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(Rough) Notes on SUSY Gauge Theories and D-Branes

Warning: Conventions, signs and prefactors are not carfully chosen and they might be incompatible between sections.

1. Extended supersymmetry in general dimensions

Recall $\mathcal{N}=1$ SYM theory:

$$S=\int \mathrm{d}^4x\,\mathrm{tr}\,(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\mathrm{i}\bar\lambda_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\alpha}\nabla_\mu\lambda_\alpha)\ , \tag{1.1}$$

$F_{\mu\nu}$ and λ take values in a Lie algebra $\mathfrak{g}$: $F_{\mu\nu}=F^a_{\mu\nu}t_a$ and $\lambda=\lambda^a t_a$, where t_a are antihermitian generators of $\mathfrak{g}$.

We use $\nabla_\mu=\partial_\mu+[A_\mu,\cdot]$ and $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]$

The SUSY transformations are $\delta A_\mu=\mathrm{i}\bar\varepsilon\sigma_\mu\lambda$ and $\delta\lambda=\sigma^{\mu\nu}F_{\mu\nu}\varepsilon$

First goal: We want to generalize this to arbitrary dimensions.

1.1. Spinors in general dimensions

Recall: To classify all unitary irreps of the Lorentz group in 4d, $\mathsf{SO}(1,3)$ up to a phase, we need the double (here: universal) cover.

General d: Have a double cover $1\to\mathbb{Z}_2\to\mathsf{Spin}(t,s)\to\mathsf{SO}(t,s)\to 1$

Construction of $\mathsf{Spin}(t,s)$: Use *Clifford algebra* $\mathcal{C}\ell(\mathbb{R}^{t,s})$: $\{\gamma_\mu,\gamma_\nu\}=-2\eta_{\mu\nu}\mathbb{1}$

Then: $\gamma_{\mu\nu}=\frac{1}{4}=[\gamma_\mu,\gamma_\nu]$ form generators of $\mathsf{Spin}(t,s)$ and satisfy the Lorentz algebra $[\gamma_{\mu\nu},\gamma^{\kappa\lambda}]=-\delta^{[\kappa}_{[\mu}\gamma_{\nu]}{}^{\lambda]}$.

Exercise: check this.

Construction for $\eta_{\mu\nu}=\delta_{\mu\nu}$ (one possible example):

$$\begin{aligned}\gamma_1&=\mathrm{i}\sigma_1\otimes\sigma_0\otimes\sigma_0\otimes\dots\\ \gamma_2&=\mathrm{i}\sigma_2\otimes\sigma_0\otimes\sigma_0\otimes\dots\\ \gamma_3&=\mathrm{i}\sigma_3\otimes\sigma_1\otimes\sigma_0\otimes\dots\\ \gamma_4&=\mathrm{i}\sigma_3\otimes\sigma_2\otimes\sigma_0\otimes\dots\\ \gamma_5&=\mathrm{i}\sigma_3\otimes\sigma_3\otimes\sigma_1\otimes\dots\\ &\dots\end{aligned} \tag{1.2}$$

The pattern should be clear. One can always truncate the tensor products such that all generators are distinct.

Exercise: check that these generators satisfy the Clifford algebra.

The generators of the Clifford algebra act on a vector space S of dimension $2^{d/2}$ for d even and $2^{[d/2]}=2^{(d-1)/2}$ for d odd:

	$\mathbb{R}^2$	$\mathbb{R}^3$	$\mathbb{R}^4$	$\mathbb{R}^5$	$\mathbb{R}^6$	$\dots$
$\dim(S)$	2	2	4	4	8	$\dots$

For metrics with different signatures, insert factors of i into the generators.

Constraints on spinors

In even dimensions: $\gamma_{\text{chiral}} := \mathrm{i}^n \gamma_1 \dots \gamma_d$ with n such that $\gamma_{\text{chiral}}^2 = \mathbb{1}$. We have

$$\{\gamma_{\text{chiral}}, \gamma_\mu\} = 0 \quad \text{and} \quad [\gamma_{\text{chiral}}, \gamma_{\mu\nu}] = 0 \tag{1.3}$$

Thus, the spinor representation S is reducible: $S = S^+ \oplus S^-$ with

$$\gamma_{\text{chiral}} \psi_\pm = \pm \psi_\pm \quad \text{with} \quad \psi_\pm \in S^\pm \ . \tag{1.4}$$

S : Dirac spinors, $S^\pm$: Weyl spinors, $\dim(S^+) = 2^{d/2-1}$. Projectors onto dimensions: $P_\pm = \frac{1}{2}(\mathbb{1} \pm \gamma_{\text{chiral}})$

Second condition: Majorana condition (reality):

$$\psi = \psi^c = \mathcal{C} \gamma_1 \psi^* \ , \tag{1.5}$$

in certain dimensions. This condition again halves the number of degrees (dofs) of a spinor. (Details: see [1], appendix B, as well as [2].)

Altogether:

d=	2	3	4	5	6	7	8	9	10	11	12
Majorana	x	x	(x)				(x)	x	x	x	(x)
Weyl	x		x		x		x		x		x
Majorana–Weyl	x								x		

In dimensions 4, 8 and 12, one can impose either Majorana or Weyl condition, but not both at the same time.

1.2. Supersymmetry algebras and their representations

Recall the four-dimensional case. We start from the 4d Poincaré algebra, generated by translations P_μ and rotations and boosts $M_{\mu\nu}$:

$$[P_\mu, P_\nu] = 0 \ , \quad [P_\mu, M_{\nu\rho}] = \eta_{\mu[\nu} P_{\rho]} \ , \quad [M_{\mu\nu}, M^{\rho\sigma} = -2\delta_{[\mu}^{[\rho} M_{\nu]}{}^{\sigma]} \tag{1.6}$$

This algebra is now extended by $\mathcal{N}$ supercharges $Q_A^i = P_+ Q_A^i$ and $Q_{iA} = P_- Q_{iA}$, where $i = 1, \dots, \mathcal{N}$ and $A = 1, \dots, 4$. Using the Weyl representation with $\gamma_{\text{chiral}} = \text{diag}(\mathbb{1}_2, -\mathbb{1}_2)$, we have

$$Q_A^i = \begin{pmatrix} Q_\alpha^i \\ 0 \end{pmatrix} \ , \quad Q_{iA} = \begin{pmatrix} 0 \\ \bar{Q}_{i\dot{\alpha}} \end{pmatrix} \ . \tag{1.7}$$

The extension of the Poincaré algebra reads as:

$$\begin{aligned} &\{Q_\alpha^i, Q_\beta^j\} = \{\bar{Q}_{i\dot{\alpha}}, \bar{Q}_{j\dot{\beta}}\} = 0 \ , \quad \{Q_\alpha^i, Q_{j\dot{\alpha}}\} = 2\delta_j^i \sigma_{\alpha\dot{\alpha}}^\mu P_\mu \ , \\ &[P_\mu, Q_\alpha^i] = [P_\mu, \bar{Q}_{i\dot{\alpha}}] = 0 \ , \\ &[M_{\mu\nu}, Q_\alpha^i] = \mathrm{i} \sigma_{\mu\nu\alpha}{}^\beta Q_\beta^i \ , \quad [M_{\mu\nu}, \bar{Q}_i^{\dot{\alpha}}] = \mathrm{i} \bar{\sigma}_{\mu\nu}{}^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}_i^{\dot{\beta}} , \end{aligned} \tag{1.8}$$

where $\lambda^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\lambda_{\dot\beta}$, $\bar\sigma^{\mu\dot\alpha\alpha} = \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}\sigma^\mu_{\beta\dot\beta}$ and $\sigma^{\mu\nu\alpha}{}_\beta = \frac{1}{4}(\sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\nu\dot\alpha\beta} - \sigma^\nu_{\alpha\dot\alpha}\bar\sigma^{\mu\dot\alpha\beta})$.

Theorem: (Haag, Lopuszanski, Sohnius) [3]: Up to introducing central charges (that is: $\{Q^i_\alpha, Q^j_\beta\} = \varepsilon_{\alpha\beta}Z^{[ij]}$, same for $\bar Q$), this is the only nontrivial extension of the Poincaré algebra compatible with the axioms of relativistic quantum field theory.

- Note that Q and $\bar Q$ are indeed complex conjugate on $\mathbb{R}^{1,3}$ due to $\{Q^i_\alpha, Q_{j\dot\alpha}\} = 2\delta^i_j\sigma^\mu_{\alpha\dot\alpha}P_\mu$ and P_μ being real.
- $\mathcal{N}=1$: 2 complex or 4 real parameters ε in $\delta\Phi = \varepsilon^\alpha Q_\alpha\Phi + \bar\varepsilon^{\dot\alpha}\bar Q_{\dot\alpha}\Phi$.
- $\mathcal{N}=2$: 4 complex or 8 real parameters ε^i, general $\mathcal{N}$: $4\mathcal{N}$ real supercharges.

