2a - Geometric structures from linear ones

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If $\rho : H \to GL(V)$ is a rep of H, we can define

$$\underset{\swarrow \text{ H-invariants in } V}{V^H} \longrightarrow \Gamma(P \times_\rho V \to M)$$
$$v \longmapsto \sigma_v$$
$$(\sigma_x)_m := [p, v] \quad \text{for } \pi(p) = m$$

Well-definition: $[ph, v] = [p, \rho(h)v]$
$= [p, v]$.

In fact, we can see this from the perspective of the familiar isomorphism

$$\Omega^k_H(P; V) \cong \Omega^k(M; P \times_\rho V)$$

In particular $\underset{(k=0)}{}$, $C^\infty_H(P; V) \longrightarrow \Gamma(P \times_\rho V)$
$$f \longmapsto \sigma_f$$

$$(\sigma_f)_x := [p; f(p)] \quad (\text{any } p \in \pi^{-1}(x))$$
$$= [p; h f(ph)]$$
$$= [ph; f(ph)]$$

For $v \in V^H$, define $f_v(p) = v \ \forall p \in P$.
Clearly $f_v \in C^\infty_H(P; V)$.

So we get $\quad V^H \longrightarrow C^\infty_H(P; V) \longrightarrow \Gamma(P \times_\rho V)$
$$v \longmapsto f_v \longmapsto \sigma_{f_v} = \sigma_v$$

$$(\sigma_{f_v})_x = [p, f(p)]$$
$$= [p, v]$$
$$= \sigma_v$$

In particular, in a Cartan geometry $\pi : P \to M$ modelled on (G, H), H-invariant tensors $\mathfrak{g}/\mathfrak{h}$ give us tensor fields on M:

$$(\otimes \mathfrak{g}/\mathfrak{h})^H \longrightarrow C^\infty_H(P; \otimes \mathfrak{g}/\mathfrak{h}) \xrightarrow{\sim} \Gamma(P \times_G \otimes \mathfrak{g}/\mathfrak{h}) \xrightarrow{\sim} \Gamma(\otimes TM)$$
$$(\otimes = \otimes^{r,s}, \Lambda^p, \odot^p, \text{etc.})$$

The first map is not an isomorphism since its image is only the constant H-invariant functions.

Clearly, this corresponds to "constant sections" of $P \times_G \otimes \mathfrak{g}/\mathfrak{h} \cong \otimes TM$ — Constant in what sense?

This is the analogue in Cartan geometry of the G-invariant tensor fields on G/H in Klein geometry.

Example: Tensor fields on M from H-invariants in $\mathfrak{g}/\mathfrak{h}$

For clarity, let $e \in \Omega^1_H(P; \mathfrak{g}/\mathfrak{h})$ be the frame field and $\tilde{e} : TM \xrightarrow{\sim} P \times_\rho \mathfrak{g}/\mathfrak{h}$ the corresponding bundle isomorphism.

Tensor fields on M from H-invariants in $\mathfrak{g}/\mathfrak{h}$

and $\tilde{e} : TM \longrightarrow P \times_H \mathfrak{g}/\mathfrak{h}$ the corresponding bundle isomorphism.

Then $\tilde{e}(\xi_x) = [p, e(\bar{\xi})]$

where $\xi \in T_xM$, $p \in \pi^{-1}(x)$ and $\bar{\xi} \in T_pP$ is any lift of ξ.

Get unique vector field $\xi_{\bar{X}} \in \mathfrak{X}(M)$ for $\bar{X} \in \mathfrak{g}/\mathfrak{h}$ defined by $\xi_{\bar{X}} := \tilde{e}^{-1} \circ \sigma_{\bar{X}}$; that is, $(\xi_{\bar{X}})_x = \tilde{e}^{-1}([p, \bar{X}])$ for any lift $p \in \pi^{-1}(x)$.

Note that for the spacetime models, $\mathfrak{h} = \mathfrak{so}(3,1)$, $\mathfrak{g}/\mathfrak{h} \cong \mathbb{R}^{3,1}$; and there are no nontrivial invariants.

Example:

Metric on M from H-inv^t IP on $\mathfrak{g}/\mathfrak{h}$

lifts of $\xi_1 X$ to P

$$
\begin{aligned}
g(\xi_1, \xi_2) &= \eta(e(\bar{\xi}_1), e(\bar{\xi}_2)) \\
&= \eta(h e(\bar{\xi}_1), h e(\bar{\xi}_2)) \qquad (H\text{-invariance of } \eta) \\
&= \eta(e(R_{h^{-1}*} \bar{\xi}_1), e(R_{h^{-1}*} \bar{\xi}_2))
\end{aligned}
$$

Equivalently,
$$
\begin{aligned}
&\odot^2(\mathfrak{g}/\mathfrak{h})^* \hookrightarrow \Gamma(P \times_H \odot^2(\mathfrak{g}/\mathfrak{h})^*) \longrightarrow \Gamma(\odot^2 T^*M) \\
&\eta \longmapsto \sigma_\eta\,(\pi(p) \longmapsto [p, \eta]) \xrightarrow{e^*} e^*\sigma_\eta\,((\xi_1, \xi_2) \mapsto \sigma_\eta(e(\xi_1), e(\xi_2)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad = \eta(e(\bar{\xi}_1, \bar{\xi}_2)))
\end{aligned}
$$