Recap

Saw last week that

- Klein geometry (G,H) reductive
- Cartan geometry $\begin{matrix} P \\ \downarrow \pi \\ M \end{matrix}$, $A \in \Omega^1(P;\mathfrak{g})$ modelled on (G,H)

$P \times_H G$

gives rise to an Ehresmann connection $\omega \in \Omega^1(P;\mathfrak{h})$
frame field $e \in \Omega^1(P;\mathfrak{g}/\mathfrak{h})$

$$A = \omega + e$$

The curvature $F[A]$ also splits into components

$$F = \hat{F} + T$$

$\hat{F} \uparrow \mathfrak{h}$, $T \uparrow \mathfrak{m} \cong \mathfrak{g}/\mathfrak{h}$

$$\hat{F} = \Omega[\omega] + \tfrac{1}{2}[e,e]_{\mathfrak{h}}$$
$$T = d_\omega e + \tfrac{1}{2}[e,e]_{\mathfrak{m}}$$

$\tfrac{1}{2}[e,e]_{\mathfrak{m}} \to 0$ symmetric case

$\alpha \in \Omega^p_H(P;V)$ ← V rep of H
$\cong \Omega^p(M; P \times_H V)$

H-Invariant horizontal forms
$h \cdot \alpha = R_{h^{-1}}^* \alpha$
$\iota_{\ker \pi_*} \alpha = 0$

$d_\omega \alpha = d\alpha + \omega \cdot \alpha$

The Fake Tangent Bundle

Claim:

$$\begin{tikzcd} TM \arrow[rr, "e", "\sim"'] \arrow[dr] & & P \times_H \mathfrak{g}/\mathfrak{h} \arrow[dl] \\ & M & \end{tikzcd}$$

Proof $e \in \Omega^1_H(P;\mathfrak{g}/\mathfrak{h}) \cong \Omega^1(M; P \times_H \mathfrak{g}/\mathfrak{h})$
$= \{\text{bdle maps } TM \to P \times_H \mathfrak{g}/\mathfrak{h}\}$

e is H-invariant ✓
e is horizontal:
For $X \in \mathfrak{h}$, get vertical vector field ξ_X on P
$e(\xi_X) = \overline{A(\xi_X)} = \overline{X} = 0 \in \mathfrak{g}/\mathfrak{h}$ ✓

Further, $e_x : T_xM \to [P \times_H \mathfrak{g}/\mathfrak{h}]_x$, $x \in M$
is a linear isomorphism.
Follows from $A_p : T_pP \to \mathfrak{g}$ is a linear isomorphism. □

We call $\mathcal{T} := P \times_H \mathfrak{g}/\mathfrak{h}$ the <u>fake tangent bundle</u>

Just as H-invariant tensors on $\mathfrak{g}/\mathfrak{h}$ give us G-inv. tensor fields on G/H, they also give us tensor fields on M in the Cartan geometry.

eg. $\overline{X} \in \mathfrak{g}/\mathfrak{h} \longmapsto \xi_{\overline{X}} \in \mathfrak{X}(M)$
$X \in \mathfrak{g}$
$\xi_{\overline{X}} := e^{-1}([p, \overline{X}])$ ← H-invariance needed for well-definition.
$= \pi_* A^{-1}(X)$

$\overline{X} = \overline{Y}$
$X - Y \in \mathfrak{h}$
$A^{-1}(X-Y)$ vertical
$\Rightarrow \pi_* A^{-1}(X-Y) = 0$
$\Rightarrow \pi_* A^{-1}(X) = \pi_* A^{-1}(Y)$

H-invariant inner product η on $\mathfrak{g}/\mathfrak{h}$
$\longmapsto$ metric g on M:

$$g(\xi_1, \xi_2) = \eta(e(\xi_1), e(\xi_2))$$

$e(\xi_1) \in \mathcal{T}$

Or in components $\quad g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$

Choose an orthonormal basis $\{P_a\}$ for $\mathfrak{g}/\mathfrak{h}$ and coordinates x^μ on M

In the reductive case, consider the covariant derivative w.r.t. Ehresmann connection $\omega \in \Omega^1(P;\mathfrak{h})$

Local trivialising sections of P $\quad s_\alpha : U_\alpha \to \pi^{-1} U_\alpha$

$B \in \Omega^k_H(P;\mathfrak{g}/\mathfrak{h})$
$\cong \Omega^k(M;\mathcal{T})$
$B_\alpha := s_\alpha^* B$

$$d_\omega B_\alpha = dB_\alpha + \rho_*(\omega_\alpha) \cdot B_\alpha$$

$\rho : H \to GL(\mathfrak{g}/\mathfrak{h})$
$\rho_* : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$

In components

$$D_\mu B^a_{\mu_1 \cdots \mu_k} := \partial_\mu B^a_{\mu_1 \cdots \mu_k} + \omega_\mu{}^a{}_b B^b_{\mu_1 \cdots \mu_k}$$
$$(d_\omega B_\alpha^a)_{\mu\mu_1 \cdots \mu_k} := D_{[\mu} B^a_{\mu_1 \cdots \mu_k]}$$

$H \circlearrowleft (\mathfrak{g}/\mathfrak{h}, \eta)$ preserving η

so $\rho : H \to O(\mathfrak{g}/\mathfrak{h}, \eta)$
$\rho_* : \mathfrak{h} \to \mathfrak{so}(\mathfrak{g}/\mathfrak{h}, \eta)$

For $\xi \in \mathfrak{X}(M)$, $e(\xi) \in \Gamma(\mathcal{T}) \cong \Omega^0_H(P;\mathfrak{g}/\mathfrak{h})$
$= C^\infty_H(P;\mathfrak{g}/\mathfrak{h})$

Define a Koszul connection on TM by

$$\nabla \xi := \underbrace{e^{-1} \underbrace{d_\omega(e(\xi))}_{\Omega^1(M;\mathcal{T})}}_{\Omega^1(M;TM)}$$

$X \in \mathfrak{X}(M)$

$$\nabla_X \xi = e^{-1} D_X(e(\xi)) \qquad D_X B = (d_\omega B)(X)$$

In components $\quad \nabla_\mu \xi^\nu = e^\nu_a D_\mu (e^a_\rho \xi^\rho)$
$= \partial_\mu \xi^\nu + \Gamma^\nu{}_{\mu\rho} \xi^\rho$

$\Gamma^\nu{}_{\mu\rho} = e^\nu_a D_\mu e^a_\rho$
$= e^\nu_a (\delta^a_b \partial_\mu + \omega_\mu{}^a{}_b) e^b_\rho$

Note $\quad \nabla$ metric-compatible $\Leftrightarrow \omega_{\lambda(bc)} = 0$ – automatic since η is H-invariant so ω takes values in $\mathfrak{so}(\mathfrak{g}/\mathfrak{h}, \eta)$

∇ torsionless $\Leftrightarrow \Gamma^\lambda{}_{[\mu\nu]} = 0$

The Spacetime Models

$AdS_4 \cong SO(3,2)/SO(3,1) \qquad \mathfrak{g} = \mathfrak{so}(3,2)$

$\mathbb{R}^{3,1} \cong ISO(3,1)/SO(3,1) \qquad \mathfrak{g} = \mathfrak{iso}(4,1)$

$dS_4 \cong SO(4,1)/SO(3,1) \qquad \mathfrak{g} = \mathfrak{so}(4,1)$