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Recap

Saw last week: A Cartan geometry modelled on <u>reductive</u> (G,H)

$$P \xrightarrow{\pi} M, \quad A \in \Omega^1(P;\mathfrak{g})$$

gives rise to an Ehresmann connection $\omega \in \Omega^1(P;\mathfrak{h})$

frame field $e \in \Omega^1(P;\mathfrak{g}/\mathfrak{h})$

$$\begin{tikzcd} & & \mathfrak{h} \\ TP \arrow[r, "A"] \arrow[urr, bend left, "\omega"] \arrow[drr, bend right, "e"'] & \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \arrow[ur] \arrow[dr] & \\ & & \mathfrak{m} \cong \mathfrak{g}/\mathfrak{h} \end{tikzcd} \qquad A = \omega + e$$

We also get a split of the curvature

$$\begin{tikzcd} & & \mathfrak{h} \\ \Lambda^2 TP \arrow[r, "F"] \arrow[urr, bend left, "\hat{F}"] \arrow[drr, bend right, "T"'] & \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \arrow[ur] \arrow[dr] & \\ & & \mathfrak{m} \cong \mathfrak{g}/\mathfrak{h} \end{tikzcd} \qquad F = \hat{F} + T$$

and we found

$$\hat{F}[A] = \Omega[\omega] + \tfrac{1}{2}[e,e]_{\mathfrak{h}}$$

$$T = d_\omega e + \cancel{\tfrac{1}{2}[e,e]_{\mathfrak{m}}}$$

0 in symm case

Saw that $\{H\text{-inv}^t \text{ tensors on } \mathfrak{g}/\mathfrak{h}\} \cong \{G\text{-inv}^t \text{ tensor fields on } G/H\}$

Can we something similar for Cartan geometries?

$$\begin{tikzcd} TM \arrow[rr, "\cong"] \arrow[dr] & & P \times_H \mathfrak{g}/\mathfrak{h} \arrow[dl] \\ & M & \end{tikzcd}$$

We're particularly interested in the following models

$AdS_4 \cong SO(3,2)/SO(3,1) \qquad \mathfrak{g} = \mathfrak{so}(4,1)$

$\mathbb{R}^{3,1} \cong ISO(3,1)/SO(3,1) \qquad \mathfrak{g} = \mathfrak{iso}(4,1)$

$dS_4 \cong SO(4,1)/SO(3,1) \qquad \mathfrak{g} = \mathfrak{so}(3,2)$

$\boxed{\mathfrak{h} = \mathfrak{so}(3,1)}$ in any case

Can parametrise in unified way

Always reductive, symmetric $\longrightarrow \begin{cases} \hat{F} = \Omega[\omega] + \tfrac{1}{2}[e,e]_{\mathfrak{h}} \\ T = d_\omega e \end{cases}$ for G/H Cartan geoms

Always get a decomp $\mathfrak{g} = \mathfrak{so}(3,1) \oplus \mathfrak{p}$

$\mathfrak{p} \cong \mathfrak{g}/\mathfrak{so}(3,1) \cong \mathbb{R}^{3,1}$

Only $[\mathfrak{p},\mathfrak{p}]$ differs in the 3 models

$\Rightarrow$ equip $\mathfrak{p}$ with η, which is $\mathfrak{so}(3,1)$-equiv^t

$\Rightarrow$ Get metric g on G/H, which is G-equiv^t

The Fake Tangent Bundle

$P \to M$ a Cartan geom w/ model (G,H)

$$\begin{tikzcd} TM \arrow[rr, "e", "\sim"'] \arrow[dr] & & P \times_H \mathfrak{g}/\mathfrak{h} \arrow[dl] \\ & M & \end{tikzcd}$$

$P \times_H \mathfrak{g}$

$\cancel{X_p \longmapsto [p, A(X_p)] = [p\cdot h, Ad(h^{-1})(A(X_p))]}$

$\cancel{= [p\cdot h, (R_h)^* A(X_p)]}$

$\cancel{TP \cong P \times_H \mathfrak{g}}$

$\cancel{= [p\cdot h, A(R_h^* X_p)] = [p\cdot h, A(X_{p\cdot h})]}$

~~if X is right-inv^t~~

$$M \cong P \times_H \mathfrak{g}/\mathfrak{h}$$

~~if X is right-inv~~

Proof $e \in \Omega^1_G(P; \mathfrak{g}/\mathfrak{h}) \cong \Omega^1(M; P\times_H \mathfrak{g}/\mathfrak{h})$

