Klein Geometry

Homogeneous space (M, G): M — manifold, G — Lie group, $G \curvearrowright M$ transitively ($\forall x,y \in M,\ \exists g \in G \mid gx = y$).
$o \in M$, $G_o = \{g \in G \mid g \cdot o = o\}$.
$G/G_o \cong M$ (G-invariant data).
$gG_o \mapsto g \cdot o$.
G a group, H is a closed subgroup, G/H is a homogeneous space.

Klein Geometry (G, H): H is a closed subgroup of G and G/H is connected.

$(G,H) \xrightarrow{\text{Lie}} (\mathfrak{g}, \mathfrak{h})$

SES: $0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$ (as H-modules)

$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ (as H-modules)
$\mathfrak{m} \cong \mathfrak{g}/\mathfrak{h}$

Con	Ref	Symm
$[\mathfrak{h},\mathfrak{h}]$	$\mathfrak{h}$	$\mathfrak{h}$
$[\mathfrak{h},\mathfrak{m}]$	$\mathfrak{m}$	$\mathfrak{m}$
$[\mathfrak{m},\mathfrak{m}]$	$\mathfrak{h} \oplus \mathfrak{m}$	$\mathfrak{h}$

$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$

A $\dim G \circlearrowright M$ ((M,G) homogeneous)
$(L_g)_*: T_oM \to T_{go}M$
If we choose $o \in M$, let $H = G_o$. Then for $h \in H$,
$(L_h)_* \in GL(T_oM)$
$h \mapsto (L_h)_*$ is a rep of H on T_oM.
"linear isotropy rep of (M,G,o)"
get iso's of H-reps
$M \cong G/H$, so
$T_oM \to T_{[e]}(G/H) \xrightarrow{\sim} \mathfrak{g}/\mathfrak{h} \cong \mathfrak{m}$

$\left[0 \to T_eH \xrightarrow{(d\iota)_e} T_eG \xrightarrow{(d\pi)_e} T_o(G/H) \to 0 \right]$
$\mathfrak{h} \qquad \mathfrak{g} \qquad \Rightarrow \mathfrak{g}/\mathfrak{h} \cong T_o(G/H)$

$T_oM \cong \mathfrak{g}/\mathfrak{h} \cong \mathfrak{m}$

Linear structures on $\mathfrak{g}/\mathfrak{h}$ induce geometric structures on M.

$$\left\{\begin{array}{c}\text{H-invariant}\\ \text{tensors of}\\ \mathfrak{g}/\mathfrak{h} \cong T_oM\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{G-invariant}\\ \text{tensor fields}\\ \text{on}\\ M\end{array}\right\}$$
$$\tau \longmapsto T_o = \tau,\quad T_{g\cdot o} = (L_g)_* \tau$$

$g \cdot o = g \cdot o$
$\Rightarrow g^{-1}g \cdot o = o$
$\Rightarrow g^{-1}g \in H$
$\Rightarrow$ By H-invariance of τ, T is well-defined.

T_o $\overset{\text{contract}}{\underset{\text{at } o}{\cdot}}$ T

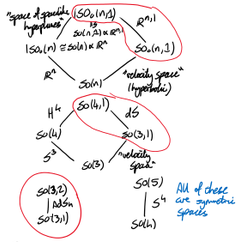
In particular, a pseudo inner product on $\mathfrak{g}/\mathfrak{h}$ gives rise to a pseudo-Riemannian metric on M.

A **metric Klein geometry** (G, H, γ)
- (G,H) is a Klein geometry
- γ is an pseudo inner product on $\mathfrak{g}/\mathfrak{h}$ which is H-invariant.

Examples

$G = ISO_+(\mathbb{R}^{d-1,1}) \cong SO(d-1,1) \ltimes \mathbb{R}^{d-1,1}$
$H = SO_+(d-1,1)$
$G/H \cong \mathbb{R}^{d-1,1}$
$\mathfrak{g}/\mathfrak{h} \cong \mathbb{R}^{d-1,1}$

$G = ISO_+(\mathbb{R}^d) = SO(d) \ltimes \mathbb{R}^d$
$H = SO(d)$
$G/H \cong \mathbb{R}^d$



