Clifford algebras & Courant algebroids

In Carlos's previous lecture(s) he introduced Courant algebroids as the intersection of "Courant spaces", "Dorfman algebras" and "vector bundles". In particular, he defined the Courant bracket on sections of the double tangent bundle $\mathbb{T}M := TM \oplus T^*M$ via

$$[X+\alpha, Y+\beta]_H = [X,Y] + \mathcal{L}_X\beta - \mathcal{L}_Y\alpha + \tfrac{1}{2}d(\alpha(Y) - \beta(X)) + \iota_Y \iota_X H$$

where $H \in \Omega^3(M)$ is closed and $[H] \in H^3_{dR}(M)$ is the Ševera class of the exact Courant algebroid E:

$$0 \to T^*M \xrightarrow{i} E \xrightarrow{\rho} TM \to 0$$

When I first met the Courant bracket, I found it very mysterious and it was only after I understood (via Hitchin) its relationship to Clifford algebras, that it was demystified for me. I hope to show you that the above Courant bracket is in a certain sense 'natural'.

① Clifford algebras and Clifford modules

Let V be a finite-dimensional $\mathbb{R}$ vector space and $B: V \times V \to \mathbb{R}$ a symmetric bilinear form and let $Q: V \to \mathbb{R}$, defined by $Q(v) = B(v,v)$ be the corresponding 'norm'. The pair (V,Q) is called a quadratic vector space and they are objects of a category **QVec** whose morphisms $(V,Q_V) \xrightarrow{\phi} (W,Q_W)$ are linear maps $\phi: V \to W$ such that $\phi^* Q_W = Q_V$. That is, $Q_W(\phi(v)) = Q_V(v)$ $\forall v \in V$.

Let A be a real associative algebra with unit 1_A. A linear map $\phi : V \to A$, where (V,Q) is a quadratic vector space, is said to be **Clifford** if $\phi(v)^2 = -Q(v)\,1_A \quad \forall v \in V$. Clifford maps from a fixed quadratic vector space (V,Q) are the objects of a category $\mathbf{Cliff(V,Q)}$ whose morphisms are commutative triangles

$$\begin{tikzcd} & V \arrow[dl, "\phi"'] \arrow[dr, "\phi'"] & \\ A \arrow[rr, "f"] & & A' \end{tikzcd}$$

where $f : A \to A'$ is an algebra homomorphism.

A **Clifford algebra** for (V,Q) is an initial object in $\mathbf{Cliff(V,Q)}$. In other words, it is an associative algebra $C\ell(V,Q)$ and a Clifford map $i : V \to C\ell(V,Q)$ such that given any Clifford map $\phi : V \to A$ there exists a unique algebra homomorphism $\Phi : C\ell(V,Q) \to A$ such that the following triangle commutes

$$\begin{tikzcd} & V \arrow[dl, "i"'] \arrow[dr, "\phi"] & \\ C\ell(V,Q) \arrow[rr, "\Phi"] & & A \end{tikzcd}$$

As usual for initial objects, if a Clifford algebra exists for (V,Q) it is unique up to a unique isomorphism. If you have not seen this before, suppose $i : V \to C$ and $i' : V \to C'$ be Clifford algebras for (V,Q). Then we get commuting triangles

$$\begin{tikzcd} & V \arrow[dl, "i"'] \arrow[dr, "i'"] & \\ C \arrow[rr, "\phi", "\exists!"'] & & C' \end{tikzcd} \qquad \begin{tikzcd} & V \arrow[dl, "i'"'] \arrow[dr, "i"] & \\ C' \arrow[rr, "\phi'", "\exists!"'] & & C \end{tikzcd}$$

which compose to

$$\begin{tikzcd} & V \arrow[dl, "i"'] \arrow[dr, "i"] & \\ C \arrow[rr, "\phi'\circ\phi"'] & & C \end{tikzcd} \qquad \begin{tikzcd} & V \arrow[dl, "i'"'] \arrow[dr, "i'"] & \\ C' \arrow[rr, "\phi\circ\phi'"'] & & C' \end{tikzcd}$$

but by uniqueness, $\phi'\circ\phi = \mathrm{id}_C$ and $\phi\circ\phi' = \mathrm{id}_{C'}$.