$\Rightarrow e$ is a bundle map $TM \to P\times_H \mathfrak{g}/\mathfrak{h}$

(3) Iso on fibres. $\square$

(1) $(R_h^* e)(\xi) = e(R_{h*}X)$

$\xi \in \mathfrak{X}(P)$
$$= \overline{A(R_{h*}X)} = \overline{R_h^* A(X)} = \overline{\mathrm{Ad}(h^{-1})\cdot A(X)} = h^{-1}\cdot \overline{A(X)} \quad \text{by def of } H \circlearrowright \mathfrak{g}/\mathfrak{h}$$
$$= h^{-1}\cdot e(X)$$

$\Rightarrow e$ is H-equivariant

For $X \in \mathfrak{h}$, $e(\xi_X) = \overline{A(\xi_X)} = \overline{X} = 0 \in \mathfrak{g}/\mathfrak{h}$

so e is horizontal

We call $T := P\times_H \mathfrak{g}/\mathfrak{h}$ the fake tangent bundle

(2) $\xi \in \mathfrak{X}(M)$
$$e(\xi_m) = [s_\alpha(m), e(s_{\alpha *}\xi_m)]$$
$$= [s_\beta(m)\, g_{\alpha\beta}(m)^{-1}, e(s_{\alpha *}\xi_m)]$$
$$= [s_\beta(m), g_{\alpha\beta}(m)^{-1}\cdot e(s_{\alpha *}\xi_m)]$$
$$= [s_\beta(m), R^*_{g_{\alpha\beta}(m)}\cdot e(s_{\alpha *}\xi_m)]$$
$$= [s_\beta(m), e((R_{g_{\alpha\beta}(m)}\circ s_\alpha)_*\xi_m)]$$
$$= [s_\beta(m), e((s_\beta)_*\xi_m)]$$

↳ Not dependent on choice of local triv.

From this, we find that any H-inv^t tensor on $\mathfrak{g}/\mathfrak{h}$ induces a tensor field on M!

eg $\overline{X} \in \mathfrak{g}/\mathfrak{h} \mapsto \xi_{\overline{X}} \in \mathfrak{X}(M)$
$$\xi_{\overline{X}} = e^{-1}(\overline{X}) \quad (= \pi_* A^{-1}(X))$$

IP η on $\mathfrak{g}/\mathfrak{h} \mapsto$ metric g on M:
$$g(\xi_1, \xi_2) = \eta(e(\xi_1), e(\xi_2))$$

Or in components, $\boxed{g_{\mu\nu} = \eta_{ab}\, e^a{}_\mu e^b{}_\nu}$

Choose (orthonormal) basis P_a for $\mathfrak{g}/\mathfrak{h}$ and coords x^μ for M

(3) $e(\xi_m) = 0 \Leftrightarrow e(s_{\alpha *}\xi_m) = 0$
$\Leftrightarrow \overline{A(s_{\alpha *}\xi_m)} = 0$
$\Leftrightarrow A(s_{\alpha *}\xi_m) \in \mathfrak{h}$
$\Leftrightarrow s_{\alpha *}\xi_m$ vertical
$\Leftrightarrow \xi_m = 0$

Write any elt of $\mathfrak{g}/\mathfrak{h}$ as $\overline{X}$ for $X \in \mathfrak{g}$

Then $e(A^{-1}(\xi_X)) = \overline{X}$ so e surjective.

