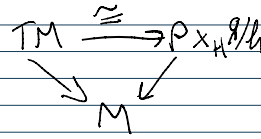


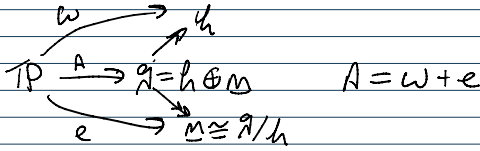
Recap

Saw last week: A Cartan geometry modelled on reductive  $(G, H)$   
 $\downarrow \pi$   
 $M$ ,  $A \in \Omega^1(P; \mathfrak{g})$

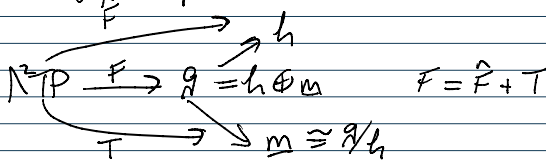
Saw that  $\{H\text{-inv}^t \text{ tensors on } \mathfrak{g}/\mathfrak{h}\} \cong \{G\text{-inv}^t \text{ tensor fields on } G/H\}$   
 Can we something similar for Cartan geometries?



gives rise to an Ehresmann connection  $\omega \in \Omega^1(P; \mathfrak{h})$   
 frame field  $e \in \Omega^1(P; \mathfrak{g}/\mathfrak{h})$



We also get a split of the curvature



and we found

$\hat{F}[A] = \Omega[\omega] + \frac{1}{2}[e, e]_{\mathfrak{h}}$   
 $T = d\omega e + \frac{1}{2}[e, e]_{\mathfrak{m}}$   
*so in symmetric case*

We're particularly interested in the following models

$AdS_4 \cong so(3,2)/so(3,1) \quad \mathfrak{g} = so(4,1)$

$\mathbb{R}^{3,1} \cong so(3,1)/so(3,1) \quad \mathfrak{g} = lso(4,1)$

$dS_4 \cong so(4,1)/so(3,1) \quad \mathfrak{g} = so(3,2)$

$\mathfrak{h} = so(3,1)$   
 in any case

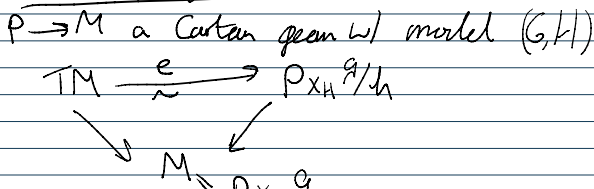
Can parametrise in unified way

Always reductive, symmetric  $\longrightarrow \begin{cases} \hat{F} = \Omega[\omega] + \frac{1}{2}[e, e]_{\mathfrak{h}} \\ T = d\omega e \end{cases}$  for  $G/H$  Cartan groups  
 Always get a decomp  $\mathfrak{g} = so(3,1) \oplus \mathfrak{p}$   
 $\mathfrak{p} \cong \mathfrak{g}/so(3,1) \cong \mathbb{R}^{3,1}$

Only  $[\mathfrak{p}, \mathfrak{p}]$  differs in the 3 models

- $\Rightarrow$  equip  $\mathfrak{p}$  with  $\mathcal{M}$ , which is  $so(3,1)$ -equiv<sup>t</sup>
- $\Rightarrow$  Get metric  $g$  on  $G/H$ , which is  $G$ -equiv<sup>t</sup>

The Fake Tangent Bundle



$X_p \mapsto [p, A(X_p)] = [p \cdot h, Ad(h^{-1})A(X_p)]$   
 $= [p \cdot h, (A \circ A^{-1})(X_p)]$   
 $TP \cong P \times_H \mathfrak{g} = [p \cdot h, A(X_p \cdot h)]$   
 if  $X$  is right-invariant

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$M = P \times_H \mathbb{R}^n$   
 Proof  $e \in \Omega_G(P; \mathbb{R}^n) \cong \Omega^1(M; P \times_H \mathbb{R}^n)$   
 $\Rightarrow e$  is a bundle map  $TM \rightarrow P \times_H \mathbb{R}^n$   
 (3) Iso on fibres.  $\square$

(1)  $(R_h^* e)(\xi) = e(R_h^* X)$   
 $\stackrel{\text{by def of } H \circ \mathbb{R}^n}{=} \frac{A(R_h^* X)}{R_h^* A(X)} = \text{Ad}(h^{-1}) \cdot A(X) = h^{-1} \cdot A(X) = h^{-1} \cdot e(X)$   
 $\Rightarrow e$  is  $H$ -equivariant  
 For  $X \in \mathfrak{h}$ ,  $e(\xi_X) = A(\xi_X) = \bar{X} = 0 \in \mathbb{R}^n$   
 $\Rightarrow e$  is horizontal