Massless representations of SUSY algebra:

Go to massless frame: $P_\mu = (E,0,0,E)$:

$$\sigma^\mu_{\alpha\dot\alpha}P_\mu = \begin{pmatrix} 0 & 0 \\ 0 & 2E \end{pmatrix}_{\alpha\dot\alpha} . \tag{1.9}$$

As $\langle\psi|\{Q^1,\bar Q_1\}|\psi\rangle = ||Q_1|\psi\rangle||^2 + ||\bar Q_1|\psi\rangle||^2 = 0$, $Q_1|\psi\rangle = \bar Q|\psi\rangle = 0$. It remains an algebra of fermionic creation and annihilation operators: $\{Q^i_2, \bar Q_{\dot 2,j}\} = 4E\delta^i_j$.

Choose the helicity operator $J_3 = M_{12}$, then $[J_3, Q^i_2] = -\frac{1}{2}Q^i_2$ and $[J_3, \bar Q_{i\dot 2}] = \frac{1}{2}Q_{i\dot 2}$. Thus, Q^i_2 and $\bar Q_{\dot 2 i}$ lower and raise, respectively, the helicity of a state by $\frac{1}{2}$: $\bar Q_{\dot 2 i}|h\rangle = |h+\frac{1}{2}\rangle$, h: helicity.

To avoid gravity, we have the condition $|h| \leq 1$ and therefore $\mathcal{N} \leq 4$. This is maximally supersymmetric Yang-Mills theory with 16 real supercharges.

To include gravity, but to avoid fields of higher helicity: $|h| \leq 2$ and therefore $\mathcal{N} \leq 8$. This is maximally supersymmetric gravity with 32 real supercharges.

Using the fermionic creation operators, we can build up all the multiplets from a lowest weight state $Q^2_i|h\rangle = 0$:

- $\mathcal{N}=1$: $|0\rangle \to \bar Q_{\dot 2}|0\rangle = |\frac{1}{2}\rangle$ "$\mathcal{N}=1$ chiral multiplet", $|\frac{1}{2}\rangle \to \bar Q_{\dot 2}|\frac{1}{2}\rangle = |1\rangle$ "$\mathcal{N}=1$ vector multiplet" (Each multiplet needs to be complemented by its CPT conjugate with opposite chirality.)
- $\mathcal{N}=2$: $|0\rangle \to 2\times|\frac{1}{2}\rangle \to |1\rangle$: "$\mathcal{N}=2$ vector multiplet", $|-\frac{1}{2}\rangle \to 2\times|0\rangle \to |\frac{1}{2}\rangle$: "$\mathcal{N}=2$ hypermultiplet"
- $\mathcal{N}=3:$ $|-\frac{1}{2}\rangle \to 3\times|0\rangle \to 3\times|\frac{1}{2}\rangle \to |1\rangle$: "$\mathcal{N}=3$ vector multiplet" $|-1\rangle \to 3\times|-\frac{1}{2}\rangle \to 3\times|0\rangle \to |\frac{1}{2}\rangle$: CPT conjugate of "$\mathcal{N}=3$ hypermultiplet"
- $\mathcal{N}=4:$ $|-1\rangle \to 4\times|-\frac{1}{2}\rangle \to 6\times|0\rangle \to 4\times|\frac{1}{2}\rangle \to |1\rangle$: "$\mathcal{N}=4$ vector multiplet". Note that this multiplet is its own CPT conjugate!

Note that Q and $\bar Q$ change bosons to fermions and vice versa. Because $\{Q,\bar Q\}$ is proportional to an (invertible) translation, there are as many fermionic states as there are bosonic ones.

R-symmetry is the automorphism group of the supersymmetry algebra. It rotates the index $i = 1, \ldots, \mathcal{N}$ labelling the generators of supersymmetry Q^i_α and $Q_{i\dot\alpha}$. In 4d, this group is given by $\mathsf{U}(\mathcal{N})$.

In general dimensions (details in [2]), the key relation is of the form

$$\{Q^i_A, Q_{Bj}\} = (\gamma^\mu \mathcal{C})_{AB} P_\mu \delta^i_j \ . \tag{1.10}$$

In even dimensions, we can restrict the supercharges to Weyl spinors $P^+ Q^i_A$ and $P^- Q^i_A$. Then the question is, if there are relations like $\{Q^+, Q^+\} \sim P$ and $\{Q^-, Q^-\} \sim P$. Going through reality conditions [2] implies that $\frac{d}{2}$ has to be odd (for Minkowski signature). That is, in $d = 2, 6, 10$ this is possible. Correspondingly, supersymmetry algebras in these dimensions are labelled by the number of relations of the first and the second type as $\mathcal{N} = (p, q)$:

d=	2	3	4	5	6	7	8	9	10	11	12
Majorana	x	x	(x)				(x)	x	x	x	(x)
Weyl	x		x		x		x		x		x
Majorana–Weyl	x								x		
(p, q) SUSY	x				x				x		

1.3. Super Yang-Mills theory in higher dimensions

We saw that a crucial requirement for SUSY is the equal number of bosonic and fermionic degrees of freedom (dof). In general, we can count states in a classical field theory as follows: We are always interested in real, on-shell degrees of freedom. A real scalar gives 1 dof. A Dirac spinor has $2 \times 2^{[d/2]}$ dofs, which are reduced by the Dirac equation ($\gamma_\mu p^\mu \psi = 0$ in momentum space) to $2^{[d/2]}$ dofs. Imposing Majorana and Weyl conditions reduce the dofs by 1/2. A massless gauge potential has a priori d dofs, which are reduced by one due to gauge symmetry. The mass-shell condition $k^\mu A_\mu = 0$ takes away another dof, and we end up with $d - 2$ dofs. The dofs of fields of higher helicity follow similary.

Let us now find out in which dimensions we can in principal have $\mathcal{N} = 1$ pure super Yang-Mills theory. That is, when the action

$$S = \int \mathrm{d}^d x \left(-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathrm{i} \bar\lambda \gamma^\mu \nabla_\mu \lambda \right) \tag{1.11}$$

is supersymmetric. We have the following numbers of states in dimensions where # bosonic dofs= # fermionic dofs:

	A_μ	λ_D	λ_M	λ_W	λ_{MW}
$d = 3$	1	2	$\boxed{1}$	-	-
$d = 4$	2	4	$\boxed{2}$	$\boxed{2}$	-
$d = 6$	4	8	-	$\boxed{4}$	-
$d = 10$	8	32	16	16	$\boxed{8}$

Other dimensions $d < 10$ do not work, higher dimensions cannot work as the bosonic dofs grow linearly, while the fermionic ones grow exponentially.

Interestingly, the fact that pure $\mathcal{N}=1$ SYM theory exists only in $d=3,4,6,10$ dimensions can be linked to properties of the four normed division algebras $\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}$: Under SUSY transformations, we have $\delta\mathcal{L}=\text{tri}\psi+\text{totaldivergence}$, where $\text{tri}\psi$ is an associator type object. Because of $\mathsf{Spin}(1,2)\cong\mathsf{SL}(2,\mathbb{R})$, $\mathsf{Spin}(1,3)\cong\mathsf{SL}(2,\mathbb{C})$, $\mathsf{Spin}(1,5)\cong\mathsf{SL}(2,\mathbb{H})$, and $\mathsf{Spin}(1,9)\cong\mathsf{SL}(2,\mathbb{O})$, this expression vanishes in $d=3,4,6,10$ [4].

1.4. Kaluza-Klein reduction

Compactify: $\mathbb{R}^{1,d}\to\mathbb{R}^{1,d-1}\times S^1$, $x^M\to(x^\mu,y)$ and let R be the radius of the circle. A scalar field can be Fourier expanded:

$$\phi(x^M)=\phi(x^\mu,y)=\sum_{n\in\mathbb{Z}}\phi_n(x^\mu)\frac{\mathrm{e}^{\mathrm{i}ny/R}}{\sqrt{2\pi R}}\ . \tag{1.12}$$

The kinetic term of the action reads as:

$$\begin{aligned}\int\mathrm{d}^dx\phi^\dagger(\Box_d-m^2)\phi\ \to\ &\int\mathrm{d}^{d-1}x\int\mathrm{d}y\phi^\dagger(\Box_{d-1}+\frac{\partial^2}{\partial y^2}-m^2)\phi\\ &=\int\mathrm{d}^{d-1}x\sum_n\phi_n^\dagger(\Box_{d-1}+\frac{\partial^2}{\partial y^2}-m^2-\frac{n^2}{R^2})\phi_n\end{aligned} \tag{1.13}$$

Thus: the Fourier modes of ϕ along the circle become massive with mass $m^2=\frac{n^2}{R^2}$. For $R\to0$, all modes except for the zero modes become infinitely massive and therefore decouple from the spectrum. The same happens for all fields.