Existence follows by construction. Let (V,Q) be a quadratic vector space and let $T^{\cdot}V = \bigoplus_{p\geq 0} V^{\otimes p}$, where $V^{\otimes 0} = \mathbb{R}$, $V^{\otimes 1} = V$, $V^{\otimes p} = V \otimes V^{\otimes p-1}$, ... be the tensor algebra with product $V^{\otimes p} \times V^{\otimes q} \xrightarrow{\otimes} V^{\otimes(p+q)}$ and $1 \in V^{\otimes 0}$ as the unit. Then $j : V = V^{\otimes 1} \hookrightarrow T^{\cdot}V$ is an initial object in the category whose objects are linear maps $V \xrightarrow{\phi} A$ to associative

algebras (and morphisms are commuting triangles as in **Cliff(V,Q)**).
In other words, given a linear map $\phi : V \to A$, there exists a unique algebra morphism $\tilde{\Phi} : T^\cdot V \to A$ extending ϕ: $\tilde{\Phi}(1) = 1_A$ and $\tilde{\Phi}(\underset{\text{“}j(v)\text{”}}{v}) = \phi(v)$ $\forall v \in V$. It follows that

$$\tilde{\Phi}(\Sigma\, v_{i_1} \otimes \cdots \otimes v_{i_p}) = \Sigma\, \phi(v_{i_1}) \cdots \phi(v_{i_p})$$

Now suppose that $\phi : V \to A$ is Clifford, then the extension $\tilde{\Phi} : T^\cdot V \to A$ annihilates the 2-sided ideal I_Q generated by $v \otimes v + Q(v) \in V^{\otimes 2} \oplus V^{\otimes 0}$ $\forall v \in V$:

$$\begin{aligned}\tilde{\Phi}(t_1 \otimes (v \otimes v + Q(v)) \otimes t_2) &= \tilde{\Phi}(t_1)\, \tilde{\Phi}(v \otimes v + Q(v))\, \tilde{\Phi}(t_2) \\ &= \tilde{\Phi}(t_1) \big(\underbrace{\phi(v)^2 + Q(v) 1_A}_{0 \text{ since } \phi \text{ is Clifford}} \big) \tilde{\Phi}(t_2)\end{aligned}$$

and hence it induces a unique algebra morphism $\Phi : T^\cdot V / I_Q \to A$ making the following diagram commute:

$$\begin{tikzcd} V \arrow[d, "j"] \arrow[dr, "\phi"] \arrow[dd, bend right, "i"'] & \\ T^\cdot V \arrow[r, "\tilde{\Phi}"] \arrow[d, "\pi"] & A \\ T^\cdot V / I_Q \arrow[ur, "\Phi"'] & \end{tikzcd}$$

By universality, $Cl(V,Q) \overset{!}{\cong} T^\cdot V / I_Q$.

Remarks

① $T^\cdot V$ is a $\mathbb{Z}$-graded algebra, but I_Q is not homogeneous (if $Q \neq 0$)
$\Rightarrow$ $Cl(V,Q)$ is not $\mathbb{Z}$-graded (unless $Q = 0$, in which case $Cl(V,0) \cong \Lambda^\cdot V$)
$\rightarrow$ $Cl(V,Q)$ is a filtered $\mathbb{Z}_2$-graded algebra and its associated graded algebra is $\Lambda^\cdot V$.

② I_Q has even parity, so that $Cl(V,Q)$ is $\mathbb{Z}_2$-graded

③ $i = \pi \circ j : V \to T^\cdot V / I_Q$ is injective,
since j is injective and if $i(v) = 0$, then $j(v) \in I_Q$, but parity forbids it unless $v = 0$. We will give a proof of injectivity below.

④ The Clifford algebra defines a functor $\mathbf{QVec} \xrightarrow{Cl} \mathbf{Assoc}$.

If (V, Q_V), (W, Q_W) are quadratic vector spaces and $f: V \to W$ is linear with $f^* Q_W = Q_V$. Let $i_V: V \to Cl(V,Q)$, $i_W: W \to Cl(W,Q)$ be the corresponding Clifford algebras.

Then $i_W \circ f : V \to Cl(W, Q_W)$ is a Clifford map:

$$(i_W \circ f)(v)^2 = -Q_W(f(v))\,\mathbb{1} = -Q_V(v)\,\mathbb{1}$$

and hence there is a unique $Cl(f): Cl(V,Q_V) \to Cl(W,Q_W)$.

Existence then implies $Cl(\mathrm{id}_V) = \mathrm{id}_{Cl(V,Q)}$ and $Cl(g \circ f) = Cl(g) \circ Cl(f)$.

showing that Cl is a functor.

Now consider the orthogonal group of (V,Q)

$$O(V,Q) = \{ g \in GL(V) \mid g^* Q = Q \}$$

Then for $g \in O(V,Q)$, $Cl(g) \in \mathrm{Aut}\, Cl(V,Q)$.

In particular, if $g = -\mathrm{id}_V$, then $\tau := Cl(-\mathrm{id}_V)$ is an involutive automorphism which is the parity automorphism:

$$a \in Cl(V,Q)_{\bar 0} \iff \tau(a) = a$$

$$a \in Cl(V,Q)_{\bar 1} \iff \tau(a) = -a.$$

This gives another explanation why $Cl(V,Q)$ is a $\mathbb{Z}_2$-graded algebra.