(2)  $e(\xi_m) = [S_x(m), e(S_x^* \xi_m)]$   
 $= [S_p(m) \cdot g_p(m)^{-1}, e(S_x^* \xi_m)]$   
 $= [S_p(m), g_p(m)^* \cdot e(S_x^* \xi_m)]$   
 $= [S_p(m), R_{g_p(m)}^* \cdot e(S_x^* \xi_m)]$   
 $= [S_p(m), e((R_{g_p(m)} \circ S_x)^* \xi_m)]$   
 $= [S_p(m), e(S_p^* \xi_m)]$   
*Not dependent on choice of local trivial.*

We call  $\mathcal{T} := P \times_H \mathbb{R}^n$  the fake tangent bundle  
 From this, we find that any  $H$ -inv't tensor on  $\mathbb{R}^n$  induces a tensor field on  $M$ !  
 eg  $\bar{X} \in \mathbb{R}^n \mapsto \xi_{\bar{X}} \in \mathcal{H}(M)$   
 $\xi_{\bar{X}} = e^{-1}(X) = (\pi^* A^{-1})(X)$

$\langle \cdot, \cdot \rangle$  on  $\mathbb{R}^n \mapsto$  metric  $g$  on  $M$ :  
 $g(\xi_1, \xi_2) = \langle e(\xi_1), e(\xi_2) \rangle$   
 Or in components,  $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$   
 Choose (orthonormal) basis  $P_a$  for  $\mathbb{R}^n$  and coords  $x^\mu$  for  $M$ .

Now in the reductive case consider the covariant derivative w.r.t. Ehresmann connection  $\omega \in \Omega^1(P; \mathfrak{g}/\mathfrak{h})$

$\forall X \in \mathcal{H}(M)$   $d\omega := d\omega + \mathcal{L}_X(\omega) \cdot B$  for  $B \in \Omega^1_G(P; \mathfrak{g}/\mathfrak{h}) \cong \Omega^1(M; \mathfrak{g}/\mathfrak{h})$   
 $D_X B := d\omega_X + \mathcal{L}_X(\omega_X) \cdot B$   
 In comp  $D_\mu B^a := \partial_\mu B^a + \omega^a_{\mu b} B^b$   
 $\mathcal{L}_X \omega := \omega_X(X) \cdot B$   
 $\omega_X := s^* \omega$   
 $B_X := s^* B$   
 for  $s$  local sections

For  $\xi \in \mathcal{H}(M)$ ,  $e(\xi) \in \Gamma(T \rightarrow M) \cong \Omega^1_G(P; \mathbb{R}^n)$   
 $(\omega_X dB + X \cdot B)(\xi_1, \dots, \xi_k)$   
 $\stackrel{\text{by def}}{=} \omega_X \cdot B + X \cdot B$

$\nabla \xi := e^{-1} d\omega e(\xi)$  i.e.  $\nabla_X \xi = e^{-1} D_X e(\xi)$

defines a Koszul connection on  $TM$ .

In comp  $\nabla_\mu \xi^\nu = e^{\nu a} D_\mu (e^a_\rho \xi^\rho)$   
 $= e^{\nu a} \partial_\mu (e^a_\rho \xi^\rho) + \omega^a_{\mu b} e^b_\rho \xi^\rho$   
 $= \partial_\mu \xi^\nu + e^{\nu a} (\partial_\mu e^a_\rho + \omega^a_{\mu b} e^b_\rho) \xi^\rho$   
 or  $\nabla_\mu \xi^\nu = \partial_\mu \xi^\nu + \Gamma^{\nu}_{\mu\rho} \xi^\rho$   
 $\Gamma^{\nu}_{\mu\rho} := e^{\nu a} (\partial_\mu e^a_\rho + \omega^a_{\mu b} e^b_\rho)$   
 $= e^{\nu a} D_\mu e^a_\rho$

$e^a_\mu e^b_\mu = \text{Id}_{TM}$   $e \cdot e^{-1} = \text{Id}_{P \times_H \mathbb{R}^n}$   
 $\Leftrightarrow (e^{-1})^a_\mu e^\mu = \delta^a_b$   $e^a_\mu (e^{-1})^\mu = \delta^a_b$   
 $\frac{\partial}{\partial x^\mu}$   $\frac{\partial}{\partial b}$

$d\omega e = 0 \Leftrightarrow D_\mu e^a_\nu = 0 \Leftrightarrow \Gamma^{\lambda}_{\mu\nu} = 0$   
 $\Leftrightarrow \partial_\mu e^a_\nu + \omega^a_{\mu b} e^b_\nu = 0$