Now in the reductive case consider the covariant derivative wrt Ehresmann connection $\omega \in \Omega^1(P; \mathfrak{g}/\mathfrak{h})$

$$d_\omega B := dB + \rho_*(\omega)\cdot B \quad \text{for } B \in \Omega^k_G(P; \mathfrak{g}/\mathfrak{h}) \cong \Omega^k(M; T)$$

$X \in \mathfrak{X}(M)$ $\quad D_X B_\alpha := dB_\alpha + \rho_*(\omega_\alpha)\cdot B_\alpha$

$\rho: H \to GL(\mathfrak{g}/\mathfrak{h})$, $\rho_*: \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$

$\omega_\alpha := s_\alpha^* \omega$, $B_\alpha := s_\alpha^* B$ for s_α local sections

In comps $\boxed{D_\mu B^a := \partial_\mu B^a + \omega_\mu{}^a{}_b B^b}$

For $\xi \in \mathfrak{X}(M)$, $e(\xi) \in \Gamma(T \to M) \cong \Omega^0_G(P; \mathfrak{g}/\mathfrak{h})$

$$(\underbrace{\iota_{\xi_X} dB}_{\mathcal{L}_{\xi_X} B} + X\cdot B)(\xi_1, \dots, \xi_{k-1})$$
$$\mathcal{L}_{\xi_X} B + X\cdot B$$

$\nabla \xi := e^{-1} d_\omega e(\xi)$ i.e. $\nabla_X \xi = e^{-1} D_X e(\xi)$

defines a Koszul connection on TM.

$e^{-1}\circ e = \mathrm{Id}_{TM}$ $\quad \Leftrightarrow (e^{-1})^\mu{}_a e^a{}_\nu = \delta^\mu_\nu$, $\quad (e^{-1})^\mu{}_a = e^\mu{}_a$

$e\circ e^{-1} = \mathrm{Id}_{P\times_H \mathfrak{g}/\mathfrak{h}}$ $\quad e^a{}_\mu (e^{-1})^\mu{}_b = \delta^a_b$, $\quad (e^{-1})^\mu{}_b = e^\mu_b$

In comps $\nabla_\mu \xi^\nu = e^\nu{}_a D_\mu(e^a{}_\rho \xi^\rho)$
$$= e^\nu{}_a \partial_\mu(e^a{}_\rho \xi^\rho) + \omega_\mu{}^a{}_b e^b{}_\rho \xi^\rho$$
$$= \partial_\mu \xi^\nu + e^\nu{}_a(\partial_\mu e^a{}_\rho + \omega_\mu{}^a{}_b e^b{}_\rho)\xi^\rho$$

or $\boxed{\nabla_\mu \xi^\nu = \partial_\mu \xi^\nu + \Gamma^\nu{}_{\mu\rho}\xi^\rho}$

$$\Gamma^\nu{}_{\mu\rho} := e^\nu{}_a(\delta^a_b \partial_\mu + \omega_\mu{}^a{}_b)e^b{}_\rho = e^\nu{}_a D_\mu e^a{}_\rho$$

$d_\omega e = 0 \Leftrightarrow D_{[\mu} e^a{}_{\nu]} = 0 \Leftrightarrow \Gamma^\lambda{}_{[\mu\nu]} = 0$
$\Leftrightarrow \partial_{[\mu} e^a{}_{\nu]} + \omega_{[\mu}{}^a{}_{|b|} e^b{}_{\nu]} = 0$

$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$

$\Rightarrow T^\lambda{}_{\mu\nu} = 2\Gamma^\lambda{}_{[\mu\nu]}$

$(\nabla_Z g)(X,Y) = \nabla_Z(g(X,Y)) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y)$
$= Z(X,Y) -$

$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\kappa{}_{\lambda\mu} g_{\kappa\nu} - \Gamma^\kappa{}_{\lambda\nu} g_{\mu\kappa}$

$$= Z(X,Y) -$$

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\alpha_{\lambda\mu} g_{\alpha\nu} - \Gamma^\alpha_{\lambda\nu} g_{\mu\alpha}$$

$$\begin{aligned}\Gamma^\alpha_{\lambda\mu} g_{\alpha\nu} &= (e^\alpha_c D_\lambda e^c_\mu)\eta_{ab} e^a_\alpha e^b_\nu \\ &= (D_\lambda e^a_\mu)\eta_{ab} e^b_\nu\end{aligned}$$

so $\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \eta_{ab}\big(\underbrace{(D_\lambda e^a_\mu)e^b_\nu + (D_\lambda e^a_\nu)e^b_\mu}_{D_\lambda(e^a_\mu e^b_\nu)}\big)$

$$D_\lambda(e^a_\mu e^b_\nu) = \partial_\lambda(e^a_\mu e^b_\nu) + \omega_\lambda{}^a{}_c e^c_\mu e^b_\nu + \omega_\lambda{}^b{}_c e^a_\mu e^c_\nu$$

$$\begin{aligned}&= -\eta_{ab}\omega_\lambda{}^a{}_c(e^c_\mu e^b_\nu + e^c_\nu e^b_\mu) \\ &= -2\omega_{\lambda(bc)} e^b_\mu e^c_\nu\end{aligned}$$

so ∇ metric $\Leftrightarrow \omega_{\lambda(bc)} = 0$ automatic here since η is G-invariant, so action of $G \to SO(\mathfrak{g}/\mathfrak{h}, \eta)$

