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Integrable Systems in Three-Dimensional Riemannian Geometry

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Summary. In this paper we introduce a new infinite-dimensional pencil of Hamiltonian structures. These Poisson tensors appear naturally as the ones governing the evolution of the curvatures of certain flows of curves in 3-dimensional Riemannian manifolds with constant curvature. The curves themselves are evolving following arclength-preserving geometric evolutions for which the variation of the curve is an invariant combination of the tangent, normal, and binormal vectors. Under very natural conditions, the evolution of the curvatures will be Hamiltonian and, in some instances, bi-Hamiltonian and completely integrable.

1. Introduction

The theory of integrable systems has traditionally made use of geometrical concepts and procedures. In particular, the majority of completely integrable PDEs, or systems of PDEs, are connected to the existence of two compatible Hamiltonian structures with respect to which the systems are Hamiltonian. When that happens we call the system a bi-Hamiltonian system. If one of the compatible Hamiltonian structures is nondegenerate, a recursion operator can be defined, which will generate a family of preserved quantities for the flow, effectively integrating the system. The field of Poisson geometry, or geometry of Hamiltonian evolutions, is thus a fundamental part in the study of completely integrable systems.

The study of the relationship between finite-dimensional differential geometry and partial differential equations, which later came to be known as integrable systems, started in the nineteenth century. Liouville found and solved the equation describing minimal surfaces in 3-dimensional Euclidean space [Lio53]. Bianchi solved the general Goursat problem for the sine-Gordon equation which arises in the theory of pseudospherical surfaces [Bia], [Bia92]. The original work of Darboux [Dar10] is still of interest; see [Zak98] for the modern developments of this line of research

Much later Hasimoto [Has72] found the relation between the equations for curvature and torsion of vortex filament flow and the nonlinear Schrödinger equation which led to many new developments, cf. [MW83], [LP91], [DS94], [LP96], [YS98], [Cal00]. The author of [MB99] subsequently defined the second Poisson structure for generalized KdV, the Adler–Gel'fand–Dikiĭ bracket [GD78], [Adl79], in terms of an invariant frame along a parametrized projective curve and its differential invariants. This close relationship between invariant evolutions and the Hamiltonian structures of integrable systems also holds for parametrized plane curves under the action of $O(3, 1)$, cf. [MB00b], and, under more restrictive conditions, for projective reparametrizations of the projective plane, cf. [MB00a].

In this paper we present a quadruplet of compatible Hamiltonian structures that arises in a natural way from the geometric arclength-preserving evolution of curves in any given 3–dimensional Riemannian manifold with constant curvature. We arrived at these structures from two different ways. First, one of the authors was asked whether a certain system of PDEs, which is currently being studied by Fels and Ivey (see [Ive01]), has a recursion operator. She found this recursion operator and then proceeded to find the Hamiltonian pair, producing in this way two of the Poisson tensors presented here, the ones we call $\mathcal{E} + \mathcal{B}$ and $\mathcal{D}$. These two were discovered independently by Ivey and they appear in [Ive01]. The other route is described in this paper, where we present the geometrical origin of the above-mentioned Hamiltonian pair, as generating the first Hamiltonian structure in a pencil indexed by the curvature of the manifold. We denote the pencil by $\mathcal{B} + \varkappa\mathcal{C} + \mathcal{D} + \mathcal{E}$. The tensor $\mathcal{C}$ in the pencil is compatible with the other three, forming this way a Hamiltonian quadruplet. Indeed we show that if a flow of curves in a 3–dimensional Riemannian manifold with constant curvature $\varkappa$ follows an arclength-preserving geometric evolution, the evolution of its Riemannian curvatures is always, under natural conditions, a Hamiltonian flow with respect to the element of the pencil corresponding to the value $\varkappa$.[1] The close geometric relationship remains here: The quadruplet can be obtained solely from the geometry of curves on 3-dimensional Riemannian manifolds with constant curvature.

Having obtained the pencil, we construct hereditary operators by composing the pencil with the inverse of a nondegenerate component. We explicitly do this by inverting $\mathcal{D}$ and $\mathcal{C}$. We show that, while $\mathcal{B} + \varkappa\mathcal{C} + \mathcal{D} + \mathcal{E}$ and $\mathcal{C}$ are the ones used to integrate the best known versions of the vortex filament equation on constant curvature manifolds, $\mathcal{B} + \varkappa\mathcal{C} + \mathcal{D} + \mathcal{E}$ and $\mathcal{D}$ are used to integrate the system studied by Ivey and its generalizations to manifolds with nonzero constant curvature. Next we describe the effect of the Hasimoto transformation on the Hamiltonian pair associated to the vortex filament flow. As is already known, the Hasimoto transform is a Poisson map from the vortex filament flow to the nonlinear Schrödinger equation. We show that the hodographic transformation used by Ivey for the Euclidean case [Ive01] has a role analogous to the Hasimoto transformation for the second hierarchy. The hodographic transformation is a Poisson map which, in the case $\varkappa \neq 0$, takes the second system in the hierarchy to a system of

[1] The study in [YS98] suggests that $\varkappa$ plays the role of a spectral parameter. But that is another story.

decoupled modified potential KdV equations. The decoupling does not hold in the flat case, which is degenerate in that sense. When the curvature $\varkappa$ of the manifold is nonzero, as the hodographic transformation is applied to the Hamiltonian pair, the transformed system lies in the integrable hierarchy of very simple equations. This relation was not at all clear prior to the application of the transformation. In fact, we can also produce another integrable equation using $\mathcal{B} + \varkappa\mathcal{C} + \mathcal{D} + \mathcal{E}$ and $\mathcal{B}$. However, at this moment we cannot find a proper transformation to simplify it, so we did not treat this case.

Section 2 introduces all concepts of Riemannian geometry needed. Since the paper relates two somehow separate subjects, we decided to include definitions that some of the readers might not be too familiar with, while others might find them very basic. We have done so in the simplest possible way, including only the necessary concepts. Section 3 contains two theorems: Theorem 2 describes the evolution followed by the Riemannian curvatures of a flow of curves satisfying an arclength-preserving geometric evolution. The calculation of this evolution can be carried out in many different ways, as one can see in the Euclidean case in [Cal00] for the vortex filament flow, in [LP91] for general flows, and in the case of spheres in [DS94]. Although the most generalizable way (and also the simplest) would involve the use of Cartan's definition of connection, we chose to describe it the way we think to be easier for a reader who is not familiar with Riemannian geometry and its Cartan interpretation. It is a longer procedure but perhaps easier to understand. Theorem 3 shows that the tensor defining the curvature evolution is a pencil of Hamiltonian structures indexed by the curvature of the manifold. We show that the pencil is formed by four compatible Poisson tensors. The proof of Theorem 3 needs many specialized definitions and formulae that we have preferred not to include here, since they are only used in the proof. We refer the reader to [Olv93] for the material needed. In Section 4 we study the two canonical evolution equations, their hereditary operators, and their integrable hierarchies. We also describe their associated transformations, which simplify them. The last section contains comments about the implications of the results in this paper as well as further open problems.

2. Definitions

2.1. Definitions in Riemannian Geometry

In this subsection we present all concepts about Riemannian manifolds that we need to use in the next section. Definitions and notations are mostly as in [Hic65] and [Pet98]. A manifold will be a C^∞-manifold.

Definition 1. (i) A *connection* on a manifold M is an operator ∇ which assigns to two C^∞ vector fields X and Y with domain Ω, a third C^∞ vector field denoted by $\nabla_X Y$ with the same domain Ω, in such a way that the following properties are satisfied:

(1) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$,
(2) $\nabla_{X+W} Y = \nabla_X Y + \nabla_W Y$,
(3) $\nabla_{fX} Y = f\nabla_X Y$,
(4) $\nabla_X(fY) = X(f)Y + f\nabla_X Y$,

for any X, W vectors at $p \in M$, Y, Z smooth fields and f a smooth function defined in a neighborhood of p.

(ii) We say an n–dimensional manifold M is a *Riemannian manifold* if M is endowed with a symmetric and positive definite 2–covariant tensor field $\langle , \rangle$. The tensor $\langle , \rangle$ is called the *Riemannian metric* of the manifold, and it allows us to define distances, length, angles, orthogonality, etc., in the natural way. In particular, the *length* of a vector X is defined as

$$|X| = \sqrt{\langle X, X \rangle}.$$

The simplest example of a Riemannian manifold is, of course, $\mathbb{R}^n$ with the usual dot product.

(iii) A *Riemannian connection* on a Riemannian manifold M is a connection ∇ on M such that

(5) $\nabla_X Y - \nabla_Y X = [X, Y]$ (the connection has zero *torsion tensor*),
(6) $Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$,
for all fields X, Y, Z with a common domain.

The fundamental theorem of Riemannian manifolds states that on any Riemannian manifold there exists a unique Riemannian connection. Riemannian manifolds are thus the natural generalization of Euclidean spaces and the Riemannian connections the natural generalization of covariant (or directional) differentiation.

(iv) The *curvature tensor* of a connection ∇ is a tensor R that assigns to each pair of vectors X, Y at a point p a linear transformation $R(X, Y)$ of the tangent space to p, T_pM, into itself. After extending X, Y, and Z to smooth vectorfields near p, $R(X, Y)Z$ is defined via the relation

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{1}$$

The value of this expression is independent of the way the vector fields were extended.

(v) The *Riemann–Christoffel curvature tensor* (of type $(0, 4)$) is the 4-covariant tensor

$$K(X, Y, Z, W) = \langle R(Z, W)Y, X\rangle$$

for any X, Y, Z, W tangent vectors at p.

Apart from being tensors (and thus multilinear with respect to $C^\infty(M)$ in all their components), curvature tensors are best known for the following properties:

Theorem 1. *The following relations are true:*

(1) $R(X, Y)Z + R(Z, X)Y + R(Y, Z)X = 0$ *(first* Bianchi identity*),*
(2) $\nabla_Z R(X, Y)W + \nabla_X R(Y, Z)W + \nabla_Y R(Z, X)W = 0$ *(second* Bianchi identity*),*
(3) $K(X, Y, Z, W) = -K(Y, X, Z, W) = -K(X, Y, W, Z)$*,*
(4) $K(X, Y, Z, W) = K(Z, W, X, Y)$*.*

Finally we give the last group of definitions.

Definition 2. (i) Given two independent vectors X, Y in T_pM, the normalized quadratic form,

$$\sec(\mathrm{X}, \mathrm{Y}) = \frac{\mathrm{K}(\mathrm{X}, \mathrm{Y}, \mathrm{X}, \mathrm{Y})}{\langle \mathrm{X}, \mathrm{X}\rangle\langle \mathrm{Y}, \mathrm{Y}\rangle - \langle \mathrm{X}, \mathrm{Y}\rangle^2},$$

is called *the sectional curvature* of X, Y. It can easily be checked that $\sec(\mathrm{X}, \mathrm{Y})$ depends only on the plane π spanned by X and Y, and so the sectional curvature is also called $K(\pi)$, the Riemannian curvature of the plane section π.