The Kaluza-Klein reduction therefore amounts to putting $\partial_m=0$ for directions, which have been compactified on a circle. Covariant derivatives ∇_M reduce to (∇_μ,ϕ_m).

The Lorentz group for the reduction $\mathbb{R}^{1,d}\to\mathbb{R}^{1,d-q}\times(S^1)^{\times}q$ is $\mathsf{SO}(1,d-q)\times\mathsf{SO}(2)^{\times q}$. In the limit $R\to0$, we have $\mathsf{SO}(1,d-q)\times\mathsf{SO}(q)$.

1.5. Example: $\mathcal{N}=1$, $d=10$ SYM $\to\mathcal{N}=4$, $d=4$ SYM

Original reference for this section with more examples: [5]

In the ten-dimensional theory, the supercharges Q_A are Majorana-Weyl spinors, and correspondingly, there are 16 real supercharges.

The double cover of the Lorentz group is broken according to

$$\mathsf{Spin}(1,9)\to\mathsf{Spin}(1,3)\times\mathsf{Spin}(6)\cong\mathsf{Spin}(1,3)\times\mathsf{SU}(4)\ . \tag{1.14}$$

The bosonic fields in 10d are given by A_M, $M=0,\dots,9$ and under the reduction, they are split up into A_μ, $\mu=0,\dots,3$ and ϕ^I, $I=1,\dots,6$. To use the identity $\mathsf{Spin}(6)\cong\mathsf{SU}(4)$, we introduce the 't Hooft tensors:

$$\eta^a_{ij}=\varepsilon_{aij4}+\delta_{ai}\delta_{4j}-\delta_{aj}\delta_{4i}\ ,\quad \eta^a_{ij}=\tfrac{1}{2}\varepsilon_{ijkl}\eta^a_{kl}\ , \tag{1.15}$$

which form a basis of self-dual antisymmetric tensors of rank two in four dimensions. They describe the coupling of the **6** of $\mathsf{Spin}(6)$ to the **6** of $\mathsf{SU}(4)$: we can define $\phi_{ij}:=\eta^a_{ij}(\phi^{2a-1}+\mathrm{i}\phi^{2a})$.

To describe the fermions, we have to split the Clifford algebra:

$$\Gamma_M = \left(\gamma_\mu \otimes \mathbb{1}_8, \gamma_{ij} = \gamma_5 \otimes \begin{pmatrix} 0 & \rho^{ij} \\ \rho_{ij} & 0 \end{pmatrix}\right) , \tag{1.16}$$

where $(\rho_{ij})_{kl} = \varepsilon_{ijkl}$, $(\rho^{ij})_{kl} = \frac{1}{2}\varepsilon^{ijmn} eps_{mnkl}$ and $\Gamma_{\text{chiral}} = \gamma_5 \otimes \mathbb{1}_8$.

The Majorana-Weyl spinor λ^A itself is split up into four Weyl-spinors in 4d: $\lambda^A = (P^+\chi^i, P^-\bar{\chi}_i)$.

Altogether, we obtain a gauge potential A_μ corresponding to states $|-1\rangle, |1\rangle$, four Weyl spinors χ^i_α and their complex conjugates $\bar{\chi}_i$ corresponding to each four states $|\frac{1}{2}\rangle$ and $-|\frac{1}{2}\rangle$ and six scalars ϕ_{ij} corresponding to six states $|0\rangle$. Altogether, we have the $\mathcal{N} = 4$ vector supermultiplet.

The action is given by

$$\begin{aligned} S = \text{ tr } \int \mathrm{d}^4x \ & -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \nabla_\mu\varphi_{ij}\nabla^\mu\varphi^{ij} + i\bar{\chi}\sigma^\mu\nabla_\mu\chi^i \\ & + 2g^2[\varphi_{ij}, \varphi_{kl}][\varphi^{ij}, \varphi^{kl}] - g\chi^i[\chi^j, \varphi_{ij}] - g\bar{\chi}_i[\bar{\chi}_j, \varphi^{ij}] \ . \end{aligned} \tag{1.17}$$

This action can also be obtained from a vector superfield and three chiral superfields:

$$\begin{aligned} \mathcal{L} = \text{ tr } & \int \mathrm{d}^4\theta \ \bar{\Phi}e^{gV}\Phi + \frac{1}{4g^2}\int \mathrm{d}^2\theta \frac{1}{4}W^\alpha W_\alpha + c.c. \\ & + i\frac{\sqrt{2}}{3!}\left(\int \mathrm{d}^2\theta \ \epsilon_{ijk}\Phi^i[\Phi^j, \Phi^k] + \int \mathrm{d}^2\bar{\theta} \ \bar{\Phi}^i[\bar{\Phi}^j, \bar{\Phi}^k]\right) . \end{aligned} \tag{1.18}$$

Properties of $\mathcal{N} = 4$ SYM theory:

Recall that Yang-Mills theory is conformally invariant (the coupling constant is dimensionless). At quantum level, however, conformal invariance is broken and the β-function is non-trivial. $\mathcal{N} = 4$ SYM theory, however, is conformal even at quantum level. This extends the supersymmetry algebra to the superconformal algebra. In particular, the Lorentz group is extended from $\mathsf{SO}(1,3)$ to $\mathsf{SO}(2,4) \simeq \mathsf{SU}(2,2)$. Altogether, the superconformal group is given by supermatrices of the form

$$\left(\begin{array}{c|c} \mathsf{SU}(2,2) & \begin{array}{c}\text{8 SUSY,}\\ \text{8 superconformal}\end{array} \\ \hline \begin{array}{c}\text{8 SUSY,}\\ \text{8 superconformal}\end{array} & \mathsf{SU}(4) \end{array}\right) , \tag{1.19}$$

yielding $\mathsf{PSU}(2,2|4)$. This is both the symmetry group of the supersymmetric extension of $AdS_5 \times S^5$ and the supertwistor space $\mathbb{C}P^{3,4}$.

Similar reductions go as follows:

$$\begin{array}{lllll} \text{16 supercharges:} & 10d, \mathcal{N}=1 \ \rightarrow & 6d, \mathcal{N}=2 \ \rightarrow & 4d, \mathcal{N}=4 \ \rightarrow & 3d, \mathcal{N}=8 \ , \\ \text{8 supercharges:} & & 6d, \mathcal{N}=1 \ \rightarrow & 4d, \mathcal{N}=2 \ \rightarrow & 3d, \mathcal{N}=4 \ , \\ \text{4 supercharges:} & & & 4d, \mathcal{N}=1 \ \rightarrow & 3d, \mathcal{N}=2 \ . \end{array} \tag{1.20}$$

1.6. Theories with 16 supercharges

Further details: [6]

In the following table, representation are labelled as r: real or p: pseudoreal. Additionally, recall the triality of $\mathsf{SO}(8)$, i.e. the fact that the vector, spinor and conjugate spinor representations v, s, c are all 8 dimensional.

total d	Representation of Q^i_A	R-symmetry group $\supseteq \mathsf{Spin}(10-d)$
9	$\mathbf{16}_r$	
8	$(\mathbf{8}_s, \mathbf{1}) \oplus (\mathbf{8}_c, \bar{\mathbf{1}})$	$\mathsf{U}(1) \cong \mathsf{Spin}(2)$
7	$(\mathbf{8}_p, \mathbf{2}_p)$	$\mathsf{SP}(1) \cong \mathsf{Spin}(3) \cong \mathsf{SU}(2)$
6	$(\mathbf{4}_p, \mathbf{2}_p) \oplus (\mathbf{4}'_p, \mathbf{2}'_p)$	$\mathsf{SP}(1) \times \mathsf{SP}(1) \cong \mathsf{Spin}(4) \cong \mathsf{SU}(2) \times \mathsf{SU}(2)$
5	$(\mathbf{4}_p, \mathbf{4}_p)$	$\mathsf{SP}(2) \cong \mathsf{Spin}(5)$
4	$(\mathbf{2}, \mathbf{4}) \oplus (\bar{\mathbf{2}}, \bar{\mathbf{4}})$	$\mathsf{U}(4) \supset\cong \mathsf{Spin}(6)$
3	$(\mathbf{2}_r, \mathbf{8}_r)$	$\mathsf{Spin}(8) \supset \mathsf{Spin}(7)$
2	$(\mathbf{1}_r, \mathbf{8}_s) \oplus (\bar{\mathbf{1}}_r, \mathbf{8}_c)$	$\mathsf{Spin}(8) \times \mathsf{Spin}(8) \supset \mathsf{Spin}(8)$

Note that the R-symmetry group always contains the flavour group as a subgroup.