A representation of $Cl(V,Q)$ is an algebra homomorphism

$$\Gamma : Cl(V,Q) \longrightarrow \mathrm{End}(S)$$

where the vector space S is a Clifford module. By definition, it is equivalent to exhibit a Clifford map $V \xrightarrow{\gamma} \mathrm{End}(S)$. (The notation stems from the fact that the image $\gamma_\mu := \gamma(e_\mu)$ of a basis for V are the so-called Dirac Γ-matrices.

NB: Same Dirac, different reason. Dirac did many things: Dirac structures come from his theory of constraints; whereas Dirac Γ-matrices comes from the relativistic Dirac equation.

Define $\phi : V \longrightarrow \mathrm{End}(\Lambda^{\cdot} V)$ by

$$\phi(v)\cdot\alpha = v\wedge\alpha - \iota_{v^\flat}\alpha$$

where $v^\flat(w) = B(v,w) = \frac{1}{2}\big(Q(v+w) - Q(v) - Q(w)\big)$, and $\iota_{v^\flat}(1) = 0$, $\iota_{v^\flat}(w) = B(v,w)$, ... Then ϕ is Clifford:

$$\begin{aligned}
\phi(v)^2\cdot\alpha &= \phi(v)\big(v\wedge\alpha - \iota_{v^\flat}\alpha\big)\\
&= v\wedge\big(v\wedge\alpha - \iota_{v^\flat}\alpha\big) - \iota_{v^\flat}\big(v\wedge\alpha - \iota_{v^\flat}\alpha\big)\\
&= -\, v\wedge\iota_{v^\flat}\alpha - B(v,v)\,\alpha + v\wedge\iota_{v^\flat}\alpha\\
&= -\,Q(v)\,\alpha
\end{aligned}$$

Let $\Phi : Cl(V,Q) \longrightarrow \mathrm{End}(\Lambda V)$ and let $\Phi_1 : Cl(V,Q) \longrightarrow \Lambda V$ be evaluation at 1. Then for all $v\in V$, $a \longmapsto \Phi(a)\cdot 1$
$\Phi_1(i(v)) = v$, so in particular $\Phi_1\circ i$ is injective and this shows that $i : V \to Cl(V,Q)$ is injective. In fact, one can show that $\Phi_1 : Cl(V,Q) \longrightarrow \Lambda^{\cdot}V$ is a vector space isomorphism whose inverse $\Phi_1^{-1} : \Lambda^{\cdot}V \longrightarrow Cl(V,Q)$ is the skew-symmetrisation map:

$$\Phi_1^{-1}(v\wedge w) = \tfrac{1}{2}\big(i(v)\,i(w) - i(w)\,i(v)\big) \quad \text{, etc.}$$

This is an explicit quantisation of $\Lambda^{\cdot}V$.

Let $\mathbb{V} = V\oplus V^*$, where $V^* = \mathrm{Hom}(V,\mathbb{R})$, and let $B : \mathbb{V}\times\mathbb{V} \longrightarrow \mathbb{R}$ denote the dual pairing $B(w_1+\alpha_1, w_2+\alpha_2) = -\frac{1}{2}\big(\alpha_1(w_2) + \alpha_2(w_1)\big)$ so that $Q(w+\alpha) = -\alpha(w)$. Let $S = \Lambda^{\cdot}V^*$ and define $\phi : \mathbb{V} \longrightarrow \mathrm{End}(S)$ by

$$\phi(w+\alpha)\,\sigma = \iota_w\sigma + \alpha\wedge\sigma$$

Then $\phi(w+\alpha)^2\sigma = \alpha(w)\,\sigma = -Q(w+\alpha)\,\sigma$, so that S is a Clifford module. Notice that $S = \Lambda^{\cdot}V^* = Cl(V,Q)/Cl(V,Q)\cdot V$ and since $Cl(V,Q)\cdot V$ is a maximal ideal, S is a simple module.

Notice that $V^*\subset\mathbb{V}$ is a lagrangian subspace. The orthogonal group $O(\mathbb{V},Q)$ acts transitively on the grassmannian of lagrangian subspaces and also acts on $Cl(\mathbb{V},Q)$ via automorphisms. So if $L\subset\mathbb{V}$ is lagrangian, we get $S_L = Cl(\mathbb{V})/Cl(\mathbb{V})\cdot L \cong \Lambda L^*$. All such S_L are isomorphic as modules.

② Exact Courant algebroids via Clifford algebras

We depart from the observation that the Courant bracket on $\mathbb{T}M$ extends the Lie bracket of vector fields. I claim that, properly interpreted, this extension is the 'natural' one. To see this, we use the Cartan calculus to relate the Lie bracket of vector fields to the operations on the dga of differential forms. Recall that if $\omega \in \Omega^1(M)$ and $X, Y \in \mathfrak{X}(M)$, then

$$d\omega(X,Y) = X\cdot\omega(Y) - Y\cdot\omega(X) - \omega([X,Y])$$

so that

$$\begin{aligned}\omega([X,Y]) &= X\cdot\omega(Y) - Y\cdot\omega(X) - d\omega(X,Y)\\ &= \iota_X d\iota_Y\omega - \iota_Y d\iota_X\omega - \iota_Y\iota_X d\omega\end{aligned}$$

So if we know d we know $[-,-]$.