(ii) A Riemannian manifold M is said to have *constant Riemannian curvature* $\varkappa$ if the Riemannian curvature of all plane sections is the constant $\varkappa$.

(iii) If S is a $(0, r)$ tensor, one can define the *covariant derivative of the tensor* along the vector field X by ensuring that the Leibniz rule holds. That is, $\nabla_X(S)$ is determined by the relation

$$\nabla_X(S(Y_1, \dots, Y_r)) = (\nabla_X S)(Y_1, \dots, Y_r) + S(\nabla_X Y_1, \dots, Y_r) + \cdots + S(Y_1, \dots, \nabla_X Y_r)$$

to hold for any vectors $Y_1, \dots, Y_r$ in T_pM.

This proposition can be found in [Pet98].

Proposition 1. *The following properties are equivalent:*

(1) $K(\pi) = \varkappa$ *for all* 2*–planes in* T_pM.
(2) $R(X, Y)Z = \varkappa(\langle Y, Z\rangle X - \langle X, Z\rangle Y)$ *for any* X, Y, Z *in* T_pM.

Corollary 1. *Assume the manifold M has constant Riemannian curvature. Then*

(a) $\nabla_X R = 0$ *along any direction determined by the vector field X. That is, the Riemann curvature tensor is* parallel*.*
(b) If Z is orthogonal to X and Y, then $R(X, Y)Z = 0$.
(c) If W is orthogonal to X and Y, then $K(W, Z, X, Y) = 0$ *for any Z.*

What follows is the description of a Frénet frame and Frénet formulae for any smooth curve on a Riemannian manifold under some nondegeneracy conditions that follow from the construction.

Let γ: $U \subset \mathbb{R} \to M$ be a smooth curve on a Riemannian manifold M with Riemannian connection ∇. From now on we will assume that all vector fields are defined on some common open subset of U. Let $V(x)$ be the tangent field at x obtained by differentiation with respect to x (also called the *velocity vector*). We will naturally say that γ is *parametrized by arclength* whenever $|V(x)| = 1$ for all x in the domain of γ. Assume that γ is nondegenerate, that is, $V(x) \neq 0$ for all $x \in U$. We can then define the first vector in the Frénet frame, the *unit tangent vector*, as

$$\mathbf{e}_1(x) = \frac{V(x)}{|V(x)|}.$$

Define the *geodesic curvature* (or first curvature) of γ to be the length of the field $\nabla_{\mathbf{e}_1}\mathbf{e}_1$, that is, $k_1 = |\nabla_{\mathbf{e}_1}\mathbf{e}_1|$.

One immediately sees from property (6) in Definition 1 of the Riemannian connection that the vector $\nabla_{\mathbf{e}_1}\mathbf{e}_1$ must be orthogonal to $\mathbf{e}_1$ with respect to the Riemannian metric. In the case for which $k_1(x) \neq 0$, we can define the *first normal* to γ at x to be the unit vector $\mathbf{e}_2(x)$ in the direction of $\nabla_{\mathbf{e}_1}\mathbf{e}_1(x)$, so that

$$\nabla_{\mathbf{e}_1}\mathbf{e}_1 = k_1\mathbf{e}_2.$$

Also using property (6) we see that

$$0 = \langle \nabla_{\mathbf{e}_1}\mathbf{e}_2, \mathbf{e}_1\rangle + \langle \mathbf{e}_2, \nabla_{\mathbf{e}_1}\mathbf{e}_1\rangle = \langle \nabla_{\mathbf{e}_1}\mathbf{e}_2, \mathbf{e}_1\rangle + k_1,$$

so that

$$\langle \nabla_{\mathbf{e}_1}\mathbf{e}_2 + k_1\mathbf{e}_1, \mathbf{e}_1\rangle = \langle \nabla_{\mathbf{e}_1}\mathbf{e}_2 + k_1\mathbf{e}_1, \mathbf{e}_2\rangle = 0.$$

We call $k_2 = |\nabla_{\mathbf{e}_1}\mathbf{e}_2 + k_1\mathbf{e}_1|$ the *torsion* of γ (or second curvature).

Whenever $k_2 \neq 0$ we can define *the second normal* to γ to be the unit vector $\mathbf{e}_3$ in the direction of $\nabla_{\mathbf{e}_1}\mathbf{e}_2 + k_1\mathbf{e}_1$, so that

$$\nabla_{\mathbf{e}_1}\mathbf{e}_2 = k_2\mathbf{e}_3 - k_1\mathbf{e}_1.$$

The process above can be continued to define k_3, the third curvature, and whenever $k_3 \neq 0$ we can define $\mathbf{e}_4$, the third normal, etc.

Definition 3. The orthonormal vectors $\mathbf{e}_i$, $i = 1, \ldots, n$ are called the *Frénet vectors* or *Frénet frame*. Equations

$$\begin{aligned} \nabla_{\mathbf{e}_1}\mathbf{e}_1 &= k_1\mathbf{e}_2, \\ \nabla_{\mathbf{e}_1}\mathbf{e}_i &= k_i\mathbf{e}_{i+1} - k_{i-1}\mathbf{e}_{i-1}, \quad i = 2, \ldots, n-1, \\ \nabla_{\mathbf{e}_1}\mathbf{e}_n &= -k_{n-1}\mathbf{e}_{n-1}, \end{aligned} \tag{2}$$

are called the *Frénet formulae*.

2.2. Definitions in the Theory of Integrable Systems

In this subsection we present the concepts about bi-Hamiltonian integrable systems that we need to use in the next section. Definitions and notations are mostly as in [Olv93]. Another good introduction is [Dor93].

Let $M \subset X \times U$ be an open subset of the space of independent and dependent variables $x = (x^1, \ldots, x^p)$ and $u = (u^1, \ldots, u^q)$. The algebra of differential functions $P[u]$ over M is denoted by $\mathcal{A}$. We define the space $\mathcal{F}$ of functionals $\mathcal{P} = \int P\,dx$ as the quotient of $\mathcal{A}$ by the image of the total divergence.

For a linear differential operator $\mathcal{D}$: $\mathcal{A}^q \to \mathcal{A}^q$, which we can think of as a $q \times q$ matrix differential operator depending on x, u, and derivatives of u, we define a bracket on $\mathcal{F}$ as follows:

$$\{\mathcal{P}, \mathcal{Q}\} = \int \delta\mathcal{P} \cdot \mathcal{D}\delta\mathcal{Q}\,dx, \tag{3}$$

where $\delta\mathcal{P}$ is the variational derivative of functional $\mathcal{P}$ and where by $\cdot$ we denote the usual dot product in $\mathbb{R}^q$.

Definition 4. A linear operator $\mathcal{D}$: $\mathcal{A}^q \to \mathcal{A}^q$ is called Hamiltonian if the bracket (3) satisfies the conditions of skew-symmetry

$$\{\mathcal{P}, \mathcal{Q}\} = -\{\mathcal{Q}, \mathcal{P}\}, \tag{4}$$

and Jacobi identity

$$\{\{\mathcal{P}, \mathcal{Q}\}, \mathcal{S}\} + \{\{\mathcal{S}, \mathcal{P}\}, \mathcal{Q}\} + \{\{\mathcal{Q}, \mathcal{S}\}, \mathcal{P}\} = 0, \tag{5}$$

for all functionals $\mathcal{P}, \mathcal{Q}, \mathcal{S} \in \mathcal{F}$. The bracket (3) is called *Poisson bracket*.

We say two Hamiltonian operators $\mathcal{D}$ and $\mathcal{E}$ form a *Hamiltonian pair* or are *compatible* if every linear combination $a\mathcal{D} + b\mathcal{E}$, $a, b \in \mathbb{R}$ is a Hamiltonian operator. In fact, we only need to check whether $\mathcal{D} + \mathcal{E}$ is a Hamiltonian operator (Lemma 7.20 in [Olv93]) to prove this. We say that four Hamiltonian operators form a *Hamiltonian quadruplet*, or are compatible, if any two of them are compatible.

An evolution system is a Hamiltonian system if for a Hamiltonian operator $\mathcal{D}$, there exists a functional $\mathcal{H}$, called Hamiltonian, such that

$$u_t = K[u] = \mathcal{D}\delta\mathcal{H}, \qquad K[u] \in \mathcal{A}^q.$$

If for a Hamiltonian pair $\mathcal{D}$ and $\mathcal{E}$, there exists corresponding Hamiltonian functionals $\mathcal{H}_1$ and $\mathcal{H}_0$ such that

$$u_t = K[u] = \mathcal{D}\delta\mathcal{H}_1 = \mathcal{E}\delta\mathcal{H}_0, \tag{6}$$

we say the evolution system is a *bi-Hamiltonian system*.

Definition 5. A differential operator $\mathcal{D}$: $\mathcal{A}^q \to \mathcal{A}^q$ is *degenerate* if there is a nonzero differential operator $\tilde{\mathcal{D}}$: $\mathcal{A}^q \to \mathcal{A}$ such that $\tilde{\mathcal{D}} \cdot \mathcal{D} = 0$.

In the field of nonlinear evolution equations, one important question to answer is whether a given equation is integrable, in the sense that it has infinitely many commuting symmetries.

Definition 6. We say $Q[u] \in \mathcal{A}^q$ is a *symmetry* of $u_t = K[u]$ if and only if

$$[K, Q] = D_Q[K] - D_K[Q] = 0,$$

where $D_Q[K]$ is the Fréchet derivative of Q in the direction of K. If

$$D_Q[K] + D_K^{\star}[Q] = 0,$$

where $D_K^{\star}$ is the formal adjoint of D_K, then $Q[u]$ is a *cosymmetry* of the equation.

For system (6), all variational derivatives of its Hamiltonian functionals are cosymmetries.

Definition 7. A linear differential operator $\mathfrak{R}$: $\mathcal{A}^q \to \mathcal{A}^q$ is a *recursion operator* of $u_t = K[u]$ if it maps a symmetry to a new symmetry.

It follows that $\mathfrak{R}$ is a recursion operator of $u_t = K[u]$ if and only if

$$\mathfrak{R}_t = [D_K, \mathfrak{R}],$$

where $\mathfrak{R}_t = \frac{\partial \mathfrak{R}}{\partial t} + D_{\mathfrak{R}}[K]$, assuming $D_x^{-1} D_x = 1$. For more details, see [SW01a], [SW01b].