Comments:

- SYM in $d > 4$: The inverse length dimensions of a scalar field are $[\phi] = \frac{d-2}{2}$ as deduced from the kinetic term $\partial_\mu \phi \partial^\mu \phi$. The interaction term $[\phi, \phi]^2$ has therefore dimension $2(d-2)$. For the theory to be renormalizable by powercounting, this term should be marginal (dimension 4) or irrelevant (dimension<4), which is true for $d \leq 4$. In higher dimensions, ϕ^4 interactions are not renormalizable and therefore irrelevant.

- In $d = 3$, the 7 scalars combine together with an eighth scalar φ obtained by dualizing the gauge potential: $F_{\mu\nu} = \varepsilon_{\mu\nu\kappa}\partial^\kappa \varphi$ to the **8** of $\mathsf{Spin}(8)$. This theory is not conformal and flows in the IR to the ABJM model. Here, the eight scalars and spinors are complemented with a topological gauge potential carrying no additional degrees of freedom.

- In $d = 6$, the $\mathcal{N} = (1,1)$ SUSY of SYM cannot be made conformal.

- In $d = 1$ and $d = 0$, one obtains the BFSS and IKKT matrix models, respectively. Both models were considered potential non-perturbative definitions of string and M-theory.

1.7. (0,2) Theory in $d = 6$

There is in fact another multiplet in $d = 6$ with 16 supercharges: Combining 5 real scalar fields with a two-form potential $B_{\mu\nu}$ giving rise to a self-dual curvature $H = \mathrm{d}B$ with $H_{\mu\nu\kappa} = \frac{1}{3!}\varepsilon_{\mu\nu\kappa\rho\sigma\tau}H^{\rho\sigma\tau}$ yields 8 bosonic degrees of freedom. This can be turned into a superconformal field theory.

By KK reduction, the (0,2) theory in $d = 6$ reduces to $\mathcal{N} = 2$ SYM in $d = 5$: the no. of scalars remains the same, we split $B_{\mu\nu} \to (B_{ij}, A_i := B_{i5})$. Because of $F_{ij} = \varepsilon ij5kmnH^{kmn}$, the degrees of freedom contained in B_{ij} are redundant, and we are left with $\mathcal{N} = 2$ SYM theory in $d = 5$.

2. Supersymmetric gauge theories from branes

References for this chapter: M-theory: [7], string theory and SUGRA: [8] and Branes and gauge theories: [9].

2.1. 11d Supergravity

Supergravity is the SUSY extension of general relativity (and therefore supersymmetry is now local). Moreover, it is the low energy limit of M- and string theories.

Field content:

The metric G_{MN}, which contains a dynamical traceless symmetric tensor of rank 2: $\frac{1}{2}(10-2)(10-1) = 44$ dofs.

The gravitini ψ_M^α, which are Majorna spinors with a vector index, satisfying the traceless condition $(\Gamma^M)^{\beta\alpha}\psi_{M\alpha} = 0$ and therefore contains $(9-1) \times 16 = 128$ dofs.

An antisymmetric 3-form A^3_{MNK} containing $\frac{1}{3!}(10-2)(10-3)(10-4) = 84$ dofs, completing the supermultiplet.

On-shell we thus have 128 bosonic and 128 fermionic degrees of freedom and 32 supercharges (Majorana spinor in 11d). The action reads as

$$S = \int \mathrm{d}^{11}x\sqrt{G}(R + |\mathrm{d}A^3|^2) + \int A^3 \wedge \mathrm{d}A^3 \wedge \mathrm{d}A^3 + \text{fermions} \ . \tag{2.1}$$

Recall couplind of a point particle to EM field: $\int \mathrm{d}\tau A_\mu \frac{\mathrm{d}}{\mathrm{d}\tau}x^\mu(\tau)$ or $\int \mathrm{d}x^\mu A_\mu$, where $x^\mu(\tau)$ is the particles worldline and A_μ is the gauge potential of the electromagnetic field. Here, we have a natural coupling of A^3 to a three-dimensional object:

$$\int \mathrm{d}^3\sigma \varepsilon^{abc} A_{MNK}\partial_a X^M \partial_b X^N \partial_c X^K \ . \tag{2.2}$$

This object is called an M2-brane. Note that we can perform the following duality: $A^3 \to F^4 = \mathrm{d}A^3 \to *F^4 = F^7 = \mathrm{d}A^6 \to A^6$, which suggests a natural coupling to a six-dimensional object, the M5-brane.

To find explicit solutions, we restrict ourselves to BPS solutions, as planar M2- and M5-branes would be expected to break half of the supersymmetries of 11d SUGRA. We have

$$\delta\psi_\mu = (\nabla_\mu + \ldots)\varepsilon \ , \tag{2.3}$$

and we introduce projectors $P_{M2} = \frac{1}{2}(\mathbb{1} + \Gamma^{012})$ and $P_{M5} = \frac{1}{2}(\mathbb{1} + \Gamma^{012345})$. The projected SUSYs we expect to be preserved are generated by parameters $\varepsilon^{M2} = P_{M2}\varepsilon$ and $\varepsilon^{M5} = P_{M5}\varepsilon$. Explicitly, the corresponding solutions are given by

$$\begin{aligned} \text{M2}: \quad & \mathrm{d}s^2 = H(\vec{y})^{-2/3}\mathrm{d}x^\mu \mathrm{d}x_\mu + H(\vec{y})^{1/3}\mathrm{d}\vec{y}^2 \ , \\ \text{M5}: \quad & \mathrm{d}s^2 = H(\vec{y})^{-1/3}\mathrm{d}x^\mu \mathrm{d}x_\mu + H(\vec{y})^{2/3}\mathrm{d}\vec{y}^2 \ . \end{aligned} \tag{2.4}$$

Here, x labels directions parallel to the brane and y directions perpendicular to it. The functions $H(\vec{y})$ are harmonic functions, i.e.

$$H(\vec{y}) = 1 + \sum_\ell \frac{C_\ell}{|\vec{y} - \vec{y}_\ell|^{p-2}} \ . \tag{2.5}$$

2.2. 10d Supergravities

Idea: Use a KK reduction as for SYM to reduce 11d SUGRA to 10d SUGRA. Recall that there are two cases: $\mathcal{N} = (1,1)$ or type IIA and $\mathcal{N} = (2,0)$ or type IIB. The first theory is obtained by KK reduction, the second one by a subsequent KK reduction.

Type IIA:

$G_{MN} \to G_{\mu\nu}, G_{\mu 10} =: A_\mu, G_{10,10} =: \Phi$

$A_{MNK} \to A_{\mu\nu\kappa}, A_{\mu\nu 10} = B_{\mu\nu}$

(In string theory, the fields $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ stem from the Neveu-Schwarz (NS)-NS-sector, the fields A_μ and $A_{\mu\nu\kappa}$ come from the Ramond-Ramond sector.)

Similarly to the coupling of the M2-branes to A^3, we have here natural couplings as follows:

$B_{\mu\nu}$ couples to fundamental strings (F1), and EM-dually ($B_{\mu\nu} \to F^3 = \mathrm{d}B \to *F^3 = F^7 = \mathrm{d}B^6 \to B^6$) to a solitonic, six-dimensional object called the NS5-brane.

The RR-field A_μ couples to D0- and EM-dually to D6-branes, while the RR-field $A_{\mu\nu\kappa}$ couples to D2- and EM-dually to D4-branes. Altogether, we arrive at the following extended objects in type IIA string theory: F1 (the fundamental string), D0-, D2-, D4-, D6- and NS5-branes.

The M-theory interpretations of these are as follows: A D0-brane is a KK mode of the graviton along the M-theory circle. The D2-brane is an M2-brane transverse to the circle. The D4-brane is an M5-brane wrapped on the circle, the NS5-brane is the M5-brane transverse to the circle. The D6 is the EM dual to the D0-brane.

There are differences between F1, NS5 and D0, D2, D4, D6-branes related to tension, which we will not discuss in detail.

Type IIB:

Here, one needs to compactify type IIA along a further special direction. Wrapping modes around this direction become massless in the $R \to 0$-limit and turn into a 10th dimension. We skip the details here. The field content is: NS-NS: $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ, RR: A_0, $A_{\mu\nu}$, $A_{\mu\nu\kappa\lambda}$. The natural coupling of the NS-NS-sectors is as in type IIA, the coupling of the RR-fields is A_0: D(-1)- and D7-branes, $A_{\mu\nu}$: D1- and D5-branes, $A_{\mu\nu\kappa\lambda}$: D3-branes. Note that the D3-branes are EM-self-dual!