A **Cartan Geometry** $(\pi: P \to M, A)$ modelled on a Klein geometry (G,H) is a principal right H-bundle $\pi: P \to M$ equipped with the Cartan connection $A \in \Omega^1(P, \mathfrak{g})$ satisfying
a) $A_p: T_pP \to \mathfrak{g}$ is a linear isomorphism $\forall p \in P$
1) $(R_h)^* A = \mathrm{Ad}_{h^{-1}} \circ A \quad \forall h \in H$
2) $A(\tilde{S}_X) = X$ for $X \in \mathfrak{h}$ $\left(\text{Hop defn } \tilde{S}_X \text{ for } X \in \mathfrak{h} \right)$
$\left(\tilde{S}_X \right)_p = \frac{d}{dt}\left(p \cdot \exp(tX) \right)|_{t=0}$

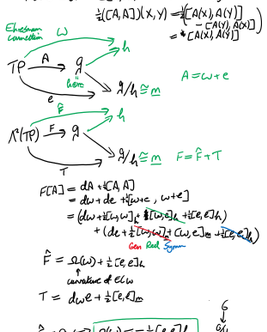
Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ for some vector space complement $\mathfrak{m}$ of $\mathfrak{h}$ (not necessarily an $\mathfrak{h}$-submodule).
Then let $p_1: \mathfrak{g} \to \mathfrak{h}$ be projection wrt the split.
When is $\omega := p_1 \circ A \in \Omega^1(P, \mathfrak{h})$ an Ehresmann connection & a Cartan connection?
$\omega(\tilde{S}_X) = p_1(A(\tilde{S}_X)) = p_1(X) = X \quad \forall X \in \mathfrak{h}$ ✓
$\omega(R_h^* \tilde{S}) = p_1(A(R_h^* \tilde{S}))$
$= p_1(\mathrm{Ad}_{h^{-1}} \circ A(\tilde{S}))$
$= p_1(\mathrm{Ad}_{h^{-1}} \circ \omega(\tilde{S}) + \mathrm{Ad}_{h^{-1}} \circ e(\tilde{S}))$ ($e = p_2 \circ A$)
$= \mathrm{Ad}_{h^{-1}} \circ \omega(\tilde{S})$ ($e(\tilde{S}) \in \mathfrak{m}$)
$\Rightarrow$ ω is Ehresmann $\iff \mathrm{Ad}_{h^{-1}} e(\tilde{S}) \in \mathfrak{m}$ $\forall h \in H$
$\iff \forall \xi \in T(P)$ s.t. ξ is parallel wrt $\mathfrak{h}$, $e(T\xi) \equiv 0$ is preserved by H
But by (2) of def of A, e is algebraic, so
ω is Ehresmann $\iff \mathfrak{m}$ is preserved by H
$\iff \mathfrak{g}/\mathfrak{h} \cong \mathfrak{m}$ is a reductive split.

Observations
$\dim P = \dim G$
$\dim M = \dim G/H$
$(A_p)^{-1}: \mathfrak{g} \to T_pP$
$(\tilde{A}_p)^{-1}: \mathfrak{g} \to \mathfrak{X}(P)$
$(A_*)^{-1}: \mathfrak{h} \to \mathfrak{X}_{\mathrm{vert}}(P)$
$X \mapsto \tilde{S}_X$

If G/H is metric Klein, M inherits a metric of same sig.
$T_xM \cong T_pP / \ker \pi_{*p} \cong \mathfrak{g}/\mathfrak{h} \cong \mathfrak{m}$?
$\pi(p) = x$

$TM \simeq P \times_H \mathfrak{g}/\mathfrak{h}$

Curvature of a cartan geometry
$F[A] = dA + \frac{1}{2}[A,A] \qquad F \in \Omega^2(P; \mathfrak{g})$
$\frac{1}{2}([A,A])(X,Y) = \frac{1}{2}[A(X), A(Y)] - [A(Y), A(X)] = [A(X), A(Y)]$



$F[A] = dA + \frac{1}{2}[A,A]$
$= d\omega + de + \frac{1}{2}[\omega + e, \omega + e]$
$= (d\omega + \frac{1}{2}[\omega,\omega] + \frac{1}{2}[e,e]_{\mathfrak{h}} + [e,e]_{\mathfrak{h}}) + (de + \frac{1}{2}[\omega,e] + \frac{1}{2}[e,\omega] + [e,e]_{\mathfrak{m}})$
Con Ref Symm

$\hat{F} = \Omega(\omega) + \frac{1}{2}[e,e]_{\mathfrak{h}}$ (curvature of ω)
$T = d_\omega e + \frac{1}{2}[e,e]_{\mathfrak{m}}$

$\hat{F} = 0 \iff \boxed{\Omega(\omega) = -\frac{1}{2}[e,e]_{\mathfrak{h}}}$
$T = 0 \iff \boxed{d_\omega e = -\frac{1}{2}[e,e]_{\mathfrak{m}}}$

$G \to G/H$