Related to the concept of the recursion operator is that of the *hereditary operator*. An operator $\mathfrak{R}$ is said to be hereditary if its Nijenhuis tensor vanishes, i.e.,

$$[\mathfrak{R}X, \mathfrak{R}Y] - \mathfrak{R}[\mathfrak{R}X, Y] - \mathfrak{R}[X, \mathfrak{R}Y] + \mathfrak{R}^2[X, Y] = 0, \quad \text{for all } X, Y \in dom\, \mathfrak{R}.$$

Many recursion operators are hereditary, but one should notice that the definition of *hereditary* does not need any specific equation; it is a geometric property of the operator defining a structure on the space. Given a Hamiltonian pair $(\mathcal{D}, \mathcal{E})$, *one constructs a hereditary operator by taking* $\mathfrak{R} = \mathcal{E}\mathcal{D}^{-1}$ *if* $\mathcal{D}$ *is nondegenerate* (cf. Theorem 7.24 in [Olv93] or Theorem 3.12 in [Dor93]).

The following is an important property: If X is a symmetry of the hereditary operator $\mathfrak{R}$, that is, $D_X \mathfrak{R} = [D_K, \mathfrak{R}]$, then for any k, l,

$$[\mathfrak{R}^k X, \mathfrak{R}^l X] = 0.$$

If there exists $\mathcal{H}_1 \in \mathcal{F}$ such that $\mathfrak{R}^\star \delta \mathcal{H}_0 = \delta \mathcal{H}_1$, then for any $n \in \mathcal{N}$, there exists $\mathcal{H}_n \in \mathcal{F}$ such that $\mathfrak{R}^{\star n} \delta \mathcal{H}_0 = \delta \mathcal{H}_n$. This explains how the infinitely many conserved densities (or Hamiltonians) arise, implying the integrability of the Hamiltonian system.

3. Hamiltonian Quadruplet

In this section we assume that we are working on a 3-dimensional Riemannian manifold M with constant curvature $\varkappa$. We remark that there are obvious generalizations to n–dimensional Riemannian manifolds, but the exact connection with integrable systems needs further study, which we plan to undertake. There are also generalizations to other homogeneous spaces. Many of these cases are still open. The following theorem generalizes results if [LP91] to the case of nonzero curvature.

Theorem 2. *Let M be a 3–dimensional Riemannian manifold with constant curvature $\varkappa$, and let $\gamma(x, t)$ be a family of curves on M satisfying a geometric evolution system of equations of the form*

$$\gamma_t = h_1 \mathbf{e}_1 + h_2 \mathbf{e}_2 + h_3 \mathbf{e}_3, \tag{7}$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the Frénet frame of γ, and where h_1, h_2, h_3 are arbitrary smooth functions of the curvatures k_1, k_2 and their derivatives with respect to x. Since we are in 3–dimensional space, from now on we use the notation κ, τ for k_1, k_2.

Assume that x is the arclength parameter and that evolution (7) is arclength preserving.

Then, the curvatures κ, τ satisfy the evolution

$$\begin{pmatrix}\kappa\\ \tau\end{pmatrix}_t = P\begin{pmatrix}h_3\\ h_1\end{pmatrix}, \tag{8}$$

where, if we denote by D_x the total differentiation operator with respect to x,

$$P = \begin{pmatrix} -\tau D_x - D_x\tau & D_x^2\frac{1}{\kappa}D_x - \frac{\tau^2}{\kappa}D_x + D_x\kappa \\ D_x\frac{1}{\kappa}D_x^2 - D_x\frac{\tau^2}{\kappa} + \kappa D_x & D_x\left(\frac{\tau}{\kappa^2}D_x + D_x\frac{\tau}{\kappa^2}\right)D_x + \tau D_x + D_x\tau \end{pmatrix} + \varkappa\begin{pmatrix} 0 & \frac{1}{\kappa}D_x \\ D_x\frac{1}{\kappa} & 0\end{pmatrix}. \tag{9}$$

Proof. A short comment on the calculations to follow: Let us denote by $T = \gamma_t$ and $\mathbf{e}_1 = \gamma_x$, assuming x to be arclength. These vectors are defined as the push-forward vectors of the vectors $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x}$, tangent to the domain of γ, through γ. That is, if $\gamma : U \subset \mathbb{R}^2 \to M$, then $\gamma_t = \gamma_*\frac{\partial}{\partial t}$ acting on functions as $\gamma_t(f) = \frac{\partial}{\partial t}f(\gamma(t,x))$; likewise for x. Thus, by applying T or $\mathbf{e}_1$ to functions defined along γ, we are indeed taking their derivatives with respect to t or x, respectively. If (7) is arclength preserving, these two vectors will commute since $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$ commute and γ_* preserves Lie brackets. The condition needed to guarantee that (7) is arclength preserving will be given below.

In order to find the evolution of the curvatures κ and τ, we will also find the evolution of the two first members of the frame, $\mathbf{e}_1$ and $\mathbf{e}_2$.

It follows from property (5) of a Riemannian connection and the fact that t and x differentiation commute, if γ is a solution of (7), that

$$\nabla_T\mathbf{e}_1 = \nabla_{\mathbf{e}_1}T = \nabla_{\mathbf{e}_1}(h_1\mathbf{e}_1 + h_2\mathbf{e}_2 + h_3\mathbf{e}_3). \tag{10}$$

Using property (4) of ∇ and Frénet formulae (2), evolution (10) can be rewritten as

$$\nabla_T\mathbf{e}_1 = (h_1' - \kappa h_2)\mathbf{e}_1 + (h_2' + h_1\kappa - \tau h_3)\mathbf{e}_2 + (h_3' + h_2\tau)\mathbf{e}_3, \tag{11}$$

where we denote by $'$ the total differentiation of the functions with respect to x. In order for the flow to be arclength preserving, it suffices to have $\nabla_T\mathbf{e}_1$ to be orthogonal to $\mathbf{e}_1$. This leads to

$$h_2 = \frac{h_1'}{\kappa}; \tag{12}$$

see [Ive01].

The evolution of κ can be found from here. On the one hand, we have

$$2\kappa\kappa_t = (\kappa^2)_t = 2\langle\nabla_T(\nabla_{\mathbf{e}_1}\mathbf{e}_1), \nabla_{\mathbf{e}_1}\mathbf{e}_1\rangle = 2\kappa\langle\nabla_T\nabla_{\mathbf{e}_1}\mathbf{e}_1, \mathbf{e}_2\rangle. \tag{13}$$

Meanwhile, from the definition of the curvature tensor

$$\nabla_T\nabla_{\mathbf{e}_1}\mathbf{e}_1 = \nabla_{\mathbf{e}_1}\nabla_T\mathbf{e}_1 + R(T,\mathbf{e}_1)\mathbf{e}_1, \tag{14}$$

where R is the curvature tensor of the manifold. Therefore

$$\kappa_t = \langle\nabla_{\mathbf{e}_1}\nabla_T\mathbf{e}_1, \mathbf{e}_2\rangle + K(\mathbf{e}_2,\mathbf{e}_1,T,\mathbf{e}_1), \tag{15}$$

where K is the Riemann-Christoffel curvature tensor of M. On the other hand, by applying $\nabla_{\mathbf{e}_1}$ to both sides of (11), one obtains

$$\begin{aligned}\nabla_{\mathbf{e}_1}\nabla_T\mathbf{e}_1 = \nabla^2_{\mathbf{e}_1}T \;=\; & [(h_1' - \kappa h_2)' - \kappa(h_2' + h_1\kappa - \tau h_3)]\mathbf{e}_1 \\ & + [(h_2' + h_1\kappa - \tau h_3)' + \kappa(h_1' - \kappa h_2) - \tau(h_3' + h_2\tau)]\mathbf{e}_2 \\ & + [(h_3' + h_2\tau)' + \tau(h_2' + h_1\kappa - \tau h_3)]\mathbf{e}_3,\end{aligned} \tag{16}$$

with the simple use of the Frénet formulae. From these two relations we obtain the evolution of κ as given by

$$\kappa_t = (h_2' + \kappa h_1 - \tau h_3)' + \kappa(h_1' - \kappa h_2) - \tau(h_3' + \tau h_2) + K(\mathbf{e}_2, \mathbf{e}_1, T, \mathbf{e}_1). \tag{17}$$

Applying the tensorial properties of K and Theorem 1, we obtain

$$\begin{aligned}\kappa_t \;=\; & (h_2' + \kappa h_1 - \tau h_3)' + \kappa(h_1' - \kappa h_2) - \tau(h_3' + \tau h_2) \\ & + h_2 K(\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1) + h_3 K(\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_1).\end{aligned} \tag{18}$$

We can now find the evolution of $\mathbf{e}_2$ from the Frénet relationship

$$\mathbf{e}_2 = \frac{1}{\kappa}\nabla_{\mathbf{e}_1}\mathbf{e}_1. \tag{19}$$

Indeed, if we apply ∇_T to (19), and we use that the result should be orthogonal to $\mathbf{e}_2$ (since it is a unit vector), this leads to

$$\nabla_T\mathbf{e}_2 = \frac{1}{\kappa}\nabla_T\nabla_{\mathbf{e}_1}\mathbf{e}_1 - \frac{1}{\kappa}\langle\nabla_T\nabla_{\mathbf{e}_1}\mathbf{e}_1, \mathbf{e}_2\rangle\mathbf{e}_2. \tag{20}$$

Substituting (14) into it, we obtain

$$\nabla_T\mathbf{e}_2 = \frac{1}{\kappa}\nabla^2_{\mathbf{e}_1}T + \frac{1}{\kappa}R(T, \mathbf{e}_1)\mathbf{e}_1 - \frac{1}{\kappa}\langle\nabla^2_{\mathbf{e}_1}T, \mathbf{e}_2\rangle\mathbf{e}_2 - \frac{1}{\kappa}K(\mathbf{e}_2, \mathbf{e}_1, T, \mathbf{e}_1)\mathbf{e}_2. \tag{21}$$

We are now in a position to find the evolution of τ. As we did in (13) for κ, it is very simple to see that

$$\tau_t = \langle\nabla_T(\nabla_{\mathbf{e}_1}\mathbf{e}_2 + \kappa\mathbf{e}_1), \mathbf{e}_3\rangle. \tag{22}$$