Solutions to the theories can be constructed as for M-theory. Here, one imposes the condition $\varepsilon_L = \Gamma^{01\dots p}\varepsilon_R$, and the preserved SUSY are $\varepsilon_L Q_L + \varepsilon_R Q_R$. Note that at the endpoints of strings (on the D-branes) the left- and right-movers are not independent.

2.3. Worldvolume descripton of D-branes

A nice motivation of the world-volume action of D-branes can be found in [10], section 4.1. Recall that the action of a particle (and a a string) can be defined as the area of its worldvolume. The generalization of this to D-branes is given by the Dirac-Born-Infeld action, which (in static gauge) is given by

$$S = T_p \int \mathrm{d}^{p+1}x \mathrm{e}^{-\Phi} \sqrt{-\det(G_{ab} + \mathcal{F}_{ab})} + \text{WZ couplings} + \text{SUSY completion} \ , \tag{2.6}$$

where

$$\mathcal{F}_{ab} = \alpha' F_{ab} + B_{ab} + \alpha' \partial_a X^m \partial_b X_m \ . \tag{2.7}$$

Note that B_{ab} is the pullback of $B_{\mu\nu}$ along $X^\mu : \Sigma \to \mathcal{M}$. F_{ab}: $\mathsf{U}(1)$ gauge field strength, T_p: tension of the brane.

Expanding S in powers of α' for $B = 0$ in flat space, we have

$$S_{\text{eff}} = \alpha' \int \mathrm{d}^{p+1}x F_{ab} F^{ab} + (\partial_a X^m \partial^a X_m) + \text{SUSY completion} \ . \tag{2.8}$$

Expection 16 supercharges, we therefore arrive at maximally supersymmetric Yang-Mills theory. The general rule is that a Dp-brane with gravity turned off ($\alpha' \to 0$) is effectively described by $p+1$-dimensional maximally SUSY YM theory. In particular, a D3-brane gives rise to $\mathcal{N} = 4$, $d = 4$ SYM theory.

Consider a stack of two Dp-branes. There are strings from brane 1 to brane 1, from brane 1 to brane 2, from brane 2 to brane 1 and from brane 2 to brane 2. These degrees of freedom can be encoded in a matrix, and one should therefore consider SYM theory with gauge group $\mathsf{U}(2)$ for a stack of two Dp-branes, gauge group $\mathsf{U}(N)$ for a stack of N Dp-branes.

Recall that the potential for $\mathcal{N} = 4$, $d = 4$ SYM theory was given by $[\phi_{ij}, \phi_{kl}][\phi^{ij}, \phi^{kl}]$. A vacuum is therefore given by $[\phi_I, \phi_J] = 0$, and we can simultaneously diagonalize the Lie-algebra valued matrices ϕ_I. The eigenvalues of these matrices describe the fluctuations of the Dp-branes in the corresponding direction: The kth eigenvalue of ϕ_1 gives the x^5-coordinate of the k-th D-brane. One can factor out a $\mathsf{U}(1)$ from the gauge group, describing the position of the center of mass.

Comment on NS5: In type IIA, we have $\varepsilon_L = \Gamma^{0\dots5}\varepsilon_R$ and $\varepsilon_R = \Gamma^{0\dots5}\varepsilon_L$, while in type IIB, we have $\varepsilon_L = \Gamma^{0\dots5}\varepsilon_R$ and $\varepsilon_R = -\Gamma^{0\dots5}\varepsilon_L$. Therefore, the NS5-brane in type IIA should have a chiral description in terms of a $\mathcal{N} = (2,0)$ theory, while in type IIB, it should be described by a $\mathcal{N} = (1,1)$ theory.

2.4. Dualities

Without giving many details, let us introduce two kinds of dualities:

T-duality relates a theory on $M \times S^1_R$ to a theory on $M \times S^1_{\frac{1}{R}}$, where the subscripts indicate the radii of the circles. This is a duality mapping type IIA configurations to IIB and vice versa:

IIA	F1 (p,w)	Dp $\|\|$ S^1	Dp $\perp$ S^1	NS5 $\|\|$ S^1	NS5 $\perp$ S^1
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
IIB	F1 (w,p)	D$(p-1)$ $\perp$ S^1	D$(p+1)$ $\|\|$ S^1	NS5 $\|\|$ S^1	KK monopole

(This can be derived, e.g. from compactifications of M-theory, cf. [9].)

S-duality is a symmetry of type IIB string theory. It is related to the electro-magnetic symmetry of $\mathcal{N}=4$, $d=4$ SYM theory. The latter theory had complex coupling constant $\tau = \frac{\theta}{2\pi} + \frac{\mathrm{i}}{g^2_{\mathrm{YM}}}$ and a symmetry group $\mathsf{SL}(2,\mathbb{Z})$ acting by $\tau \to \frac{a\tau+b}{c\tau+d}$. Here, S-duality corresponds to $\tau \to -\frac{1}{\tau}$, a strong-weak coupling symmetry.

Under S-duality, F1$\leftrightarrow$D1, D3$\leftrightarrow$D3, and NS5$\leftrightarrow$D5. (Remark: The full $\mathsf{SL}(2,\mathbb{Z})$-symmetry is recovered, once (p,q)-1 and 5-branes are included in the picture. These are bound states of p F1 and q D1-branes and p D5- and q NS5-branes, respectively.)

2.5. Branes ending on other branes

F1 can end on all Dp-branes.

F1 on D5 $\stackrel{S}{\to}$ D1 on NS5

F1 on D3 $\stackrel{S}{\to}$ D1 on D3 $\stackrel{T\dots}{\to}$ Dp on D$p+2$ ($\to$ Monopoles)

D3 on D5 $\stackrel{S}{\to}$ D3 on NS5 $\stackrel{T}{\to}$ D4 on NS5 ($\to$ Hanany-Witten)

Any Dp-brane with $p \leq 6$ can end on an NS5-brane

In M-theory, most of this corresponds to M2-branes ending on M5-branes

SUSY analysis: For each stack of branes with worldvolume in the directions $i_0, \dots, i_p$, the condition $\varepsilon_L = \Gamma^{i_0}\dots\Gamma^{i_p}\varepsilon_R$ has to be satisfied. This yields for each stack of branes one condition, reducing supersymmetry in general by $\frac{1}{2}$.

Example: A $D1$-brane ending on a $D3$-brane preserves 8 supercharges ($\mathcal{N}=2$ in 4d)

2.6. D1-D3-branes and Monopoles

Warm-up exercise: D1-brane ending on a D3-brane

IIB	0	1	2	3	4	5	6	7	8	9
D3	$\times$	$\times$	$\times$	$\times$	0	0	0	0	0	0
D1	$\times$	0	0	0	0	0	$\times$	0	0	0

(2.9)

Analysis of SUSY: $\varepsilon_L = \Gamma^{0\cdots 3}\varepsilon_R = \Gamma^{06}\varepsilon_R$, so 8 supercharges are preserved.

Perspective of D1-brane: 2d SYM with fields A_0, A_6, $X^1, \dots, X^5, X^7, X^8, X^9$.

A_0, A_6 and X^1, X^2, X^3 combine into vector superfield

D1-brane ending on D3s: X^7, X^8, X^9 satisfy Dirichlet boundary conditions. That is, at $x^6 = 0$ we have $X^7, X^8, X^9 = 0$

The D,F-flatness conditions in a superfield formulation of this theory amounts to $\frac{\mathrm{d}x^6}{\mathrm{d}X}^i + \frac{1}{2}\varepsilon^{ijk}[X^j, X^k] = 0$, $i = 1,2,3$. This is the so-called *Nahm equation*, which is also the BPS equation in the worldvolume description of the D3-brane: $\delta\psi \sim (\frac{\mathrm{d}x^6}{\mathrm{d}X}^i + \frac{1}{2}\varepsilon^{ijk}[X^j, X^k])\varepsilon$.

A simple solution is obtained from the ansatz $X^i = f(x^6)G^i$. Plugging this into the Nahm equation results in $f(x^6) = \frac{1}{x^6}$ and $[G^i, G^j] = \varepsilon^{ijk}G^k$. The G^i form actually irreducible representations of $\mathsf{SU}(2)$, and at each x^6, the worldvolume of the D1-brane therefore polarizes into a fuzzy or noncommutative sphere, gaining two noncommutative spatial dimensions. The radius of this sphere is given by $f(x^6)$ and diverges as $x^6 \to 0$. This configuration therefore describes the smooth transition between a D1-brane and a D3-brane with partially noncommutative volume.

The shape of this "fuzzy funnel" can be derived balancing currents, cf. section 16.3 in [11].

