)_g^{-1}(\mathrm{Ad}(g^{-1})X)) \\ \Rightarrow \varpi_{(p,g)}(\xi_X) &= \varpi_{(p,g)}(\omega_p^{-1}(X), -(\vartheta_G)_g^{-1}(\mathrm{Ad}(g^{-1})X)) \\ &= (\mathrm{Ad}(g^{-1}) \cdot (\pi_P^* \circ \omega) + \pi_G^* \vartheta_G)(\omega_p^{-1}(X), -(\vartheta_G)_g^{-1}(\mathrm{Ad}(g^{-1})X)) \\ &= \mathrm{Ad}(g^{-1})X = \mathrm{Ad}(g^{-1})X = 0 \end{aligned}$$

Therefore ϖ descends to $\varpi \in \Omega^1(P \times_H G, \mathfrak{g})$ and satisfies the properties of an Ehresmann connection which in addition obeys $\ker \varpi \cap \varphi_*(TP) = 0$.
Finally, we need to show that φ^* and j are mutual inverses:

$$\begin{aligned} \varphi^*(j(\omega_p)) = \varphi^* \varpi_{(p,e)} &= \mathrm{Ad}(e)^{-1} \circ \varphi^* \pi_P^* \omega_p + \varphi^* \pi_G^* \vartheta_{Ge} \\ &= (\pi_P \circ \varphi)^* \omega_p + 0 \quad (\text{since } \pi_G \circ \varphi \text{ is constant}) \qquad = \omega_p \end{aligned}$$

shows that $\varphi^* \circ j = \mathrm{id}$. To do the other direction, it suffices to show φ^* is injective. Now if $\varphi^* \varpi = \varphi^* \varpi_2$ then ϖ_1, ϖ_2 agree on the image $\varphi_*(TP)$ and hence on all the right translations. But ϖ_1, ϖ_2 agree on ξ_X and these two kinds of vectors span TQ □