But

$$\nabla_T\nabla_{\mathbf{e}_1}\mathbf{e}_2 = \nabla_{\mathbf{e}_1}\nabla_T\mathbf{e}_2 + R(T, \mathbf{e}_1)\mathbf{e}_2,$$

so that, applying $\nabla_{\mathbf{e}_1}$ to (21) we obtain, after some short calculations,

$$\langle\nabla_{\mathbf{e}_1}\nabla_T\mathbf{e}_2, \mathbf{e}_3\rangle = \left(\frac{1}{\kappa}\langle\nabla^2_{\mathbf{e}_1}T, \mathbf{e}_3\rangle\right)' + \left(\frac{1}{\kappa}K(\mathbf{e}_3, \mathbf{e}_1, T, \mathbf{e}_1)\right)', \tag{23}$$

and from here we obtain

$$\begin{aligned}\tau_t \;=\; & [\tfrac{1}{\kappa}(h_3' + h_2\tau)' + \tfrac{\tau}{\kappa}(h_2' + h_1\kappa - \tau h_3)]' + \kappa(h_3' + h_2\tau) \\ & + [\tfrac{h_2}{\kappa}K(\mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1) + \tfrac{h_3}{\kappa}\mathrm{sec}(\mathbf{e}_1, \mathbf{e}_3)]' \\ & + h_2 K(\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_1) + h_3 K(\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1).\end{aligned} \tag{24}$$

If we finally impose arclength preserving condition (12) to evolutions (17) and (24), we obtain that the evolution of κ and τ can be written as

$$\begin{pmatrix}\kappa\\ \tau\end{pmatrix}_t = \hat{P}\begin{pmatrix}h_3\\ h_1\end{pmatrix},$$

where, if D_x is the total derivative with respect to x, and we denote $K_{ijkr} = K(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_r)$,

$$\hat{P} = \begin{pmatrix} -\tau D_x - D_x\tau & D_x^2\frac{1}{\kappa}D_x - \frac{\tau^2}{\kappa}D_x + D_x\kappa \\ D_x\frac{1}{\kappa}D_x^2 - D_x\frac{\tau^2}{\kappa} + \kappa D_x & D_x\left(\frac{\tau}{\kappa^2}D_x + D_x\frac{\tau}{\kappa^2}\right)D_x + \tau D_x + D_x\tau \end{pmatrix}$$
$$+ \begin{pmatrix} K_{2131}\frac{1}{\kappa}\mathrm{sec}(\mathbf{e}_1, \mathbf{e}_2)\mathrm{D_x} \\ K_{3231} + D_x\frac{1}{\kappa}\mathrm{sec}(\mathbf{e}_1, \mathbf{e}_3)\frac{1}{\kappa}\mathrm{K_{3221}D_x} + \mathrm{D_x}\frac{\mathrm{K_{3121}}}{\kappa^2}\mathrm{D_x} \end{pmatrix}. \tag{25}$$

If the manifold M has constant curvature $\varkappa$, Proposition 1 and Corollary 1 provide the values

$$K(\mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1) = K(\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_2) = K(\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_3) = 0,$$

and if $i \neq j$,

$$\mathrm{sec}(\mathbf{e}_\mathrm{i}, \mathbf{e}_\mathrm{j}) = \varkappa,$$

for any $i, j = 1, \ldots, 3$. We simply need to substitute the values in (25), to obtain the result of this theorem. □

The general scheme to obtain the evolution equations in the n-dimensional arclength-preserving case runs as follows. Let us define $k_0 = 0$. Then we use the following formulae to inductively compute all derivatives with respect to t.

$$\nabla_T\mathbf{e}_i = \begin{cases} \nabla_{\mathbf{e}_1}T & \text{if } i = 1, \\ (\nabla_{\mathbf{e}_1}\nabla_T\mathbf{e}_{i-1} - k_{i-1,t}\mathbf{e}_i + k_{i-2,t}\mathbf{e}_{i-2}, \\ +k_{i-2}\nabla_T\mathbf{e}_{i-2} + R(T, \mathbf{e}_1)\mathbf{e}_{i-1})/k_{i-1} & \text{if } i = 2, \ldots, n, \end{cases}$$

and

$$k_{it} = \langle\nabla_{\mathbf{e}_1}\nabla_T\mathbf{e}_i + k_{i-1}\nabla_T\mathbf{e}_{i-1}, \mathbf{e}_{i+1}\rangle + K(\mathbf{e}_{i+1}, \mathbf{e}_i, T, \mathbf{e}_1).$$

We now turn back to our 3-dimensional problem.

Theorem 3. *The skew-symmetric operators*

$$\mathcal{B} = \begin{pmatrix} -\tau D_x - D_x\tau & -\frac{\tau^2}{\kappa}D_x \\ -D_x\frac{\tau^2}{\kappa} & 0 \end{pmatrix}, \tag{26}$$

$$\mathcal{C} = \begin{pmatrix} 0 & \frac{1}{\kappa}D_x \\ D_x\frac{1}{\kappa} & 0 \end{pmatrix}, \tag{27}$$

$$\mathcal{D} = \begin{pmatrix} 0 & D_x\kappa \\ \kappa D_x & \tau D_x + D_x\tau \end{pmatrix}, \tag{28}$$

$$\mathcal{E} = \begin{pmatrix} 0 & D_x^2\frac{1}{\kappa}D_x \\ D_x\frac{1}{\kappa}D_x^2 & D_x(\frac{\tau}{\kappa^2}D_x + D_x\frac{\tau}{\kappa^2})D_x \end{pmatrix}, \tag{29}$$

form a quadruplet of compatible Hamiltonian operators (where $P = \mathcal{B} + \varkappa\mathcal{C} + \mathcal{D} + \mathcal{E}$*, cf. (9)).*

Proof. We prove this by checking the conditions of Theorem 7.8 and Corollary 7.21 in [Olv93]. Here we give only the details to show that $\mathcal{D}$ and $\mathcal{E}$ are compatible and leave the remaining, identical computations to the reader.

First we check that the operators $\mathcal{D}$ and $\mathcal{E}$ are indeed Hamiltonian operators. The associated bivector of $\mathcal{D}$ is by definition

$$\begin{aligned}\Theta_{\mathcal{D}} &= \tfrac{1}{2}\int (\theta \dot{\wedge} \mathcal{D}\theta)\, dx \\ &= \tfrac{1}{2}\int (\kappa\vartheta \wedge \zeta_1 + \kappa_1\vartheta \wedge \zeta + \kappa\zeta \wedge \vartheta_1 + 2\tau\zeta \wedge \zeta_1)\, dx \\ &= \int (\kappa\zeta \wedge \vartheta_1 + \tau\zeta \wedge \zeta_1)\, dx,\end{aligned}$$

where $\theta = (\vartheta, \zeta)$ and $\vartheta_i = \frac{\partial^i \vartheta}{\partial x^i}$, etc. Here $\dot{\wedge}$ means that one needs to take the ordinary inner product between the vectors θ and $\mathcal{D}\theta$. The elements of these vectors are then multiplied using the ordinary wedge product. We need to check the vanishing of the Schouten bracket $[\mathcal{D}, \mathcal{D}]$ which is equivalent to the Jacobi identity for the Lie bracket defined by $\mathcal{D}$.

$$\begin{aligned}[][\mathcal{D}, \mathcal{D}] &= \mathrm{pr}\, \mathbf{v}_{\mathcal{D}\theta}(\Theta_{\mathcal{D}}) \\ &= \int ((\kappa_1\zeta + \kappa\zeta_1) \wedge \zeta \wedge \vartheta_1 + (\kappa\vartheta_1 + 2\tau\zeta_1 + \tau_1\zeta) \wedge \zeta \wedge \zeta_1)\, dx \\ &= \int (\kappa\zeta_1 \wedge \zeta \wedge \vartheta_1 + \kappa\vartheta_1 \wedge \zeta \wedge \zeta_1)\, dx \\ &= 0.\end{aligned}$$

The proof that $\mathcal{E}$ is Hamiltonian is a rather laborious calculation. Its associated bivector is

$$\begin{aligned}\Theta_{\mathcal{E}} &= \tfrac{1}{2}\int \left(\theta \dot{\wedge} \mathcal{E}\theta\right) dx \\ &= \tfrac{1}{2}\int \left(\vartheta \wedge (D_x^2 \frac{\zeta_1}{\kappa}) + \zeta \wedge (D_x \frac{\vartheta_2}{\kappa}) + \zeta \wedge (D_x \frac{\tau\zeta_2}{\kappa^2}) + \zeta \wedge (D_x^2 \frac{\tau\zeta_1}{\kappa^2})\right) dx \\ &= \int \left(\frac{1}{\kappa}\vartheta_2 \wedge \zeta_1 + \frac{\tau}{\kappa^2}\zeta_2 \wedge \zeta_1\right) dx.\end{aligned}$$

The vanishing of $\mathrm{pr}\, \mathbf{v}_{\mathcal{E}\theta}(\Theta_{\mathcal{E}})$ can be proved by the fact that

$$\begin{aligned}\mathrm{pr}\, \mathbf{v}_{\mathcal{E}\theta}(\kappa) &= D_x^2 \frac{1}{\kappa}\zeta_1 \equiv -\frac{2\kappa_1}{\kappa^2}\zeta_2 + \frac{1}{\kappa}\zeta_3 \bmod \zeta_1, \\ \mathrm{pr}\, \mathbf{v}_{\mathcal{E}\theta}(\tau) &= D_x \frac{1}{\kappa}\vartheta_2 + D_x\left(\frac{\tau}{\kappa}\zeta_2 + D_x \frac{\tau}{\kappa^2}\zeta_1\right) \\ &\equiv -\frac{\kappa_1}{\kappa^2}\vartheta_2 + \frac{1}{\kappa}\vartheta_3 + \frac{2\tau}{\kappa^2}\zeta_3 \bmod \zeta_1 \wedge \zeta_2,\end{aligned}$$

and by integration by parts:

$$\begin{aligned}
[\mathcal{E},\mathcal{E}] &= \mathrm{pr}\,\mathbf{v}_{\mathcal{E}\theta}(\Theta_{\mathcal{E}}) \\
&= \int -\frac{1}{\kappa^2}\mathrm{pr}\,\mathbf{v}_{\mathcal{E}\theta}(\kappa)\vartheta_2\wedge\zeta_1 - \frac{2\tau}{\kappa^3}\mathrm{pr}\,\mathbf{v}_{\mathcal{E}\theta}(\kappa)\zeta_2\wedge\zeta_1 + \frac{1}{\kappa^2}\mathrm{pr}\,\mathbf{v}_{\mathcal{E}\theta}(\tau)\zeta_2\wedge\zeta_1\,dx \\
&= \int -\frac{1}{\kappa^2}(-\frac{2\kappa_1}{\kappa^2}\zeta_2 + \frac{1}{\kappa}\zeta_3)\wedge\vartheta_2\wedge\zeta_1 - \frac{2\tau}{\kappa^4}\zeta_3\wedge\zeta_2\wedge\zeta_1 \\
&\quad + \frac{1}{\kappa^2}(-\frac{\kappa_1}{\kappa^2}\vartheta_2 + \frac{1}{\kappa}\vartheta_3 + \frac{2\tau}{\kappa^2}\zeta_3)\wedge\zeta_2\wedge\zeta_1\,dx \\
&= \int \frac{3\kappa_1}{\kappa^4}\zeta_2\wedge\vartheta_2\wedge\zeta_1 - \frac{1}{\kappa^3}\zeta_3\wedge\vartheta_2\wedge\zeta_1 + \frac{1}{\kappa^3}\vartheta_3\wedge\zeta_2\wedge\zeta_1\,dx \\
&= \int D_x\left(\frac{1}{\kappa^3}\vartheta_2\wedge\zeta_2\wedge\zeta_1\right)dx = 0.
\end{aligned}$$