Lemma Let $\omega \in \Omega^{\bullet}(M)$ and $X, Y \in \mathfrak{X}(M)$. Then

$$2\iota_{[X,Y]}\omega = 2\iota_X d\iota_Y\omega - 2\iota_Y d\iota_X\omega + (\iota_X\iota_Y - \iota_Y\iota_X)d\omega + d((\iota_X\iota_Y - \iota_Y\iota_X)\omega)$$

Proof Follows from Cartan formula:

$$\begin{aligned}2\iota_{[X,Y]} &= [\mathcal{L}_X, \iota_Y] - [\mathcal{L}_Y, \iota_X]\\ &= [[d,\iota_X],\iota_Y] - [[d,\iota_Y],\iota_X]\\ &= (d\iota_X + \iota_X d)\iota_Y - \iota_Y(d\iota_X + \iota_X d) - (d\iota_Y + \iota_Y d)\iota_X + \iota_X(d\iota_Y + \iota_Y d)\\ &= d\iota_X\iota_Y + \iota_X d\iota_Y - \iota_Y d\iota_X - \iota_Y\iota_X d - d\iota_Y\iota_X - \iota_Y d\iota_X + \iota_X d\iota_Y + \iota_X\iota_Y d\\ &= d(\iota_X\iota_Y - \iota_Y\iota_X) + (\iota_X\iota_Y - \iota_Y\iota_X)d + 2\iota_X d\iota_Y - 2\iota_Y d\iota_X \quad \blacksquare\end{aligned}$$

The next observation is that on $\mathbb{T}M$ we have a split signature inner product $\langle -,- \rangle$, given in terms of the dual pairing between TM and T^*M:

$$\langle X+\alpha, Y+\beta\rangle := \tfrac{1}{2}(\alpha(Y) + \beta(X)).$$

This allows us to define a Clifford bundle $Cl(\mathbb{T}M)$. Its fibre at $p \in M$ is $Cl(T_pM \oplus T_p^*M)$ of which ΛT_p^*M is a simple Clifford module.

We have a Clifford map $\Gamma(\mathbb{T}M) \longrightarrow \mathrm{End}\, \Omega^{\bullet}(M)$ given by

this is more conventional, so + in Cl!

$$(X+\alpha)\cdot\omega = \iota_X\omega + \alpha\wedge\omega$$

Notice in particular that $X\cdot\omega = \iota_X\omega$ and the Clifford action extends to $\Gamma(\mathbb{T}M)$ the contraction ι_X on $\mathfrak{X}(M)$.

The idea is then to extend the Lie bracket from $\mathfrak{X}(M)$ to $\Gamma(\mathbb{T}M)$ by recognising the instances of "ι_X" as Clifford action. So we have

$$2\,[X,Y]\cdot\omega = 2\,X\cdot d(Y\cdot\omega) - 2\,Y\cdot d(X\cdot\omega) + (X\cdot Y - Y\cdot X)\cdot d\omega + d\big((X\cdot Y - Y\cdot X)\cdot\omega\big)$$

and now we replace $X \mapsto X+\alpha$, $Y \mapsto Y+\beta$

$$\underbrace{2\,[X+\alpha, Y+\beta]}_{\text{(Courant bracket)}}\cdot\omega = 2(X+\alpha)\cdot d\big((Y+\beta)\cdot\omega\big) - 2\,(Y+\beta)\cdot d\big((X+\alpha)\cdot\omega\big)$$
$$+ \big((X+\alpha)\cdot(Y+\beta) - (Y+\beta)\cdot(X+\alpha)\big)\cdot d\omega$$
$$+ d\Big(\big((X+\alpha)\cdot(Y+\beta) - (Y+\beta)\cdot(X+\alpha)\big)\cdot\omega\Big)$$

We know (by setting $\alpha=\beta=0$) that $[X+\alpha, Y+\beta] = [X,Y] + \cdots$

The remarkable thing is that the RHS is of the form $2\big(\iota_{[X,Y]}\omega + \gamma\wedge\omega\big)$

$\exists\,\gamma\in\Omega^1(M)$ and, after a calculation $\gamma = \mathcal{L}_X\beta - \mathcal{L}_Y\alpha + \frac{1}{2}(d\iota_Y\alpha - d\iota_X\beta)$

In summary,

$$[X+\alpha, Y+\beta] = [X,Y] + \mathcal{L}_X\beta - \mathcal{L}_Y\alpha + \tfrac{1}{2}\,d\big(\alpha(Y) - \beta(X)\big)$$

which is precisely the Courant bracket.