$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$

$\Rightarrow T^{\lambda}_{\mu\nu} = 2\Gamma^{\lambda}_{[\mu\nu]}$

$(\nabla_X g)(X, Y) = \nabla_X (g(X, Y)) - g(\nabla_X X, Y) - g(X, \nabla_X Y)$   
 $= 2(X, Y) -$   
 $\nabla_X g_{\mu\nu} = \partial_X g_{\mu\nu} - \Gamma^{\lambda}_{X\mu} g_{\lambda\nu} - \Gamma^{\lambda}_{X\nu} g_{\mu\lambda}$

$$= Z(X, Y) -$$

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\alpha_{\lambda\mu} g_{\alpha\nu} - \Gamma^\alpha_{\lambda\nu} g_{\mu\alpha}$$

$$\Gamma^\alpha_{\lambda\mu} g_{\alpha\nu} = (e^c \partial_\lambda e_\mu^c) M_{ab} e^a e^b = (\partial_\lambda e_\mu^a) M_{ab} e^b$$

$$\begin{aligned} \Rightarrow \nabla_\lambda g_{\mu\nu} &= \partial_\lambda g_{\mu\nu} - \underbrace{M_{ab} ((\partial_\lambda e_\mu^a) e^b + (e^a \partial_\lambda e_\mu^b))}_{\partial_\lambda (e_\mu^a e^b)} \\ &= -M_{ab} \omega_\lambda^a (e_\mu^c e^b + e^c e_\mu^b) \\ &= -2\omega_\lambda^a (e_\mu^c e^b) \end{aligned}$$

So  $\nabla$  metric  $\Leftrightarrow \omega_\lambda^a (e_\mu^c) = 0$  automatic here since  $M$  is  $G$ -invariant, so  
 torsionless  $\Leftrightarrow \Gamma^\alpha_{\lambda\mu} = 0$  action of  $G \rightarrow \text{SST}(\mathcal{M}, M)$

### Trivial Cartan connection:

$$\begin{aligned} G &\downarrow \\ A &= \Theta_G \text{ MC 1-form on } G \\ G/H & \quad F = d\Theta_G + \frac{1}{2}[\Theta_G, \Theta_G] \\ & \quad = 0 \text{ by MC structure eq} \end{aligned}$$

If  $(G, H)$  symmetric,  $\Theta_G = \omega + e$

$$\Omega(\omega) + \frac{1}{2}[e, e] = 0$$

$$\Rightarrow \Omega(\omega) = -\frac{1}{2}[e, e]$$

$$d\omega e = 0 \quad \text{torsionless}$$

In any of our cases

$$\Omega(\omega)^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b = R^a_b$$

$$\frac{1}{2}[e, e] = e^a \wedge e^b [P_a, P_b] \sim \frac{1}{2} e^a \wedge e^b M_{ab}$$

$$\leadsto R^{ab} \sim \frac{1}{2} e^a \wedge e^b$$

$$\Rightarrow R^{ab}_{\mu\nu} = \frac{1}{2} (e_\mu^a e_\nu^b - e_\nu^a e_\mu^b)$$

$$\Rightarrow R_{\mu\nu\sigma} = \frac{1}{2} (g_{\mu\sigma} g_{\nu\rho} - g_{\nu\sigma} g_{\mu\rho})$$

In a non-trivial Cartan geometry

$$\hat{F} = \Omega(\omega) + \frac{1}{2}[e, e]_h$$

$$T = d\omega e + \frac{1}{2}[e, e]_m \text{ if sym}$$

$$\text{If } T=0, \hat{F}^{ab} = R^{ab} - R^{ab}_{\text{triv}}$$

Bianchi:  $d_A F = 0$

$$\begin{aligned} d_A F &= dF_h + [A, F] \\ &= dF_h + dT_m + [\omega, F]_h + [e, F]_m + [\omega, T]_m + [e, T]_h \\ &= d\omega F_h + d\omega T_m + [e, F]_m + [e, T]_h \end{aligned}$$

$$\begin{aligned} \Rightarrow 0 &= d\omega F_h + [e, T]_h \\ &= d\omega \Omega + \frac{1}{2} d\omega [e, e]_h + [e, d\omega e]_h + \frac{1}{2} [e, [e, e]_m]_h \\ &= 0? \end{aligned}$$

$$\begin{aligned} 0 &= d\omega T_m + [e, F]_h + [e, T]_m \\ &= d\omega e + d\omega [e, e]_m + [e, \omega]_m + [e, \frac{1}{2}[e, e]_h]_m + [e, d\omega e]_m \\ & \quad + [e, [e, e]_m]_m \end{aligned}$$