∇ torsionless $\Leftrightarrow \Gamma^\lambda_{[\mu\nu]} = 0$

Trivial Cartan connection:

$G \downarrow G/H$

$A = \Theta_G$ MC 1-form on G

$F = d\Theta_G + \frac{1}{2}[\Theta_G, \Theta_G]$
$= 0$ by MC structure eq

If (G,H) symmetric, $\Theta_G = \omega + e$

$$\Omega(\omega) + \tfrac{1}{2}[e,e] = 0$$

$$\Rightarrow \Omega(\omega) = -\tfrac{1}{2}[e,e]$$

$$d_\omega e = 0$$ Torsionless

In any of our cases,

$$\Omega(\omega)^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_d = R^a{}_b$$

$$\tfrac{1}{2}[e,e] = e^a \wedge e^b [P_a, P_b]$$
$$\sim \tfrac{\Lambda}{3} e^a \wedge e^b M_{ab}$$

$$\rightsquigarrow \quad R^{ab} \sim \tfrac{\Lambda}{3} e^a \wedge e^b$$

$$\Rightarrow R^{ab}{}_{\mu\nu} = \frac{\Lambda}{3}(e^a_\mu e^b_\nu - e^a_\nu e^b_\mu)$$

$$\Rightarrow R_{\mu\nu\rho\sigma} = \frac{\Lambda}{3}(g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma})$$

In a nontrivial Cartan geometry,

$$\hat{F} = \Omega(\omega) + \tfrac{1}{2}[e,e]_{\mathfrak{h}}$$

$$T = d_\omega e + \cancel{\tfrac{1}{2}[e,e]_{\mathfrak{m}}}$$ 0 if Sym

If $T = 0$, $\hat{F}^{ab} = R^{ab} - R^{ab}_{\text{triv}}$.

Bianchi: $d_A F = 0$

$$\begin{aligned} d_A F &= dF + [A,F] \\ &= d\hat{F} + dT + [\omega, \hat{F}] + [e, \hat{F}] + [\omega, T] + [e,T] \\ &= \underset{\mathfrak{h}}{d_\omega \hat{F}} + \underset{\mathfrak{m}}{d_\omega T} + \underset{\mathfrak{m}}{[e,\hat{F}]} + \underset{\mathfrak{g}\ \mathfrak{h}}{[e,T]} \end{aligned}$$

$$\begin{aligned}\Rightarrow 0 &= d_\omega \hat{F} + [e,T]_{\mathfrak{h}} \\ &= d_\omega \Omega + \underbrace{\tfrac{1}{2} d_\omega [e,e]_{\mathfrak{h}} + [e, d_\omega e]_{\mathfrak{h}}}_{=0?} + \cancel{\tfrac{1}{2}[e,[e,e]_{\mathfrak{m}}]_{\mathfrak{h}}}\end{aligned}$$

$$\begin{aligned} 0 &= d_\omega T + [e,\hat{F}] + \cancel{[e,T]_{\mathfrak{m}}} \\ &= d_\omega^2 e + \cancel{d_\omega [e,e]_{\mathfrak{m}}} + [e, \Omega] + \underbrace{[e, \tfrac{1}{2}[e,e]_{\cancel{\mathfrak{h}}}]_{\mathfrak{m}}}_{0 \text{ by Bianchi}} + \cancel{[e, d_\omega e]_{\mathfrak{m}}} \\ &\quad + \cancel{[e,[e,e]_{\mathfrak{m}}]_{\mathfrak{m}}} \end{aligned}$$