From the D3-brane perspective, this configuration corresponds to a magnetic monopole, and the Nahm equation is a nice way of encoding the moduli space of monopoles.

2.7. Higgs and Coulomb branches in gauge theory

Reference: [12], cf. Arjun's lectures.

Gauge theories in 4d: different couplings can lead to different phases (Higgs, Coulomb, confining), characterized by an effective potential between electric test charges:

Coulomb: $V(r) \sim \frac{1}{r}$, Higgs: $V(r) \sim const.$, Confining: $V(r) \sim r$.

In SUSY gauge theories: moduli space of ground states, ground states can be in different phases.

Moduli space consists of separate branches touching each other at transition point.

Consider a SUSY gauge theory with gauge group G, vector superfield $V = (A_\mu, \lambda, D)$ and matter superfields $Q_i = (q_i, \psi, F)$. The superpotential gives rise to the interaction terms:

$$\mathcal{L}_{int} = q_i^\dagger \lambda \psi_i + c.c. + (q_i^\dagger t_a q^i)^2 \ , \tag{2.10}$$

where t_a are generators of G.

D-flatness condition: Ex: two superfields (from $\mathcal{N} = 2$ SUSY, one hypermultiple=two chiral multiplets)

$D = (Q^\dagger Q - \tilde{Q}^\dagger \tilde{Q})$, potential: D^2.

Continuum of degenerate vacua: $\langle Q \rangle = \langle \tilde{Q} \rangle = a \in \mathbb{C}$.

Classical moduli space of vacua parameterized by $\langle q_i \rangle$ with $D = 0$. gauge superfield gets mass $|a|$ by "eating" one chiral superfield degree of freedom.

Consider now SUSY QCD: $G = \mathsf{SU}(N_c)$, # flavors= N_f.

If $N_f < N_c$:

$$Q = \tilde{Q} = \begin{pmatrix} a_1 & 0 & \cdots & & & \\ 0 & a_2 & & & & \\ & & \ddots & & & \\ & & & a_{N_f} & 0 & \cdots \end{pmatrix} \ . \tag{2.11}$$

Gauge inv. description: *mesons* $M^i_j = Q^i \tilde{Q}_j$

If $N_f \geq N_c$:

$$Q = \tilde{Q} = \begin{pmatrix} a_1 & 0 & \cdots & \\ 0 & a_2 & & \\ & & \ddots & \\ & & & a_{N_f} \\ 0 & \cdots & & 0 \\ \vdots & & & \end{pmatrix} . \tag{2.12}$$

Gauge inv. description: mesons and *baryons*: $B = \varepsilon_{i_1 \dots i_{N_c}} Q^{i_1} \dots Q^{i_{N_c}}$ and similarly $\tilde{B}$.

Gauge group *higgsed*: gauge group broken to subgroup: $B = \tilde{B} = 0$, $\text{rk}(M) = k$: $\mathsf{SU}(N_c) \to \mathsf{SU}(N_c - k)$

Coulomb phase: $\mathsf{U}(1)$-factors in the gauge theory present

2.8. D4-D6-NS5-configurations

Consider the following brane configuration (identify: $x^6 = s$)

IIA	0	1	2	3	4	5	6
NS5	×	×	×	×	×	×	s_0
NS5	×	×	×	×	×	×	s_1
D4	×	×	×	×			$s_0 \leq s \leq s_1$

(2.13)

with $s_1 - s_0 = L_s$.

The Lorentz group is broken down: $\mathsf{SO}(1,9) \to \mathsf{SO}(1,3) \times \mathsf{SO}(2) \times \mathsf{SO}(3)$

NS5-branes heavy, modes decouple, "frozen"

L_s to be considered small: Consequently, fluctuations in the s-direction are of momentum and energy $\sim \frac{1}{L_s}$, they are very difficult to excite and therefore decouple. We can then integrate:

$$\frac{1}{g_{YM}^2} \int \mathrm{d}^5 x \ \text{SYM}_{5d} \quad \to \quad \frac{L_s}{g_{YM}^2} \int \mathrm{d}^4 x \ \text{SYM}_{4d} \tag{2.14}$$

Field content on D4-brane: A_μ, $\mu = 0, \dots, 3, 6$, scalars X^I with $I = 4, 5, 7, 8, 9$, $\mathcal{N} = 2$ SYM theory in 5d or rather 4d.

Assume now that we separate the N_c D4s in the 4,5-directions, at positions $a_1, \dots, a_{N_c} \in \mathbb{C}^{N_c}$. The strings connecting D4s with a finite distance in the 4,5-direction become massive and can be decoupled. Thus, the gauge group is broken in general from $\mathsf{U}(N_c) \to \mathsf{U}(1)^{\times N_c}$, and we are in the *Coulomb branch* of the underlying gauge theory. A point in the Coulomb branch is characterized by the N_c complex numbers a_i.

To add matter in the fundamental representation, we can add infinitely long D4s for $s \leq s0$ to the left of the above configuration. As the D4s are infinitely long, they are very massive and correspondingly do not fluctuate. Strings connecting these D4s to the D4s suspended between the NS5s give therefore rise to a hypermultiplet containing chiral superfields Q and $\tilde{Q}$ in the fundamental and antifundamental representation. The distances of the infinite D4s to the suspended ones gives rise to masses for these hypermultiplets. The number of infinite D4s is the number of flavors.

To enter the Higgs branch, we have to add D6-branes at the ends of the D4-branes as follows:

$$\begin{array}{lccccccccc}
\text{IIA} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\text{NS5} & \times & \times & \times & \times & \times & \times & s_0 & & & \\
\text{NS5} & \times & \times & \times & \times & \times & \times & s_1 & & & \\
\text{D4} & \times & \times & \times & \times & a_i^x & a_i^y & s_0 \leq s \leq s_1 & & & \\
\text{D4} & \times & \times & \times & \times & m_i^x & m_i^y & s_i \leq s \leq s_0 & & & \\
\text{D6} & \times & \times & \times & \times & m_i^x & m_i^y & s_i \leq s \leq s_0 & \times & \times & \times
\end{array} \tag{2.15}$$

The formerly infinite D4s now end each on a D6-brane. The distance between the NS5s and the D6s, $s_0 - s_i$, is assumed to be small again, so that there are no fluctuations. This, together with the boundary conditions on the NS5 together with the boundary conditions on the D6s forces all fluctuations to vanish. Thus, there are no new degrees of freedom. Note however, that the positions s_i of the D6-branes can be different.

For D4-branes suspended between NS5 and D6-branes, there is the so-called s-rule, which is complicated to derive, and cannot be observed with our techniques so far: More than one D4-brane suspended between the same NS5-brane and the same D6-brane is no longer supersymmetric.

To enter the Higgs-branch while respecting the s-rule, we move two D6-branes to the same position: $m_1 = m_2$. We also move one of the D4-branes to this position: $a_1 = m_1$. Assume that $s_1 < s_2$. We can then break the D4 connecting the NS5-brane with the D6-brane at s_1 and move the part between the D6-branes at s_1 and s_2 away. The position of this D4-brane in the $x^{7,8,9}$-direction will give the coordinates on the Higgs-branch.

2.9. Hanany-Witten configurations

References: [13], conventions used here: [14]

Consider the following configuration in type IIB string theory:

	0	1	2	3	4	5	6	7	8	9
Coordinates	x^0	x^1	x^2	$\vec{z}$			s	$\vec{y}$		
Symmetries	$\mathsf{SO}(1,2)$			$\mathsf{SO}(3)_Z$				$\mathsf{SO}(3)_Y$		
NS5	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	p_σ	$\vec{\nu}_\sigma$		
D5	$\times$	$\times$	$\times$	$\vec{0}$			λ_j	$\times$	$\times$	$\times$
D3	$\times$	$\times$	$\times$	$\vec{0}$			$\times$	$\vec{y}^{\,D3}$		
$\mathcal{N}=1$ fields/Ψ	$\mathcal{V}$			$\mathcal{Z}$		$\mathcal{V}$				
Ψ components	v_0	v_1	v_2	Z		Z_3				
$\mathcal{N}=1$ fields/Υ							$\mathcal{X}$		$\mathcal{Y}$	
Υ components							v_6	Y_1	Y	

Here, the vector superfield $\mathcal{V}$ and the chiral superfield $\mathcal{Z}$ are combined into a $\mathcal{N}=2$, $d=4$ vector multiplet, while the two chiral superfields $\mathcal{X}$ and $\mathcal{Y}$ form a hypermultiplet. A picture of this configuration is given below in figure 1.