$0$  by Bianchi

So in sym case,  $d_A F = 0$

Olay Bianchi  $+ [e, e, e]_{\text{in}}$

So in sym case,  $dsF=0$

$$\Leftrightarrow \begin{cases} d\omega \wedge \Omega = 0 \\ d^2 e = -[e, \Omega] \end{cases} \quad \begin{array}{l} \text{Bianchi for } \omega \\ \text{Integrability?} \end{array}$$

Palatini Action G/A a (symmetric) spacetime model

Consider the  $\Lambda^4 T$ -valued 4-form  $\alpha = \Omega^4(M) \wedge \Lambda^4 T$

$$\alpha = \frac{1}{26} (e^a \wedge e^b \wedge R - \frac{\Lambda}{6} e^a \wedge e^b \wedge e^c \wedge e^d)$$

*$\wedge$  acts on both  $\mathbb{R}^4$  indices a,b... and  $M$  indices  $\mu, \nu, \dots$*

If  $*$  is the Hodge operator on  $(\mathbb{R}^4, \eta) \rightsquigarrow$  on  $T = P_x \mathbb{R}^4$ ,  
 $*x \in \Omega^4(M)$   
 and we can define

$$\begin{aligned} & \int_M (e^a \wedge e^b \wedge R) = \int_M (e^a \wedge e^b \wedge R_{cd}) (dx^c \wedge dx^d \wedge e^1 \wedge e^2 \wedge e^3 \wedge e^4) \\ & = \int_M (e^a \wedge e^b \wedge R_{cd}) (\underbrace{dx^c \wedge dx^d}_{\wedge \text{ of } \mathbb{R}^4} \wedge \underbrace{e^1 \wedge e^2 \wedge e^3 \wedge e^4}_{\wedge \text{ of } \mathbb{R}^4(M)}) \\ & = \int_M (e^a \wedge e^b \wedge e^c \wedge e^d) (R_{ab} \wedge R_{cd} \wedge e^1 \wedge e^2 \wedge e^3 \wedge e^4) \\ & \quad \wedge \text{ of } \mathbb{R}^4(M) \quad \wedge \text{ of } \mathbb{R}^4 \end{aligned}$$

$$\begin{aligned} \text{Spal}[\omega, e] &:= \int_M * \alpha \\ &= \frac{1}{26} \int_M *(e^a \wedge e^b \wedge R - \frac{\Lambda}{6} e^a \wedge e^b \wedge e^c \wedge e^d) \\ &= \frac{1}{26} \int_M (e^a \wedge e^b \wedge R^{cd} - \frac{\Lambda}{6} e^a \wedge e^b \wedge e^c \wedge e^d) \varepsilon_{abcd} \end{aligned}$$

Equs are  $d\omega(e) = 0 \Leftrightarrow d\omega$  for  $e$  isomorphism  
 $e \wedge R - \frac{\Lambda}{3} e \wedge e \wedge e = 0 \Leftrightarrow \text{Einstein?}$

We now seek to write Spal more "naturally" in terms of  $A$ .

Recall  $\Lambda^2 \mathbb{R}^{3,1} \cong \mathfrak{so}(3,1)$  via  $\eta$

The hodge star on the left is an endo of this space, denote it  $\otimes$ . We find  $\varepsilon_{abcd} \lambda^{ab} = (-2i \otimes \lambda)_{cd}$

$$\begin{aligned} * \alpha &= -\frac{1}{6} \text{tr} (e^a \wedge e^b \otimes R - \frac{\Lambda}{6} e^a \wedge e^b \otimes (e^c \wedge e^d)) \\ & \quad \uparrow \text{matrix trace in fundamental of } \mathfrak{so}(3,1) \\ &= -\frac{1}{6} \text{tr} (\underbrace{e^a \wedge e^b}_{\frac{3}{\Lambda} (R - \hat{F})} \otimes \underbrace{(R - \frac{\Lambda}{6} e^c \wedge e^d)}_{\frac{1}{2} (R + \hat{F})}) \quad \begin{array}{l} \hat{F} = R - \frac{\Lambda}{6} e^c \wedge e^d \\ \Rightarrow \frac{\Lambda}{6} e^c \wedge e^d = \frac{1}{2} (R - \hat{F}) \end{array} \end{aligned}$$

$$\begin{aligned} &= -\frac{3}{26\Lambda} \text{tr} ((R - \hat{F}) \wedge \otimes (R + \hat{F})) \\ &= -\frac{3}{26\Lambda} \text{tr} (R \wedge \otimes R - \hat{F} \wedge \otimes \hat{F} + \underbrace{R \wedge \otimes \hat{F} - \hat{F} \wedge \otimes R}_{\text{Claim: topological term} = 0}) \end{aligned}$$

$$\sim \frac{3}{26\Lambda} \text{tr} (\hat{F} \wedge \otimes \hat{F})$$