The twisted Courant bracket $[\,,\,]_H$ by $H\in\Omega^3(M)_{cl}$ is obtained in the same way but replacing d by d_H, where $d_H\omega = d\omega - H\wedge\omega$, which is again a differential if $dH=0$:

$$d_H^2\omega = d_H(d\omega - H\wedge\omega) = d(d\omega - H\wedge\omega) - H\wedge(d\omega - H\wedge\omega) = -dH\wedge\omega.$$

After a bit of calculation this results in an additional term

$$[X+\alpha, Y+\beta]_H = [X+\alpha, Y+\beta] + \iota_Y\iota_X H$$

③ Pure spinors

Let $\mathbb{V} = V \oplus V^*$ be a quadratic vector space relative to the dual pairing and let S be a simple $C\ell(\mathbb{V})$ module. Recall that $S \cong \Lambda^\cdot L^*$ where $L \subset \mathbb{V}$ is a lagrangian subspace, e.g. $L = V$. Given a nonzero spinor $s \in S$, we define $\mathbb{V}_s \subset \mathbb{V}$ to consist of $v \in \mathbb{V}$ s.t. $v \cdot s = 0$. It is easy to see that $\mathbb{V}_s \subset \mathbb{V}$ is isotropic: if $v, w \in \mathbb{V}_s$, then

$$0 = v \cdot w \cdot s + w \cdot v \cdot s = -2B(v,w)\, s \Rightarrow B(v,w) = 0 \quad \text{if } s \neq 0.$$

We say that $s \in S$ is a pure spinor if $\mathbb{V}_s$ is maximally isotropic (i.e. lagrangian).

Examples

① $1 \in \Lambda^\cdot V^*$

$$\{(X+\alpha) \in \mathbb{V} \text{ s.t. } (X+\alpha)\cdot 1 = \iota_X 1 + \alpha \wedge 1 = 0\} = V$$

② Let $\theta \in V^*$ be nonzero.

$$\{X+\alpha \text{ s.t. } (X+\alpha)\cdot\theta = \iota_X \theta + \alpha\wedge\theta = 0\} = \ker\theta \oplus \mathbb{R}\theta$$

③ Let $B \in \Lambda^2 V^*$ and let $\varphi = e^B \wedge 1 = e^B$.

Then

$$\{X+\alpha \in \mathbb{V} \text{ s.t. } (X+\alpha)\cdot e^B = \iota_X B \wedge e^B + \alpha \wedge e^B = 0\}$$
$$= \{X - \iota_X B \mid X \in V\} = \text{graph of } -B^\flat$$

Similarly $e^B \wedge \theta$ is also a pure spinor.

If s is a pure spinor and $\lambda \neq 0$, λs is also pure: indeed, $\mathbb{V}_s = \mathbb{V}_{\lambda s}$. So we have a map $\mathbb{P}(PS) \longrightarrow \mathrm{Lag}$ associating with every pure spinor line, a lagrangian subspace of $\mathbb{V}$. This map is equivariant under $SO(\mathbb{V})$ and is a diffeomorphism.

Lagrangians in $\mathbb{V}$

Let $E \subseteq V \subset \mathbb{V}$ be any subspace and $E^\circ \subseteq V^*$ the annihilator.
Then $E \oplus E^\circ \subset V \oplus V^* = \mathbb{V}$ is lagrangian.

Example $E \subseteq V$ and $\varepsilon \in \Lambda^2 E^*$ (recall, $E^* \cong V^*/E^\circ$)

Let $\varepsilon^\flat : E \to E^*$ be the linear map $X \mapsto \iota_X \varepsilon$ for $X \in E$.

Define

$$L(E,\varepsilon) = \{ X + \alpha \in E \oplus V^* \mid \alpha|_E = \iota_X \varepsilon \}$$

Check: $X+\alpha,\ Y+\beta \in L(E,\varepsilon)$

$$\langle X+\alpha, Y+\beta \rangle = \tfrac{1}{2}\big(\alpha(Y) + \beta(X)\big) = \tfrac{1}{2}\big(\varepsilon(X,Y) + \varepsilon(Y,X)\big) = 0$$

Proposition Every lagrangian in $\mathbb{V}$ is of the form $L(E,\varepsilon)$.