So in sym case, $d_A F = 0$

$+ \cancel{[e,[e,e]_{\mathfrak{m}}]_{\mathfrak{m}}}$

0 by Bianchi

So in sym case, $d_A F = 0$

$$\Leftrightarrow \begin{cases} d_\omega \Omega = 0 & \text{Bianchi for } \omega \\ d_\omega^2 e = -[e, \Omega] & \text{Integrability?} \end{cases}$$

<u>Palatini Action</u> G/H a (symmetric) spacetime model

Consider the $\Lambda^4 T$-valued 4-form $\alpha = \Omega^4(M; \Lambda^4 T)$

$$\alpha = \frac{1}{2G}\left(e \wedge e \wedge R - \frac{\Lambda}{6} e \wedge e \wedge e \wedge e\right)$$

$\wedge$ acts on both $\mathfrak{g}/\mathfrak{h}$ indices $a, b \ldots$ and M indices $\mu, \nu, \ldots$

ie $(e \wedge e \wedge R)$

$$= (e_\mu \wedge e_\nu \wedge R_{\rho\sigma})(dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma)$$

$\wedge$ of $\mathfrak{g}/\mathfrak{h}$ $\qquad$ $\wedge$ of $\Omega^1(M)$

$$= (e^a \wedge e^b \wedge e^c \wedge e^d)(P_a \wedge P_d \wedge P_c \wedge P_d)$$

$\wedge$ of $\Omega^1(M)$ $\qquad$ $\wedge$ of $\mathfrak{g}/\mathfrak{h}$

If $*$ is the Hodge operator on $(\mathfrak{g}/\mathfrak{h}, \eta) \rightsquigarrow$ on $\mathcal{T} = P \times_H \mathfrak{g}/\mathfrak{h}$,
$*\alpha \in \Omega^4(M)$
and we can define

$$S_{pal}[\omega, e] := \int_M *\alpha$$

$$= \frac{1}{2G}\int_M *\left(e \wedge e \wedge R - \frac{\Lambda}{6} e \wedge e \wedge e \wedge e\right)$$

$$= \frac{1}{2G}\int_M \left(e^a \wedge e^b \wedge R^{cd} - \frac{\Lambda}{6} e^a \wedge e^b \wedge e^c \wedge e^d\right)\varepsilon_{abcd}$$

Eqns are $\quad d_\omega(e \wedge e) = 0 \quad \Leftrightarrow d_\omega e$ for e isomorphism

$$e \wedge R - \frac{\Lambda}{3} e \wedge e \wedge e = 0 \quad \Leftrightarrow \text{Einstein?}$$

We now seek to write S_{pal} more "naturally" in terms of A.

Recall $\Lambda^2 \mathbb{R}^{3,1} \cong \mathfrak{so}(3,1)$ via η
The Hodge star on the left is an endo of this space, denote it $\circledast$. We find $\varepsilon_{abcd}\lambda^{cd} = (-2 i \circledast \lambda)_{ab}$

$$*\alpha = -\frac{1}{G}\operatorname{tr}\left(e \wedge e \wedge \circledast R - \frac{\Lambda}{6} e \wedge e \wedge \circledast(e \wedge e)\right)$$

matrix trace in fundamental of $\mathfrak{so}(3,1)$

$$= -\frac{1}{G}\operatorname{tr}\Big(\underbrace{e \wedge e}_{\frac{3}{\Lambda}(R - \hat{F})} \wedge \circledast \underbrace{\left(R - \frac{\Lambda}{6} e \wedge e\right)}_{\frac{1}{2}(R + \hat{F})}\Big)$$

$$\hat{F} = R - \frac{\Lambda}{3} e \wedge e$$

$$\Rightarrow \frac{\Lambda}{6} e \wedge e = \frac{1}{2}(R - \hat{F})$$

$$= -\frac{3}{2G\Lambda}\operatorname{tr}\left((R - \hat{F}) \wedge \circledast(R + \hat{F})\right)$$

$$= -\frac{3}{2G\Lambda}\operatorname{tr}\Big(\underbrace{R \wedge \circledast R}_{\text{Chern: topological term}} - \hat{F} \wedge \circledast \hat{F} + \underbrace{R \wedge \circledast \hat{F} - \hat{F} \wedge \circledast R}_{=0?}\Big)$$

$$\sim \frac{3}{2G\Lambda}\operatorname{tr}(\hat{F} \wedge \circledast \hat{F})$$