Now we prove that $\mathcal{D}$ and $\mathcal{E}$ form a Hamiltonian pair by checking

$$[\mathcal{D},\mathcal{E}] + [\mathcal{E},\mathcal{D}] = \mathrm{pr}\,\mathbf{v}_{\mathcal{D}\theta}(\Theta_{\mathcal{E}}) + \mathrm{pr}\,\mathbf{v}_{\mathcal{E}\theta}(\Theta_{\mathcal{D}}) = 0.$$

We compute

$$\begin{aligned}
&[\mathcal{D},\mathcal{E}] + [\mathcal{E},\mathcal{D}] = \mathrm{pr}\,\mathbf{v}_{\mathcal{D}\theta}(\Theta_{\mathcal{E}}) + \mathrm{pr}\,\mathbf{v}_{\mathcal{E}\theta}(\Theta_{\mathcal{D}}) \\
&= \int\Bigg(\frac{1}{\kappa^2}\mathrm{pr}\,\mathbf{v}_{\mathcal{D}\theta}(\kappa)\wedge\zeta_1\wedge\vartheta_2 + \frac{2\tau}{\kappa^3}\mathrm{pr}\,\mathbf{v}_{\mathcal{D}\theta}(\kappa)\wedge\zeta_1\wedge\zeta_2 - \frac{1}{\kappa^2}\mathrm{pr}\,\mathbf{v}_{\mathcal{D}\theta}(\tau)\wedge\zeta_1\wedge\zeta_2 \\
&\qquad + \mathrm{pr}\,\mathbf{v}_{\mathcal{E}\theta}(\kappa)\wedge\zeta\wedge\vartheta_1 + \mathrm{pr}\,\mathbf{v}_{\mathcal{E}\theta}(\tau)\wedge\zeta\wedge\zeta_1\Bigg)\,dx \\
&= \int\Bigg(\frac{\kappa_1}{\kappa^2}\zeta\wedge\zeta_1\wedge\vartheta_2 + \frac{2\tau\kappa_1}{\kappa^3}\zeta\wedge\zeta_1\wedge\zeta_2 - \frac{1}{\kappa^2}(\kappa\vartheta_1 + \tau_1\zeta)\wedge\zeta_1\wedge\zeta_2 \\
&\qquad + \left(D_x^2\frac{1}{\kappa}\zeta_1\right)\wedge\zeta\wedge\vartheta_1 + \left(D_x\frac{1}{\kappa}\vartheta_2\right)\wedge\zeta\wedge\zeta_1 + \left(D_x\frac{\tau}{\kappa^2}\zeta_2\right)\wedge\zeta\wedge\zeta_1 \\
&\qquad + \left(D_x^2\frac{\tau}{\kappa^2}\zeta_1\right)\wedge\zeta\wedge\zeta_1\Bigg)\,dx \\
&= \int\Bigg(\frac{\kappa_1}{\kappa^2}\zeta\wedge\zeta_1\wedge\vartheta_2 + \frac{2\tau\kappa_1}{\kappa^3}\zeta\wedge\zeta_1\wedge\zeta_2 - \frac{1}{\kappa^2}(\kappa\vartheta_1 + \tau_1\zeta)\wedge\zeta_1\wedge\zeta_2 \\
&\qquad + \frac{1}{\kappa}\zeta_1\wedge(\zeta_2\wedge\vartheta_1 + \zeta\wedge\vartheta_3) - \frac{1}{\kappa}\vartheta_2\wedge\zeta\wedge\zeta_2 - \frac{\tau}{\kappa^2}\zeta\wedge\zeta_1\wedge\zeta_3\Bigg)\,dx \\
&= \int\Bigg(\frac{2\tau\kappa_1}{\kappa^3}\zeta\wedge\zeta_1\wedge\zeta_2 - \frac{\tau}{\kappa^2}\zeta\wedge\zeta_1\wedge\zeta_3 - \frac{\tau_1}{\kappa^2}\zeta\wedge\zeta_1\wedge\zeta_2 \\
&\qquad + \frac{\kappa_1}{\kappa^2}\zeta\wedge\zeta_1\wedge\vartheta_2 + \frac{1}{\kappa}\zeta_1\wedge\zeta\wedge\vartheta_3 - \frac{1}{\kappa}\zeta\wedge\zeta_2\wedge\vartheta_2\Bigg)\,dx \\
&= -\int D_x\left(\frac{\tau}{\kappa^2}\zeta\wedge\zeta_1\wedge\zeta_2 + \frac{1}{\kappa}\zeta\wedge\zeta_1\wedge\vartheta_2\right)dx = 0.
\end{aligned}$$

Thus the result follows. □

4. Integrable Evolutions

4.1. Two Integrable Canonical Evolution Equations

Having a Hamiltonian quadruplet, with three of its members, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$, being nondegenerate, allows us to produce two hereditary operators. We insist again on the fact that these are tensors linked to the intrinsic geometry of Riemannian curves and not to any integrable system in particular. On the other hand, they are indeed recursion operators for two canonical integrable systems.

The Vortex Filament Flow

The Hamiltonian pair $P = \mathcal{B} + \varkappa\mathcal{C} + \mathcal{D} + \mathcal{E}$ and $\mathcal{C}$ gives us the hereditary operator:

$$\begin{aligned}\mathfrak{R}_1 &= P\mathcal{C}^{-1}\\ &= \begin{pmatrix} D_x^2 - \tau^2 + \kappa^2 + \varkappa & -2\kappa\tau \\ 2D_x^2\frac{\tau}{\kappa} - D_x(\frac{\tau_1}{\kappa} - \frac{2\kappa_1\tau}{\kappa^2}) + 2\kappa\tau & D_x^2 + 2D_x\frac{\kappa_1}{\kappa} + \frac{\kappa_2}{\kappa} - \tau^2 + \kappa^2 + \varkappa \end{pmatrix}\\ &\quad + \begin{pmatrix} -2\tau\kappa_1 - \kappa\tau_1 \\ \frac{\kappa_3}{\kappa} - \frac{\kappa_1\kappa_2}{\kappa^2} - 2\tau\tau_1 + \kappa\kappa_1 \end{pmatrix} D_x^{-1}(0 \quad 1)\\ &\quad + \begin{pmatrix} \kappa_1 \\ \tau_1 \end{pmatrix} D_x^{-1}(\kappa \quad 0).\end{aligned}$$

From its expression, one can identify equations

$$\begin{cases} \kappa_{t_1} = -2\tau\kappa_1 - \kappa\tau_1, \\ \tau_{t_1} = \frac{\kappa_3}{\kappa} - \frac{\kappa_1\kappa_2}{\kappa^2} - 2\tau\tau_1 + \kappa\kappa_1, \end{cases} \tag{30}$$

and

$$\begin{cases} \kappa_{t_3} = \kappa_3 + \frac{3}{2}\kappa^2\kappa_1 - 3\kappa_1\tau^2 - 3\kappa\tau\tau_1 + \varkappa\kappa_1, \\ \tau_{t_3} = D_x(\tau_2 + \frac{3}{2}\kappa^2\tau - \tau^3 + 3\frac{\kappa_2\tau}{\kappa} + 3\frac{\kappa_1\tau_1}{\kappa} + \varkappa\tau), \end{cases} \tag{31}$$

the latter being $\mathfrak{R}_1\binom{\kappa_1}{\tau_1}$, as symmetries of the operator $\mathfrak{R}_1$. One can also identify their cosymmetries to be $(\kappa, 0)$ and $(0, 1)$, deriving from conservation laws with densities $\frac{\kappa^2}{2}$ and τ, respectively. These (commuting) equations have the operator $\mathfrak{R}_1$ as recursion operator ([SW01a]), which generates a hierarchy of symmetries, cosymmetries, and conservation laws for the equations.

Equations (30) and (31) are the evolutions of curvature and torsion associated to the best-known versions of the vortex filament equations. Vortex filament equations are the nonlinear evolution of curves describing the time development of a very thin vortex tube. The ones associated to our integrable systems are

$$\gamma_{t_1} = \kappa\mathbf{e}_3; \tag{32}$$

$$\gamma_{t_3} = \frac{1}{2}\kappa^2\mathbf{e}_1 + \kappa'\mathbf{e}_2 + \kappa\tau\mathbf{e}_3. \tag{33}$$

In the Euclidean case, Hasimoto found the transformation from (32) to the nonlinear Schrödinger equation, which enabled him to study its geometry and integrability, cf.

[Has72]. Some generalizations have been extensively studied in [LP91], [DS94], and [YS98]. The curve evolution (33) and its relation to (32) was also studied in [LP91].

Evolution equation (30) is easily seen to be a bi-Hamiltonian system since

$$u_{t_1} = \mathcal{C}\delta\mathcal{H}_1 = (\mathcal{B} + \varkappa\mathcal{C} + \mathcal{D} + \mathcal{E})\delta\mathcal{H}_0,$$

where $u = (\kappa, \tau)$, $\mathcal{H}_0 = \int \frac{\kappa^2}{2} dx$, and $\mathcal{H}_1 = \int \left(\frac{1}{8}\kappa^4 - \frac{1}{2}\kappa_1^2 - \frac{1}{2}\tau^2\kappa^2\right) dx$. In fact, notice that $\mathcal{H}_0$ is in the kernel of $\mathcal{C}$. This fact allows us to drop $\varkappa\mathcal{C}$ from the Hamiltonian pair of this particular equation and to consider the simpler recursion operator $(\mathcal{B} + \mathcal{D} + \mathcal{E})\mathcal{C}^{-1}$, corresponding to the flat case, as valid for the general case. This simplification does not hold for equation (31). However, it is also bi-Hamiltonian, as shown in [LP91], since

$$u_{t_3} = \mathcal{C}\delta\mathcal{H}_3 = (\mathcal{B} + \varkappa\mathcal{C} + \mathcal{D} + \mathcal{E})\delta\mathcal{H}_2,$$

with $\mathcal{H}_2 = \int \frac{1}{2}\kappa^2\tau dx$ and $\mathcal{H}_3 = \int(\kappa\kappa_2\tau - \frac{1}{2}\kappa_1^2\tau - \frac{1}{2}\tau^3\kappa^2 + \frac{3}{8}\kappa^4\tau)dx + \varkappa\mathcal{H}_2$.