Proof $L \subset \mathbb{V}$ lagrangian and let $E = \pi_V L$. Since L is lagrangian, $L \cap V^* = E^\circ$. Define $\varepsilon^\flat : E \to E^*$ by $\varepsilon^\flat(X) = \pi_{V^*}(\pi_V^{-1}(X) \cap L) \in V^*/E^\circ$

Then $L = L(E,\varepsilon)$. ∎

If $B \in \Lambda^2 V^*$, $e^B L(E,\varepsilon) = \{ X + \alpha + \iota_X B \mid \alpha|_E = \iota_X \varepsilon \}$

so $\alpha|_E + \iota_X B = \iota_X \varepsilon + \iota_X B = \iota_X(\varepsilon + i^* B)$

$= L(E, \varepsilon + i^* B)$ $\quad i : E \hookrightarrow V$

Every lagrangian is of the form $e^B L(E,0) = e^B(E \oplus E^\circ)$, $\exists E \subset V$, $B \in \Lambda^2 V^*$.

In terms of pure spinors, the lagrangian $L(E,0)$ is associated to the pure spinor line $\det E^\circ$. Indeed, if $(X+\alpha)\cdot \det E^\circ = 0$, $\iota_X \cdot \det E^\circ + \alpha \wedge \det E^\circ = 0$ hence $X \in E$ and $\alpha \in E^\circ$. Then it follows that the pure spinor line associated with $e^B L(E,0)$ is $e^B(\det E^\circ)$.

Lagrangians $L \subset \mathbb{W}$ can be pulled back & pushed forward along linear maps $f : V \to W$:

$L \subset \mathbb{V}$ lag. $\Rightarrow$ $f_* L = \{ f(X) + \alpha \in \mathbb{W} \mid X + f^*\alpha \in L \} \subset \mathbb{W}$ lag.

$M \subset \mathbb{W}$ lag. $\Rightarrow$ $f^* M = \{ X + f^*\beta \in \mathbb{V} \mid f(X) + \beta \in M \} \subset \mathbb{V}$ lag.

4) Dirac structures and pure spinors

Let $\mathbb{T}M = TM \oplus T^*M$ and $[-,-]$ the Courant bracket on $\Gamma(\mathbb{T}M)$ corresponding to the standard Courant algebroid:

$$[X+\alpha, Y+\beta]_H = [X,Y] + \mathcal{L}_X\beta - \mathcal{L}_Y\alpha + \tfrac{1}{2}d(\alpha(Y)-\beta(X)) + \iota_Y\iota_X H$$

Let $B \in \Omega^2(M)$ and let $e^B(X+\alpha) := X+\alpha+\iota_X B$.

Then

$$[e^B(X+\alpha), e^B(Y+\beta)]_H = [X+\alpha, Y+\beta]_{H+dB}$$

So it is an automorphism of the Courant bracket iff $dB=0$.

Recall that a Dirac structure $L \subset \mathbb{T}M$ is a lagrangian sub-bundle, whose sections are in involution under the Courant bracket. The fundamental example is $TM \subset \mathbb{T}M$ for $H=0$. Then also $e^\omega TM$ for $d\omega=0$ (i.e., a presymplectic structure) or $e^\omega TM$ for $H=-d\omega$:

$$\begin{aligned}
[X+\iota_X\omega, Y+\iota_Y\omega]_H &= [X,Y] + \mathcal{L}_X\iota_Y\omega - \mathcal{L}_Y\iota_X\omega + \tfrac{1}{2}d(\omega(X,Y)-\omega(Y,X)) + \iota_Y\iota_X H \\
&= [X,Y] + \iota_{[X,Y]}\omega + \iota_Y\mathcal{L}_X\omega - \mathcal{L}_Y\iota_X\omega + d\,\omega(X,Y) + \iota_Y\iota_X H \\
&= ([X,Y] + \iota_{[X,Y]}\omega) + \iota_Y\iota_X d\omega + \iota_Y d\iota_X\omega - \iota_Y d\iota_X\omega \\
&\qquad - d\iota_Y\iota_X\omega + d\iota_Y\iota_X\omega + \iota_Y\iota_X H \\
&= ([X,Y] + \iota_{[X,Y]}\omega) + \iota_Y\iota_X(\underbrace{d\omega+H}_{0}).
\end{aligned}$$

Such a 2-form is a twisted pre-symplectic structure.

Lagrangian sub-bundles of $\mathbb{T}M$ are in one-to-one correspondence with pure spinor lines $U \subset \Lambda^\bullet T^*M$, via

$$L = \{ X+\alpha \in \Gamma(\mathbb{T}M) \mid (X+\alpha)\cdot U = 0 \}$$

The Clifford algebra $Cl(\mathbb{V})$ is filtered :

$2m = \dim \mathbb{V}$

$$\mathbb{R} = Cl^0 \subset Cl^2 \subset Cl^4 \subset \cdots \subset Cl^{2m} = Cl_+(\mathbb{V})$$
$$\mathbb{V} = Cl^1 \subset Cl^3 \subset Cl^5 \subset \cdots \subset Cl^{2m-1} = Cl_-(\mathbb{V})$$