The bulk theory on the D3-branes extending in the $x^{0,1,2,6}$-directions is given by $\mathcal{N}=4$ SYM theory. The two kinds of fivebranes (NS5 and D5) can be at the same positions as

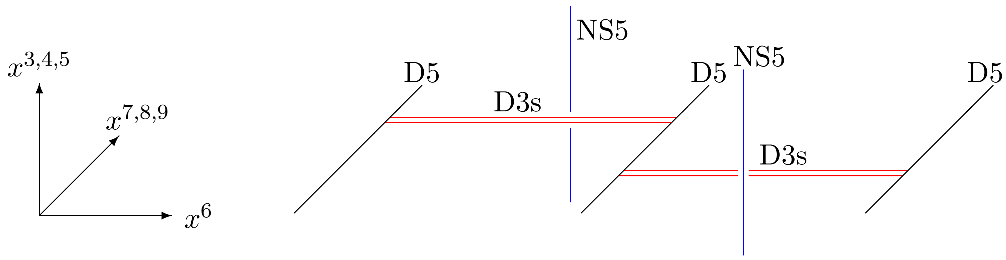

Figure 1: A picture of a Chalmers-Hanany-Witten configuration in the Higgs branch.

the D3-branes, in which case they form a defect of boundary in the 4d bulk theory. To make the superfields compatible with the three-dimensional boundaries, one rearranges degrees of freedom: One component of the vector superfield, Z_3 is turned into a scalar field by dimensional reduction. Another component of the chiral superfield $\mathcal{X}$ is turned into a gauge potential along x^6. All superfields will depend on the variable s parameterizing the x^6-direction. The field expansion is:

$$\begin{aligned} \mathcal{V} &= \mathrm{i}\theta_\alpha\bar{\theta}^\alpha Z_3 - \theta\sigma^\mu_{3\mathrm{d}}\bar{\theta}v_\mu + \mathrm{i}\theta^2\bar{\theta}\bar{\lambda} - \mathrm{i}\bar{\theta}^2\theta\lambda + \tfrac{1}{2}\theta^2\bar{\theta}^2 D \ , \\ \mathcal{X} &= v_6(y) + \mathrm{i}Y_1(y) + \sqrt{2}\theta\psi(y) + \theta^2 G(y) \ , \\ \mathcal{U}^1 &= \mathcal{Z} = Z(y) + \sqrt{2}\theta\chi^i(y) + \theta^2 F^1(y) \ , \\ \mathcal{U}^2 &= \mathcal{Y} = Y(y) + \sqrt{2}\theta\chi^i(y) + \theta^2 F^2(y) \ . \end{aligned} \tag{2.16}$$

What happens at defects?

Consider an NS5-brane at $s = p_1$ with two stacks of D3-branes ending on either side. N_L: # of D3-branes on the left, N_R: # of D3-branes on the right. Both stacks are connected by strings localized at the NS5-branes. These give rise to a hypermultiplet $(B, \tilde{B})$ with fields in the bifundamental of $\mathsf{U}(N_L) \times \overline{\mathsf{U}(N_R)}$ and the complex conjugate. The distance between the stacks ending on the NS5-branes is proportional to the mass of the hypermultiplet.

A stack of N_f D5-brane at $s = \lambda_1$, with different coordinates $x^{3,4,5}$ than a stack of N_c D3-branes leads to N_f fundamental/antifundamental hypermultiplets: The D5-branes are heavy and its fluctuations can again be considered as frozen. The result is a hypermultiplet $(\mathcal{Q}, \tilde{\mathcal{Q}})$ corresponding to D3-D5- and D5-D3-strings, where $\mathcal{Q}$ is in the fundamental of $\mathsf{U}(N_c)$, while $\tilde{\mathcal{Q}}$ is in the corresponding antifundamental representation.

The total action for a HW configuration consists of the following pieces: In 3d, the place of the chiral curvature superfield W_α is taken by

$$\Sigma := \epsilon^{\alpha\beta}\bar{D}_\alpha(e^{2\mathrm{i}\mathcal{V}}D_\beta e^{-2\mathrm{i}\mathcal{V}}) \ . \tag{2.17}$$

The bulk action is

$$\begin{aligned} S_{\mathrm{bulk}} = \int \mathrm{d}s\,\mathrm{d}^3x\,\mathrm{tr}\Big[\int \mathrm{d}^4\theta\,\big(-\tfrac{1}{16}\Sigma^2 - \tfrac{1}{4}(\mathrm{e}^{2\mathrm{i}\mathcal{V}}(\partial_s - \bar{\mathcal{X}})e^{-2\mathrm{i}\mathcal{V}} - \mathcal{X})^2 + \tfrac{1}{2}\mathrm{e}^{2\mathrm{i}\mathcal{V}}\bar{\mathcal{U}}_i e^{-2\mathrm{i}\mathcal{V}}\mathcal{U}^i\big) \\ + \tfrac{\mathrm{i}}{2}\int \mathrm{d}^2\theta\ \epsilon_{ij}\mathcal{U}^i[\partial_s + \mathcal{X}, \mathcal{U}^j] - \tfrac{\mathrm{i}}{2}\int \mathrm{d}^2\bar{\theta}\ \epsilon_{ij}\bar{\mathcal{U}}^i[\partial_s - \bar{\mathcal{X}}, \bar{\mathcal{U}}^j]\Big] \ . \end{aligned} \tag{2.18}$$

Potential Fayet-Iliopoulos terms have to be added in an $\mathcal{N}=2$ invariant manner:

$$S_{\rm FI} = \int \mathrm{d}s\, \mathrm{d}^3x\ \mathrm{tr}\left(\mathrm{i}\hat{\nu}_3(s)\int \mathrm{d}^4\theta\, \mathcal{V} - \tfrac{1}{2}\hat{\nu}(s)\int \mathrm{d}^2\theta\, \mathcal{Z} - \tfrac{1}{2}\bar{\hat{\nu}}(s)\int \mathrm{d}^2\bar{\theta}\, \bar{\mathcal{Z}}\right)\ . \tag{2.19}$$

The contribution of D5-defects at n positions λ_j is

$$\begin{aligned} S_{D5} = \tfrac{1}{2}\sum_{j=1}^{n}\int \mathrm{d}^3x\, \mathrm{d}^4\theta\, \Big(&\bar{\mathcal{Q}}_{1j}\mathrm{e}^{-2\mathrm{i}\mathcal{V}(\lambda_j)}\mathcal{Q}_{1j} + \mathcal{Q}_{2j}\mathrm{e}^{2\mathrm{i}\mathcal{V}(\lambda_j)}\bar{\mathcal{Q}}_{2j}) + \\ &\int \mathrm{d}^2\theta\, \mathcal{Q}_{2j}\mathcal{U}^1(\lambda_j)\mathcal{Q}_{1j} + \int \mathrm{d}^2\bar{\theta}\, \bar{\mathcal{Q}}_{1j}\bar{\mathcal{U}}^1(\lambda_j)\bar{\mathcal{Q}}_{2j}\Big)\ . \end{aligned} \tag{2.20}$$

The contribution of NS5-branes at k positions p_σ is

$$\begin{aligned} S_{b,1} = \tfrac{1}{2}\sum_{\sigma=1}^{k}\int \mathrm{d}^3x\ \mathrm{tr}\int \mathrm{d}^4\theta\Big(&\mathrm{e}^{2\mathrm{i}\mathcal{V}(p^L_\sigma)}\bar{\mathcal{B}}_{1\sigma}\mathrm{e}^{-2\mathrm{i}\mathcal{V}(p^R_{\sigma-1})}\mathcal{B}_{1\sigma} + \mathrm{e}^{2\mathrm{i}\mathcal{V}(p^R_{\sigma-1})}\bar{\mathcal{B}}_{2\sigma}\mathrm{e}^{-2\mathrm{i}\mathcal{V}(p^L_\sigma)}\mathcal{B}_{2\sigma}\Big) \\ &+ \int \mathrm{d}^2\theta(\mathcal{B}_{2\sigma}\mathcal{U}^1(p^R_{\sigma-1})\mathcal{B}_{1\sigma} - \mathcal{B}_{1\sigma}\mathcal{U}^1(p^L_\sigma)\mathcal{B}_{2\sigma}) + \\ &\int \mathrm{d}^2\bar{\theta}(\bar{\mathcal{B}}_{1\sigma}\bar{\mathcal{U}}^1(p^R_{\sigma-1})\bar{\mathcal{B}}_{2\sigma} - \bar{\mathcal{B}}_{2\sigma}\bar{\mathcal{U}}^1(p^L_\sigma)\bar{\mathcal{B}}_{1\sigma})\Big)\ . \end{aligned} \tag{2.21}$$