Therefore, evolutions (30) and (31) are completely integrable systems. Geometric evolutions (32) and (33) would also be integrable in the sense that their associated κ, τ evolutions are, and given that $\kappa(t, x)$, $\tau(t, x)$ determine $\gamma(t, x)$ up to the action of the group.

After minor calculations, one can easily see that the operator $\mathcal{D}$ is not a Hamiltonian operator for either of these two equations.

A Second Integrable System

Since the operator $\mathcal{D}$ is also nondegenerate, the different choice of Hamiltonian pair $\mathcal{J} = \mathcal{B} + \varkappa\mathcal{C} + \mathcal{E}$ and $\mathcal{D}$ (we will see shortly the reason why we drop $\mathcal{D}$ in this pair) gives us the hereditary operator:

$$\begin{aligned}
\mathfrak{R}_2 &= \mathcal{J}\mathcal{D}^{-1} \\
&= \begin{pmatrix} \frac{1}{\kappa^2}D_x^2 - \frac{5\kappa_1}{\kappa^3}D_x - \frac{4\kappa_2}{\kappa^3} + \frac{12\kappa_1^2}{\kappa^4} + \frac{3\tau^2}{\kappa^2} & -\frac{2\tau}{\kappa} \\ -\frac{2\tau_1}{\kappa^3}D_x + \frac{2\tau^3}{\kappa^3} - \frac{3\tau_2}{\kappa^3} + \frac{9\kappa_1\tau_1}{\kappa^4} & \frac{1}{\kappa^2}D_x^2 - \frac{3\kappa_1}{\kappa^3}D_x - \frac{\kappa_2}{\kappa^3} + \frac{3\kappa_1^2}{\kappa^4} - \frac{\tau^2}{\kappa^2} \end{pmatrix} \\
&\quad + \varkappa \begin{pmatrix} \frac{1}{\kappa^2} & 0 \\ -2\frac{\tau}{\kappa^3} & \frac{1}{\kappa^2} \end{pmatrix} \\
&\quad - \begin{pmatrix} \frac{\kappa_3}{\kappa^3} - 9\frac{\kappa_1\kappa_2}{\kappa^4} + 12\frac{\kappa_1^3}{\kappa^5} - 3\frac{\tau\tau_1}{\kappa^2} + 3\frac{\tau^2\kappa_1}{\kappa^3} + \varkappa\frac{\kappa_1}{\kappa^3} \\ \frac{\tau_3}{\kappa^3} - 6\frac{\kappa_1\tau_2}{\kappa^4} - 3\frac{\kappa_2\tau_1}{\kappa^4} - 3\frac{\tau^2\tau_1}{\kappa^3} + 3\frac{\kappa_1\tau^3}{\kappa^4} + 12\frac{\kappa_1^2\tau_1}{\kappa^5} + \varkappa\left(\frac{\tau}{\kappa^3}\right)_1 \end{pmatrix} D_x^{-1}(1 \quad 0) \\
&\quad - \begin{pmatrix} \tau_1 \\ \frac{2\tau\tau_1}{\kappa} - \frac{\tau^2\kappa_1}{\kappa^2} + \varkappa\frac{\kappa_1}{\kappa^2} \end{pmatrix} D_x^{-1} \begin{pmatrix} -\frac{\tau}{\kappa^2} & \frac{1}{\kappa} \end{pmatrix}.
\end{aligned}$$

Again, from its expression we can identify the cosymmetries in $\mathfrak{R}_2$, that is, $(1, \quad 0)$ and $(-\frac{\tau}{\kappa^2}, \quad \frac{1}{\kappa})$, deriving from conservation laws, with densities κ and $\frac{\tau}{\kappa}$, respectively.

The equations

$$\begin{cases} \kappa_{t_1} = \tau_1, \\ \tau_{t_1} = \frac{2\tau\tau_1}{\kappa} - \frac{\tau^2\kappa_1}{\kappa^2} + \varkappa\frac{\kappa_1}{\kappa^2}, \end{cases} \tag{34}$$

and

$$\begin{cases}\kappa_{t_3} = \frac{\kappa_3}{\kappa^3} - 9\frac{\kappa_1\kappa_2}{\kappa^4} + 12\frac{\kappa_1^3}{\kappa^5} - 3\frac{\tau\tau_1}{\kappa^2} + 3\frac{\tau^2\kappa_1}{\kappa^3} + \varkappa\frac{\kappa_1}{\kappa^3},\\ \tau_{t_3} = \frac{\tau_3}{\kappa^3} - 6\frac{\kappa_1\tau_2}{\kappa^4} - 3\frac{\kappa_2\tau_1}{\kappa^4} - 3\frac{\tau^2\tau_1}{\kappa^3} + 3\frac{\kappa_1\tau^3}{\kappa^4} + 12\frac{\kappa_1^2\tau_1}{\kappa^5} + \varkappa(\frac{\tau}{\kappa^3})_1,\end{cases} \tag{35}$$

are symmetries of the operator, and so they have $\mathfrak{R}_2$ as recursion operator.

Equation (35) is a generalization of the equations studied by Ivey to the case of manifolds with constant curvature.

Again, evolution equation (34) is easily seen to be a bi-Hamiltonian system. This is true since

$$u_{t_1} = \mathcal{D}\delta\mathcal{H}_1 = \mathcal{J}\delta\mathcal{H}_0,$$

where $u = (\kappa, \tau)$, $\mathcal{H}_0 = -\int \kappa\, dx$, and $\mathcal{H}_1 = \int(\frac{\tau^2}{2\kappa} + \varkappa\frac{1}{2\kappa})dx$. We also have that equation (35) is bi-Hamiltonian:

$$u_{t_3} = \mathcal{D}\delta\mathcal{H}_3 = \mathcal{J}\delta\mathcal{H}_2,$$

with $\mathcal{H}_2 = -\int \frac{\tau}{\kappa}dx$ and $\mathcal{H}_3 = \int\left(\frac{\tau\kappa_1^2}{\kappa^5} - \frac{\kappa_1\tau_1}{\kappa^4} - \frac{\tau^3}{2\kappa^3} - \varkappa\frac{\tau}{2\kappa^3}\right)dx$. However, operator $\mathcal{C}$ is not a Hamiltonian operator for either of these two equations. This implies that $\mathcal{C}$ *is not in the hierarchy generated by* $\mathcal{D}$ *and* $\mathcal{E}$. Therefore, *the quadruplet* $\mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ *is a true Hamiltonian quadruplet*.

Notice that both $\mathcal{H}_0$ and $\mathcal{H}_2$ lie on the kernel of the operator $\mathcal{D}$. This explains why from the beginning we dropped $\mathcal{D}$ from the Hamiltonian pair, considering $(\mathcal{B} + \varkappa\mathcal{C} + \mathcal{E}, \mathcal{C})$ rather than the expected, but more complicated, $(\mathcal{B} + \varkappa\mathcal{C} + \mathcal{D} + \mathcal{E}, \mathcal{C})$. Thus, (34) and (35) still have geometric evolutions of curves associated to them. Namely,

$$\gamma_{t_1} = -\mathbf{e}_3$$

is associated with (34), and the one associated with (35) is

$$\gamma_{t_3} = -\frac{1}{\kappa}\mathbf{e}_1 + \frac{\kappa_1}{\kappa^3}\mathbf{e}_2 + \frac{\tau}{\kappa^2}\mathbf{e}_3.$$

They would also be integrable in the same sense that was mentioned above for the previous set of integrable equations.

4.2. Some Transformations

In this section we describe the transformations that simplify our integrable hierarchies, the well–known Hasimoto transformation, which takes the vortex filament flow to the nonlinear Schrödinger equation, and the hodographic transformation taking the second hierarchy in the nonflat case to a decoupled potential KdV system. We first describe briefly how a change of variable will affect the Poisson brackets. Suppose we have coordinates (x, u) and a Poisson bracket in these coordinates given by

$$\{f, g\} = \int \frac{\delta f}{\delta u} \cdot \mathcal{D}\frac{\delta g}{\delta u}\, dx.$$

We map (y, v) by an invertible map Φ to (x, u) coordinates. The effect of this change of variables on the variational derivatives of f and g is known to be given by $\left(\frac{\delta v}{\delta u}\right)^* \frac{\delta \Phi^* f}{\delta v}$ and

$(\frac{\delta v}{\delta u})^* \frac{\delta \Phi^* g}{\delta v}$, respectively, where $(\frac{\delta v}{\delta u})^*$ denotes the operator conjugate to $\frac{\delta v}{\delta u}$ and $\Phi^* f = f(\Phi(y, v))$. From here we obtain the transformation of our Poisson bracket under the map Φ

$$\Phi^*\{f, g\} = \int \left[\left(\frac{\delta v}{\delta u}\right)^* \frac{\delta \Phi^* f}{\delta v} \right] \cdot \left[\mathcal{D} \left(\frac{\delta v}{\delta u}\right)^* \frac{\delta \Phi^* g}{\delta v} \right] \left| \frac{\delta x}{\delta y} \right| dy.$$

Let $\Phi^* \mathcal{D} = \frac{\delta v}{\delta u} \mathcal{D} (\frac{\delta v}{\delta u})^* |\frac{\delta x}{\delta y}|$. Then we define

$$\{\hat{f}, \hat{g}\}' = \int \frac{\delta \hat{f}}{\delta v} \cdot \Phi^* \mathcal{D} \frac{\delta \hat{g}}{\delta v} dy \tag{36}$$

to be the new bracket, with associated tensor $\Phi^* \mathcal{D}$. Since we have

$$\Phi^*\{f, g\} = \{\Phi^* f, \Phi^* g\}',$$

the skew-symmetry and Jacobi identity follow automatically.

The Hasimoto Transformation

We describe how the first integrable group of equations changes under the Hasimoto transformation:

$$\phi = \kappa e^{i \int \tau dx}.$$

Notice that

$$\begin{aligned}
\phi_t &= (\kappa_t + i\kappa \int \tau_t dx) e^{i \int \tau dx}, \\
\phi_1 &= (\kappa_1 + i\kappa\tau) e^{i \int \tau dx}, \\
\phi_2 &= (\kappa_2 + 2i\kappa_1\tau + i\kappa\tau_1 - \kappa\tau^2) e^{i \int \tau dx}, \\
\phi_3 &= (\kappa_3 + 3i\kappa_2\tau + 3i\kappa_1\tau_1 + i\kappa\tau_2 - 3\kappa\tau\tau_1 - 3\kappa_1\tau^2 - i\kappa\tau^3) e^{i \int \tau dx}.
\end{aligned}$$

We can now rewrite (30) as

$$\phi_{t_2} = i\phi_2 + \frac{i}{2}|\phi|^2\phi, \tag{37}$$

which is the nonlinear Schrödinger equation, and (31) as

$$\phi_{t_3} = \phi_3 + \frac{3}{2}|\phi|^2\phi_1,$$

which is the modified KdV equation.