Cl^{2k} spanned by an even number ($\leq 2k$) of elements in $\mathbb{V}$

Cl^{2k-1} spanned by an odd number ($\leq 2k-1$) of elements in $\mathbb{V}$

We Clifford multiply by $U \subset \Lambda^{\cdot} T^*M$ we obtain filtrations of $\Lambda^{\mathrm{even/odd}} T^*M$

$$U = U_0 \subseteq U_2 \subseteq \cdots \subseteq U_{2n} = \Lambda^{\mathrm{even/odd}} T^*M \qquad U_k := Cl^k \cdot U$$
$$L^* \cdot U = U_1 \subseteq U_3 \subseteq \cdots \subseteq U_{2n-1} = \Lambda^{\mathrm{odd/even}} T^*M \qquad U_1 := \mathbb{V} \cdot U = \mathbb{V}/_L \cdot U$$

$L^* \cong \mathbb{V}^*/_{L^0}$ using $\langle , \rangle$.

Proposition $L \subset \mathbb{T}M$ lagrangian, is involutive (rel. to $[\cdot,\cdot]_H$) iff

$$d_H \Gamma(U_0) \subset \Gamma(U_1)$$

or if ρ is a local trivialisation of U, then $\exists\ X + \alpha \in \Gamma(\mathbb{T}M)$ such that $d_H \rho = \iota_X \rho + \alpha \wedge \rho$.

This is well-defined because if $\hat{\rho} = f\rho$ $\exists f \in C^\infty(M)$ is nowhere vanishing, then

$$d_H \hat{\rho} = df \wedge \rho + f\, d\rho - H \wedge \hat{\rho} \quad (= df \wedge \rho + f\, d_H \rho)$$
$$= df \wedge \rho + f(\iota_X \rho + \alpha \wedge \rho)$$
$$= \iota_X \hat{\rho} + (f^{-1} df + \alpha) \wedge \hat{\rho}.$$

Proof Let $L \subset \mathbb{T}M$ be a lagrangian sub-bundle and let ρ be a local trivialisation for $U < \Lambda^{\cdot} T^*M$. Then the result follows in two steps :

① for $A, B \in \Gamma(L)$, $[A,B] \cdot \rho = A \cdot B \cdot d\rho$, and

② $A \cdot B \cdot d\rho = 0$ $\forall A, B \in \Gamma(L)$ iff $d\rho \in \Gamma(U_1)$.

Proof of ① Since L is isotropic, Dorfman & Courant brackets agree on $\Gamma(L)$.

Write $A = X + \alpha$, $B = Y + \beta$. Since $\Gamma(L) \cdot \rho = 0$, $\iota_X \rho = -\alpha \wedge \rho$ and $\iota_Y \rho = -\beta \wedge \rho$

We calculate

$$
\begin{aligned}
\iota_{[X,Y]}\rho &= [\mathcal{L}_X,\iota_Y]\rho \\
&= \mathcal{L}_X(\iota_Y\rho) - \iota_Y(d\iota_X\rho + \iota_X d\rho) \\
&= -\mathcal{L}_X(\beta\wedge\rho) - \iota_Y d(-\alpha\wedge\rho) + \iota_X\iota_Y d\rho \\
&= -\mathcal{L}_X\beta\wedge\rho - \beta\wedge(d(-\alpha\wedge\rho)+\iota_X d\rho) + \iota_Y(d\alpha\wedge\rho - \alpha\wedge d\rho) + \iota_X\iota_Y d\rho \\
&= -\mathcal{L}_X\beta\wedge\rho + \beta\wedge d\alpha\wedge\rho - \beta\wedge\alpha\wedge d\rho - \beta\wedge\iota_X d\rho + \iota_Y d\alpha\wedge\rho + d\alpha\wedge\iota_Y\rho - \alpha(Y)d\rho + \alpha\wedge\iota_Y d\rho \\
&\qquad + \iota_X\iota_Y d\rho \\
&= -\mathcal{L}_X\beta\wedge\rho + \beta\wedge d\alpha\wedge\rho + \alpha\wedge\beta\wedge d\rho - \beta\wedge\iota_X d\rho + \iota_Y d\alpha\wedge\rho + d\alpha\wedge(-\beta\wedge\rho) \\
&\qquad - \alpha(Y)d\rho + \alpha\wedge\iota_Y d\rho + \iota_X\iota_Y d\rho \\
&= -\mathcal{L}_X\beta\wedge\rho + \iota_Y d\alpha\wedge\rho + (\iota_X\iota_Y + \alpha\wedge\iota_Y - \alpha(Y) - \beta\iota_X + \alpha\wedge\beta\wedge)d\rho
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ [X+\alpha, Y+\beta]_D\cdot\rho &= -(\iota_Y+\beta\wedge)(\iota_X+\alpha\wedge)\cdot d\rho \\
&= -(\iota_Y\iota_X + \alpha(Y) - \alpha\wedge\iota_Y + \beta\wedge\iota_X + \beta\wedge\alpha\wedge)d\rho \\
&= -(Y+\beta)\cdot(X+\alpha)\cdot d\rho \\
&= +(X+\alpha)\cdot(Y+\beta)\cdot d\rho \qquad \text{since } \langle X+\alpha, Y+\beta\rangle = 0.
\end{aligned}
$$