Computing the D- and F-flatness conditions yields

$$\begin{aligned} \nabla_s Y_1 + \tfrac{\mathrm{i}}{2}[Z,\bar{Z}] + \tfrac{\mathrm{i}}{2}[Y,\bar{Y}] + \tfrac{\mathrm{i}}{2}\sum_{j=1}^{n}(Q_{1j}\bar{Q}_{1j} - \bar{Q}_{2j}Q_{2j})\delta(s-\lambda_j) \\ + \tfrac{\mathrm{i}}{2}\sum_{\sigma=1}^{k}\left(B_{2\sigma}\bar{B}_{2\sigma} - \bar{B}_{1\sigma}B_{1\sigma} - \nu_{3\sigma}\mathbb{1}\right)\delta(s-p^L_\sigma) \\ + \left(B_{1\sigma}\bar{B}_{1\sigma} - \bar{B}_{2\sigma}B_{2\sigma} + \nu_{3\sigma}\mathbb{1}\right)\delta(s-p^R_{\sigma-1}) = 0\ , \end{aligned} \tag{2.22}$$

$$\begin{aligned} \nabla_s Y + \mathrm{i}[Y_1,Y] + \tfrac{\mathrm{i}}{2}\sum_{j=1}^{n}Q_{1j}Q_{2j}\delta(s-\lambda_j) \\ + \tfrac{\mathrm{i}}{2}\sum_{\sigma=1}^{k}\left(B_{1\sigma}B_{2\sigma} + \nu_\sigma\mathbb{1}\right)\delta(s-p^R_{\sigma-1}) - \left(B_{2\sigma}B_{1\sigma} + \nu_\sigma\mathbb{1}\right)\delta(s-p^L_\sigma) = 0\ , \end{aligned} \tag{2.23}$$

$$\nabla_s Z + \mathrm{i}[Y_1,Z] = 0\ , \tag{2.24}$$

and

$$[Z,Y] = 0\ . \tag{2.25}$$

Recall that our description is only $\mathcal{N}=1$ SUSY invariant, which is embedded in the actual $\mathcal{N}=2$. To complete the equations, we have to make them invariant under $\mathsf{SO}(3)_{Y,Z}$. From (2.24) and (2.25), it follows that

$$[Z,Y] = [Z_3,Y] = [Z,Y_1] = [Z_3,Y_1] = 0 \qquad \text{and} \qquad \nabla_s Z = \nabla_s Z_3 = 0\ , \tag{2.26}$$

The remaining eoms on Y are:

$$\nabla_s Y_1 + \tfrac{\mathrm{i}}{2}[Y,\bar{Y}] + \tfrac{\mathrm{i}}{2}\sum_{j=1}^{n}(Q_{1j}\bar{Q}_{1j} - \bar{Q}_{2j}Q_{2j})\delta(s-\lambda_j)$$
$$+ \tfrac{\mathrm{i}}{2}\sum_{\sigma=1}^{k}\left(B_{2\sigma}\bar{B}_{2\sigma} - \bar{B}_{1\sigma}B_{1\sigma} - \nu_{3\sigma}\mathbb{1}\right)\delta(s-p^L_\sigma)$$
$$+ \left(B_{1\sigma}\bar{B}_{1\sigma} - \bar{B}_{2\sigma}B_{2\sigma} + \nu_{3\sigma}\mathbb{1}\right)\delta(s-p^R_{\sigma-1}) = 0\ , \quad (2.27)$$

$$\nabla_s Y + \mathrm{i}[Y_1, Y] + \tfrac{\mathrm{i}}{2}\sum_{j=1}^{n} Q_{1j}Q_{2j}\delta(s-\lambda_j)$$
$$+ \tfrac{\mathrm{i}}{2}\sum_{\sigma=1}^{k}\left(B_{1\sigma}B_{2\sigma} + \nu_\sigma\mathbb{1}\right)\delta(s-p^R_{\sigma-1}) - \left(B_{2\sigma}B_{1\sigma} + \nu_\sigma\mathbb{1}\right)\delta(s-p^L_\sigma) = 0\ . \quad (2.28)$$

With $B = Q = 0$, this is just the complex form of the Nahm equation!

Many phenomena of the CHW configuration are fully reflected in the gauge theory description.

Masses for the D5 are given by the distance in $x^{3,4,5}$-directions, encoded in $\vec{Z}_{aa}$ for D5-branes at $\vec{z} = 0$. This appears directly in the action. EOM for auxiliary field J in the hypermuliplet $(\mathcal{Q}, \tilde{\mathcal{Q}})$: $\bar{J} + QZ = 0$. The term $\bar{J}J$ in the action turns into $QZ\bar{Z}\bar{Q}$, a mass term with mass squared $Z\bar{Z}$, as expected. The masses for the bifundamental hypermultiplets can be similarly derived.

Breaking of D3-branes on D5-branes: If the stack of D3-brane breaks in the y^1-direction, the field Y_1 has a discontinuity at the position $s = \lambda_1$ of the D5-brane. Eqn. (2.22) allows this, if $Q_{1j}\bar{Q}_{1j} - \bar{Q}_{2j}Q_{2j} \neq 0$. The flatness condition on $\bar{J}$ implies that

$$A\begin{pmatrix} Q_{1p} \\ \bar{Q}_{2p} \end{pmatrix} := \begin{pmatrix} Z_3 - z_3^{D5} & \bar{Z} - z^{D5} \\ Z - \bar{z}^{D5} & -(Z_3 - z_3^{D5}) \end{pmatrix}\begin{pmatrix} Q_{1p} \\ \bar{Q}_{2p} \end{pmatrix} = 0\ , \quad (2.29)$$

where we modified the action to allow for arbitrary positions of the D5-brane in the $\vec{z}$-directions. We see that we need $\det(A) = 0$, which amounts to the distance between the D3-branes and the D5-branes being zero.

Comments:

- The Coulomb branch is again given by HW-configurations, in which the D3-branes have a finite distance in the $x^{3,4,5}$-direction.
- The Higgs-branch is obtained by moving two NS5-branes relatively to each other in the $x^{7,8,9}$-directions. To connect these two by D3-branes, we add a D5-brane in the middle and break the stack of D3-branes on them, cf. figure 1.
- S-duality, or gauge theory mirror symmetry, interchanges NS5- and D5-branes, the Coulomb and the Higgs branch etc.

- One can add (p, q)-fivebranes, this will give rise to a Chern-Simons gauge theory on the boundary.
- The ABJM model can be obtained in this way from a HW-configuration: Compactify the x^6-direction on a circle and put an NS5-brane and an (p, q)-brane at opposite points on the circle. Connect both by N D3-branes in both directions. Assume that the size of the circle is small so that fluctuations along the circle direction is suppressed. This gives a 3d SUSY gauge theory with bifundamental matter and a Chern-Simons gauge interaction. Flowing to the IR, one recovers the ABJM model.

References

[1] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond,* Cambridge, UK: Univ. Pr. (1998).

[2] A. Van Proeyen, *Tools for supersymmetry,* hep-th/9910030.

[3] R. Haag, J. T. Lopuszanski, and M. Sohnius, *All possible generators of supersymmetries of the S-matrix,* Nucl. Phys. B **88** (1975) 257.

[4] J. C. Baez and J. Huerta, *Division algebras and supersymmetry,* 0909.0551 [hep-th].

[5] L. Brink, J. H. Schwarz, and J. Scherk, *Supersymmetric Yang-Mills theories,* Nucl. Phys. B **121** (1977) 77.

[6] N. Seiberg, *Notes on theories with 16 supercharges,* Nucl. Phys. Proc. Suppl. **67** (1998) 158 [hep-th/9705117 [hep-th]].

[7] D. S. Berman, *M-theory branes and their interactions,* Phys. Rept. **456** (2008) 89 [0710.1707 [hep-th]].

[8] E. D'Hoker and D. Z. Freedman, *Supersymmetric gauge theories and the AdS/CFT correspondence,* hep-th/0201253.

[9] A. Giveon and D. Kutasov, *Brane dynamics and gauge theory,* Rev. Mod. Phys. **71** (1999) 983 [hep-th/9802067].

[10] C. V. Johnson, *D-brane primer,* hep-th/0007170.

[11] B. Zwiebach, *A first course in string theory,* Cambridge, UK: Univ. Pr. (2004) 558 p.

[12] K. A. Intriligator and N. Seiberg, *Lectures on supersymmetric gauge theories and electric-magnetic duality,* Nucl. Phys. Proc. Suppl. **45BC** (1996) 1 [hep-th/9509066].

[13] A. Hanany and E. Witten, *Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics,* Nucl. Phys. B **492** (1997) 152 [hep-th/9611230].

[14] S. A. Cherkis, C. O'Hara, and C. Saemann, *Super Yang-Mills theory with impurity walls and instanton moduli spaces,* Phys. Rev. D **83** (2011) 126009 [1103.0042 [hep-th]].