To show how the transformation affects the Hamiltonian pair, we write it as

$$u + iv = \kappa e^{i \int \tau dx},$$

which gives the matrix

$$Q = \begin{pmatrix} \cos(\int \tau dx) & -\kappa \sin(\int \tau dx) D_x^{-1} \\ \sin(\int \tau dx) & \kappa \cos(\int \tau dx) D_x^{-1} \end{pmatrix}$$

describing the action of the transformation on the vector fields. The Hamiltonian operator, say $\mathcal{C}$, transforms as $\mathcal{C} \mapsto Q\mathcal{C}Q^*$, according to (36), since $x = y$ so that $\frac{\delta x}{\delta y} = 1$. We find that

$$Q\mathcal{C}Q^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and

$$Q\mathcal{D}Q^* = \begin{pmatrix} -vD_x^{-1}uD_x + D_xuD_x^{-1}v & -D_xuD_x^{-1}u - vD_x^{-1}vD_x \\ uD_x^{-1}uD_x + D_xvD_x^{-1}v & uD_x^{-1}vD_x - D_xvD_x^{-1}u \end{pmatrix},$$

$$Q(\mathcal{B}+\mathcal{E})Q^* = \begin{pmatrix} 0 & -D_x^2 \\ D_x^2 & 0 \end{pmatrix}.$$

The new recursion operator, which $\mathfrak{R}_1$ transforms to, is

$$\begin{aligned} \bar{\mathfrak{R}}_1 &= \varkappa \mathrm{I} \\ &\quad + \begin{pmatrix} D_x^2+u^2+v^2+u_1D_x^{-1}u-vD_x^{-1}v_1 & vD_x^{-1}u_1+u_1D_x^{-1}v \\ uD_x^{-1}v_1+v_1D_x^{-1}u & D_x^2+u^2+v^2-uD_x^{-1}u_1+v_1D_x^{-1}v \end{pmatrix} \\ &= \varkappa \mathrm{I} - \begin{pmatrix} -vD_x^{-1}u & -D_x - vD_x^{-1}v \\ D_x+uD_x^{-1}u & uD_x^{-1}v \end{pmatrix}^2 = \varkappa \mathrm{I} - \mathfrak{R}_{nls}^2, \end{aligned}$$

where I denotes the identity matrix. It is interesting to notice that $\mathfrak{R}_{nls}$ is a recursion operator of (37), as found by Magri [Mag78]. In fact, if we look at the recursion operator prior to the Hasimoto transformation, we now realize that

$$\mathfrak{R}_1 = \varkappa \mathrm{I} - \begin{pmatrix} -\tau & -D_x\kappa D_x^{-1} \\ D_x\frac{1}{\kappa}D_x + \kappa & -D_x\tau D_x^{-1} \end{pmatrix}^2.$$

Furthermore, if we let

$$\mathcal{F} = \begin{pmatrix} D_x & \frac{\tau}{\kappa}D_x \\ D_x\frac{\tau}{\kappa} & -D_x \end{pmatrix},$$

and

$$\mathcal{G} = \begin{pmatrix} 0 & 0 \\ 0 & D_x\frac{1}{\kappa}D_x\frac{1}{\kappa}D_x \end{pmatrix},$$

then $\mathcal{F}$ and $\mathcal{G}$ are compatible Hamiltonian operators, such that

$$((\mathcal{F}-\mathcal{G})\mathcal{C}^{-1})^2 = \varkappa I - \mathfrak{R}_1.$$

Indeed, we can now see that $\mathcal{B}+\mathcal{D}+\mathcal{E} = -(\mathcal{F}-\mathcal{G})\mathcal{C}^{-1}(\mathcal{F}-\mathcal{G})$ and hence, evolution (30) is bi-Hamiltonian with respect to the simpler Hamiltonian pair $\{\mathcal{G}-\mathcal{F}, \mathcal{C}\}$. Its new associated Hamiltonian is given by $\hat{\mathcal{H}}_0 = \frac{1}{2}\int \kappa^2\tau dx$. This is exactly the Hamiltonian functional corresponding to (31). Thus *in the flat case* the second evolution (31) is now in the hierarchy of (30) with respect to the new pair, as is already known (see [LP91]).

Proposition 2. *$\{\mathcal{B}, \mathcal{C}, \mathcal{F}, \mathcal{G}\}$ form a Hamiltonian quadruplet.*

Proof. The proof of this proposition is a straightforward calculation of the kind performed in Theorem 3. □

Notice that the classical filament flow equation (30) is identically induced for both flat and nonflat cases, since $\mathcal{H}_0$ is in the kernel of $\mathcal{C}$. We consider the Hamiltonian pair $(\mathcal{E}+\mathcal{D}, \mathcal{C})$, and indeed $(\mathcal{G}-\mathcal{F}, \mathcal{C})$, as integrating it. The second evolution in the hierarchy would *not* be (31) for the nonflat case, but rather its flat analogue.

The Hodographic Transformation

We describe first the transformation, used in [Ive01], that affects the second integrable group of equations. We will see later the effect on the associated Hamiltonian pair.

The first step in this transformation is to define $\kappa = p_1$ and $\tau = q_1$, so that both systems (34) and (35) become

$$\begin{cases} p_{t_1} = q_1, \\ q_{t_1} = \frac{q_1^2}{p_1} - \varkappa \frac{1}{p_1}, \end{cases} \tag{38}$$

and

$$\begin{cases} p_{t_3} = \frac{p_3}{p_1^3} - 3\frac{p_2^2}{p_1^4} - \frac{3}{2}\frac{q_1^2}{p_1^2} - \varkappa \frac{1}{2p_1^2}, \\ q_{t_3} = \frac{q_3}{p_1^3} - 3\frac{p_2 q_2}{p_1^4} - \frac{q_1^3}{p_1^3} + \varkappa \frac{q_1}{p_1^3}. \end{cases} \tag{39}$$

We now define the *hodograph* transformation (cf. [Olv93]) by

$$u = x, \qquad v = q, \qquad y = p.$$

Later on we refer to the composition of taking the potential and the hodograph transformation simply as the hodographic transformation. If the old independent variables are (x, t), we call the new $(y, t^{'})$ and we find that

$$\begin{aligned} 1 &= \frac{\partial x}{\partial x} = \frac{\partial u}{\partial y}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial t'}\frac{\partial t}{\partial x} = \frac{\partial u}{\partial y}\frac{\partial p}{\partial x} = u_1 p_1, \\ 0 &= \frac{\partial x}{\partial t} = \frac{\partial u}{\partial y}\frac{\partial p}{\partial t} + \frac{\partial u}{\partial t'}\frac{\partial t}{\partial t} = \frac{\partial u}{\partial y}\frac{\partial p}{\partial t} + \frac{\partial u}{\partial t'} = u_1 p_t + u_{t'}, \\ q_1 &= \frac{\partial q}{\partial x} = \frac{\partial v}{\partial y}\frac{\partial p}{\partial x} + \frac{\partial v}{\partial t'}\frac{\partial t}{\partial x} = \frac{\partial v}{\partial y}\frac{\partial p}{\partial x} = v_1 p_1, \\ q_t &= \frac{\partial q}{\partial t} = \frac{\partial v}{\partial y}\frac{\partial p}{\partial t} + \frac{\partial v}{\partial t'}\frac{\partial t}{\partial t} = v_1 p_t + v_{t'}. \end{aligned}$$

This implies

$$\begin{aligned} u_1 &= \frac{1}{p_1}, \qquad u_2 = \frac{\partial u_1}{\partial x}\frac{\partial x}{\partial y} = -\frac{p_2}{p_1^2}u_1 = -\frac{p_2}{p_1^3}, \\ u_3 &= \frac{\partial u_{yy}}{\partial x}\frac{\partial x}{\partial y} = -u_1\frac{\partial}{\partial x}\frac{p_2}{p_1^3} = -\frac{p_3}{p_1^4} + 3\frac{p_2^2}{p_1^5}, \end{aligned}$$

$$u_{t'} = -u_1 p_t = -\frac{p_t}{p_1}, \qquad v_1 = \frac{q_1}{p_1},$$
$$v_{t'} = q_t - v_1 p_t = q_t - \frac{q_1}{p_1} p_t.$$

We can now rewrite the equations as

$$\begin{cases} u_{t_1'} = -v_1, \\ v_{t_1'} = -\varkappa u_1, \end{cases} \tag{40}$$

and

$$\begin{cases} u_{t_3'} = u_3 + \frac{3}{2} u_1 v_1^2 + \frac{1}{2}\varkappa u_1^3, \\ v_{t_3'} = v_3 + \frac{1}{2} v_1^3 + \frac{3}{2}\varkappa v_1 u_1^2. \end{cases} \tag{41}$$

We show later that the second equation is in the hierarchy of the first, also in the nonflat case!

Let us look at equation (41) a little bit closer. Assume $\varkappa \neq 0$ and let $\bar{u} = u$ and $\bar{v} = \frac{1}{\sqrt{\varkappa}} v$. Then

$$\begin{cases} \bar{u}_{t_3'} = \bar{u}_3 + \frac{3}{2}\varkappa \bar{u}_1 \bar{v}_1^2 + \frac{1}{2}\varkappa \bar{u}_1^3 \\ \bar{v}_{t_3'} = \bar{v}_3 + \frac{1}{2}\varkappa \bar{v}_1^3 + \frac{3}{2}\varkappa \bar{u}_1^2 \bar{v}_1 \end{cases}.$$

Let $X = \bar{u} + \bar{v}$ and $Y = \bar{u} - \bar{v}$. Then

$$\begin{cases} X_{t_3'} = X_3 + \frac{1}{2}\varkappa X_1^3 \\ Y_{t_3'} = Y_3 + \frac{1}{2}\varkappa Y_1^3 \end{cases}, \tag{42}$$

which is a decoupled set of potential modified KdV equations. This simplification only occurs in the nonflat case, while the flat case cannot be decoupled. In this sense, the flat case is a singular case in the family.