Proof of ② Recall that $U_1 = Cl^1\cdot U = \mathbb{V}\cdot U$. Then if $A, B\in L$, and $\varphi\in\mathbb{V}$,

$$
\begin{aligned}
A\cdot\varphi\cdot\rho &= \langle A,\varphi\rangle\rho - \varphi\cdot\cancelto{0}{A\rho} \\
A\cdot B\cdot\varphi\cdot\rho &= A\cdot\langle B,\varphi\rangle\rho = \langle B,\varphi\rangle A\cdot\rho = 0.
\end{aligned}
$$

Proof of Proposition (cont'd)

$$
[A,B]\cdot\rho = 0 \overset{①}{\Leftrightarrow} A\cdot B\cdot d\rho = 0 \overset{②}{\Leftrightarrow} d\rho\in\Gamma(U_1)
$$

hence $\exists\ X+\alpha\in\Gamma(\mathbb{T}M)$ such that $d\rho = (X+\alpha)\cdot\rho = \iota_X\rho + \alpha\wedge\rho$. ■

Remark Replacing d with d_H we find that similarly to the proof of ①,

$$
[X+\alpha, Y+\beta]_H\cdot\rho = (X+\alpha)\cdot(Y+\beta)\cdot d_H\rho
$$

and hence $L\subset\mathbb{T}M$ is involutive under the twisted Courant bracket if and only if $d_H\rho = \iota_X\rho + \alpha\wedge\rho$ $\exists\ X+\alpha\in\Gamma(\mathbb{T}M)$.

In particular, $d\rho = 0$ or $d_H\rho = 0$ for a pure spinor ρ implies that the annihilator of ρ defines a Dirac structure.

Examples

$\rho = e^{\omega}$, $\omega \in \Omega^2(M)$.

$$(X+\alpha)\cdot\rho = \iota_X e^{\omega} + \alpha\wedge e^{\omega} = (\iota_X\omega + \alpha)\wedge e^{\omega} = 0 \Leftrightarrow \alpha = -\iota_X\omega$$

$\therefore$ L = graph of $-\omega^\flat : TM \to T^*M$
$\qquad X \mapsto -\iota_X\omega$

$d\rho = d\omega\wedge e^{\omega}$

$d_H\rho = (d\omega - H)\wedge e^{\omega}$

$d\rho = 0 \Leftrightarrow d\omega = 0$ presymplectic

$d_H\rho = 0 \Leftrightarrow d\omega = H$ twisted presymplectic

Where's Poisson? Let $\pi \in \Gamma(\Lambda^2 TM)$ be a bivector. Then $L_\pi := e^{\pi}(T^*M) = \{\iota_\alpha\pi + \alpha \in \mathbb{T}M\}$ is Lagrangian. Given $f,g \in C^\infty(M)$, define $\{f,g\} = \pi(df,dg)$. Then L_π is involutive iff $\{,\}$ is a Poisson bracket. Let $\mu \in \Omega^{top}(M)$. Then $L_\mu = T^*M \subset \mathbb{T}M$. Then $e^{-\pi}\mu = \mu - \iota_\pi\mu + \frac{1}{2}\iota_\pi\circ\iota_\pi\mu + \cdots$

$$\begin{aligned}(X+\alpha)\cdot e^{-\pi}\mu &= \iota_X(e^{-\pi}\mu) + \alpha\wedge e^{-\pi}\mu \\ &= \iota_X\left(\mu - \iota_\pi\mu + \tfrac{1}{2}\iota_\pi^2\mu + \cdots\right) + \alpha\wedge\left(\mu - \iota_\pi\mu + \tfrac{1}{2}\iota_\pi^2\mu + \cdots\right) \\ &= \iota_X\mu - \iota_X\iota_\pi\mu + \tfrac{1}{2}\iota_X\iota_\pi^2\mu + \cdots - \alpha\wedge\iota_\pi\mu + \tfrac{1}{2}\alpha\wedge\iota_\pi^2\mu + \cdots \\ &= (\iota_X\mu - \alpha\wedge\iota_\pi\mu) + \cdots \\ &= 0 \Leftrightarrow X = \iota_\alpha\pi\end{aligned}$$

$\rho = e^{-\pi}\mu$ obeys $d\rho = \iota_X\rho \;\exists X\in\mathfrak{X}(M) \Leftrightarrow \pi$ is Poisson bivector

$d_H\rho = \iota_X\rho \;\exists X\in\mathfrak{X}(M) \Leftrightarrow \pi$ is a twisted Poisson bivector