Notice that if we take $\tau = 0$, the 2-dimensional case, system (35) reduces to

$$\kappa_{t_3} = \frac{\kappa_3}{\kappa^3} - 9\frac{\kappa_1 \kappa_2}{\kappa^4} + 12\frac{\kappa_1^3}{\kappa^5} + \varkappa\frac{\kappa_1}{\kappa^3}, \tag{43}$$

the generalization of the evolution found in [Ive01] to nonflat Riemannian manifolds with constant curvature. Using the hodographic transformation, this equation would become

$$u_{t_3'} = u_3 + \frac{1}{2}\varkappa u_1^3. \tag{44}$$

Next we describe the transformation as applied to the Hamiltonian pairs. The matrices

$$Q = \begin{pmatrix} D_x^{-1} & 0 \\ 0 & D_x^{-1} \end{pmatrix},$$

and

$$R = \begin{pmatrix} -\frac{1}{p_1} & 0 \\ -\frac{q_1}{p_1} & 1 \end{pmatrix},$$

describe the action of the transformations on the vector fields

$$\begin{pmatrix} \kappa_t \\ \tau_t \end{pmatrix}, \qquad \begin{pmatrix} p_t \\ q_t \end{pmatrix},$$

respectively, the Hamiltonian operator, say $\mathcal{D}$, transforms as

$$\mathcal{D} \mapsto Q\mathcal{D}Q^* \mapsto RQ\mathcal{D}Q^*R^*.$$

We remark that $D_x = \frac{1}{u_1} D_y$, as can easily be checked from the definitions of the second transformation. We find that

$$\bar{\mathcal{B}} = RQ\mathcal{B}Q^*R^* \left|\frac{\delta x}{\delta y}\right| = \begin{pmatrix} u_1 D_y^{-1} v_1 + v_1 D_y^{-1} u_1 & v_1 D_y^{-1} v_1 \\ v_1 D_y^{-1} v_1 & 0 \end{pmatrix},$$

$$\bar{\mathcal{C}} = RQ\mathcal{C}Q^*R^* \left|\frac{\delta x}{\delta y}\right| = \begin{pmatrix} 0 & u_1 D_y^{-1} u_1 \\ u_1 D_y^{-1} u_1 & v_1 D_y^{-1} u_1 + u_1 D_y^{-1} v_1 \end{pmatrix},$$

$$\bar{\mathcal{D}} = RQ\mathcal{D}Q^*R^* \left|\frac{\delta x}{\delta y}\right| = \begin{pmatrix} 0 & D_y^{-1} \\ D_y^{-1} & 0 \end{pmatrix},$$

and

$$\bar{\mathcal{E}} = RQ\mathcal{E}Q^*R^* \left|\frac{\delta x}{\delta y}\right| = \begin{pmatrix} 0 & D_y \\ D_y & 0 \end{pmatrix}.$$

The hodographic transformation is a Poisson transformation between the quadruplet $\{\mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}\}$ and the new quadruplet $\{\bar{\mathcal{B}}, \bar{\mathcal{C}}, \bar{\mathcal{D}}, \bar{\mathcal{E}}\}$. The following corollary is thus obvious.

Corollary 2. *The operators $\bar{\mathcal{B}}, \bar{\mathcal{C}}, \bar{\mathcal{D}}, \bar{\mathcal{E}}$ form a Hamiltonian quadruplet. Equation (41) is bi-Hamiltonian with respect to the Hamiltonian pair $(\bar{\mathcal{B}} + \varkappa\bar{\mathcal{C}} + \bar{\mathcal{E}}, \bar{\mathcal{D}})$, with associated Hamiltonian functionals*

$$\bar{\mathcal{H}}_2 = -\int u_1 v_1 \, dy \text{ and } \bar{\mathcal{H}}_3 = \int \left(u_2 v_2 - \frac{1}{2} v_1^3 u_1 - \frac{1}{2} \varkappa v_1 u_1^3 \right) dy,$$

respectively.

Note: Strictly speaking, the definition of Poisson brackets given in Section 2.2 does not include these tensors, since their entries are not defined in terms of differential operators. Nevertheless, they are if we allow nonlocal terms and the formalization of this kind of Poisson geometry.

Our final theorem states that the hodographic transformation, followed by the decoupling map, is in fact a Poisson map between the space endowed with the Poisson bracket defined by $\mathcal{E} + \mathcal{B} + \varkappa\mathcal{C}$ and the space endowed with a decoupled pair of the known potential modified KdV Hamiltonian structure. We are not entering here into the details of how these spaces should actually be defined, to avoid further complications.

Theorem 4. *Consider the decoupling transformation* $\sigma(y, u, v) = (y, X, Y)$, *where* X *and* Y *are defined as in (42). Denote by* H *the hodographic transformation. Then, if* $\varkappa \neq 0$, H *followed by* σ *is a Poisson map taking* $\mathcal{E} + \mathcal{B} + \varkappa\mathcal{C}$ *to a decoupled pair of modified potential KdV Hamiltonian structures*

$$\hat{\mathcal{B}} + \varkappa\hat{\mathcal{C}} + \hat{\mathcal{E}} = \frac{2}{\sqrt{\varkappa}} \begin{pmatrix} X_1 D_y^{-1} X_1 + \varkappa D_y & 0 \\ 0 & -Y_1 D_y^{-1} Y_1 - \varkappa D_y \end{pmatrix}. \tag{45}$$

Proof. It is a straightforward calculation to show that

$$\sigma^* H^*(\mathcal{B} + \varkappa\mathcal{C} + \mathcal{E}) = \frac{2}{\sqrt{\varkappa}} \begin{pmatrix} X_1 D_y^{-1} X_1 + \varkappa D_y & 0 \\ 0 & -Y_1 D_y^{-1} Y_1 - \varkappa D_y \end{pmatrix},$$

by conjugation, as usual (see (36)). The tensor $X_1 D_y^{-1} X_1 + \varkappa D_y$ is well-known to be a Hamiltonian structure for the potential KdV. Potential modified KdV equation is obtained with the particular choice of Hamiltonian $\mathcal{H} = -\frac{\sqrt{\varkappa}}{4} \int X_1^2 dy$. □

The new recursion operator, which $\mathfrak{R}_2$ transforms to, is

$$\begin{aligned} \bar{\mathfrak{R}}_2 \; = \; & \begin{pmatrix} u_1 D_y^{-1} v_1 + v_1 D_y^{-1} u_1 & v_1 D_y^{-1} v_1 + D_y \\ v_1 D_y^{-1} v_1 + D_y & 0 \end{pmatrix} \begin{pmatrix} 0 & D_y \\ D_y & 0 \end{pmatrix} \\ & + \varkappa \begin{pmatrix} 0 & u_1 D_y^{-1} u_1 \\ u_1 D_y^{-1} u_1 & v_1 D_y^{-1} u_1 + u_1 D_y^{-1} v_1 \end{pmatrix} \begin{pmatrix} 0 & D_y \\ D_y & 0 \end{pmatrix} \\ = \; & \begin{pmatrix} {D_y}^2 + v_1^2 + \varkappa u_1^2 & 2u_1 v_1 \\ 2\varkappa u_1 v_1 & {D_y}^2 + v_1^2 + \varkappa u_1^2 \end{pmatrix} \\ & - \mathfrak{N} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} D_y^{-1} (v_2 \quad u_2) - \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} D_y^{-1} (v_2 \quad u_2) \mathfrak{N}, \end{aligned}$$

where

$$\mathfrak{N} = \begin{pmatrix} 0 & 1 \\ \varkappa & 0 \end{pmatrix}.$$

Applying $\bar{\mathfrak{R}}_2$ to

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix},$$

we obtain

$$\begin{pmatrix} u_3 + \frac{3}{2} u_1 v_1^2 + \frac{1}{2} \varkappa u_1^3 \\ v_3 + \frac{1}{2} v_1^3 + \frac{3}{2} \varkappa u_1^2 v_1 \end{pmatrix},$$

so now this third-order equation is in the image of $\bar{\mathfrak{R}}_2$. This was not the case for the corresponding equation (35). The present coordinates give us a clearer idea of the interrelation of the two integrable systems (34) and (35) as finally belonging to the same hierarchy.

We notice that apparently we can multiply a symmetry with $\mathfrak{N}$ and obtain a new symmetry. This is because $\mathfrak{N}$ and $\bar{\mathfrak{R}}_2$ commute. Thus we can consider $\mathfrak{N}$ as another recursion operator. It is easy to check that it is hereditary. The complete hierarchy is generated by applying powers of $\bar{\mathfrak{R}}_2$ to the trivial symmetry

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix},$$

and then adding the image of these under $\mathfrak{N}$ to this collection.

5. Conclusions

In this paper we associate a quadruplet of compatible Hamiltonian structures to the intrinsic geometry of curves in 3-dimensional Riemannian manifolds. From Theorems 2 and 3 we can conclude the following interesting fact: If a flow of curves evolves following an arclength-preserving evolution of the form (7) with (h_3, h_1) being the gradient of a certain functional $\mathcal{H}$ with respect to κ and τ, then the associated flow of the curvatures is Hamiltonian with respect to the Hamiltonian structure (9) presented here, with Hamiltonian functional $\mathcal{H}$. The condition of being a gradient amounts to the Fréchet derivative of (h_3, h_1) being self-adjoint, and it is a rather natural condition one needs to impose to have a Hamiltonian. Furthermore, even if no Hamiltonian functional is associated to (h_3, h_1), the flow of the curvature evolution (8) will still lie on the Poisson leaves corresponding to the Poisson structure defined by (9), so that any possible Casimir element would be constant along the flow. This would be true for *any* arclength-preserving evolution of curvatures induced by geometric evolutions of Riemannian curves.

If in addition to (h_3, h_1) representing the gradient of a Hamiltonian functional, the associated (κ, τ) evolution is also Hamiltonian with respect to either (28) or (29), then the PDE evolution would be completely integrable. We presented several examples of such evolutions, namely the well-known vortex filament equations and the system of equations announced in [Ive01]. We unified the study of the vortex filament flow equations on Riemannian manifolds with constant curvature and we studied the geometry of the second group of integrable systems in detail, showing that, in the nonflat case, it is Poisson equivalent to a system of decoupled potential KdV equations. To our knowledge, a complete classification of integrable systems associated to this Hamiltonian quadruplet has not yet been found and it is an interesting problem in itself. Ivey sets out to classify integrable arclength-preserving geometric evolution equations for the planar Euclidean case in [Ive01].

The Poisson geometry of infinite-dimensional Poisson brackets is largely unexplored, except for some special examples. The implications of the geometrical relationship presented here for the Poisson geometry of the associated brackets is a problem which has not been studied and that could have very important consequences. The presence of the projective group is essential in the understanding of the Poisson leaves of the Adler–Gel'fand–Dikiĭ bracket, and so one could expect a similar role in the case of the Euclidean group as related to these Riemannian manifolds. The geometrical reasons that allow the appearance of these tensors are not at all understood—not only which geometrical facts produce a skew-symmetric tensor but, more interestingly, which geometrical properties allow one to obtain the Jacobi identity. The resolution of this question would be of the highest interest, and it might link these Hamiltonian evolutions to evolutions of the Drinfel'd and Sokolov type (cf. [DS84]). The study of other homogeneous spaces is still open. Some cases are currently under consideration by the authors.

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