R. D'Auria, Geometric Supergravity
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"In 1978 Y. Ne'eman and T. Regge proposed a new approach to the formulation of supergravity, based on the formalism of E. Cartan. As the adopted formalism relies on the use of differential forms, it is invariant under the group of general coordinate transformations (GCTG), and, at the same time, it gives a prominent role to the gauge invariance under the Lorentz group $H = SO(1,3)$."

Cartan connection on bundle P of oriented orthonormal frames of Lorentzian 4-mnfd (M,g)

$\pi : P \xrightarrow{H} M$

$$\mathcal{N} = \omega + V \in \Omega^1(P,\mathfrak{g}) \quad \text{where} \quad \underbrace{\mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{m} = so(1,3) \ltimes \mathbb{R}^{1,3}}_{\text{Poincaré algebra}}$$

$$\omega \in \Omega^1(P,\mathfrak{h})$$

$$V \in \Omega^1(P,\mathfrak{m})$$

$$\{\mathcal{N}^A\} = \{\omega^{ab}, V^a\} \quad 1 \le A \le 10,\ 0 \le a,b \le 3$$

<u>FACT1:</u> The curvature $F = d\mathcal{N} + \frac{1}{2}[\mathcal{N},\mathcal{N}]$ is horizontal and equivariant, i.e., $F \in \Omega^2_H(P,\mathfrak{g})$.

Pf. Take fundamental vector field A^* associated to $A \in \mathfrak{h}$, then

$$R^*_{\exp(tA)}\mathcal{N} = \mathrm{Ad}_{\exp(-tA)} \circ \mathcal{N}$$

by property 1 of Cartan connections, so deriving at $t=0$ we get $\mathcal{L}_{A^*}\mathcal{N} = -\mathrm{ad}_A \circ \mathcal{N}$

and then $-\mathrm{ad}_A \circ \mathcal{N} = d\underbrace{i_{A^*}\mathcal{N}}_{=A \text{ by property 2 of Cartan connections}} + i_{A^*} d\mathcal{N} = i_{A^*} d\mathcal{N}$. Now compute

$$(i_{A^*}F)(Y) = d\mathcal{N}(A^*,Y) + [\mathcal{N}(A^*),\mathcal{N}(Y)] = -\mathrm{ad}_A(\mathcal{N}(Y)) + [A,\mathcal{N}(Y)] = 0,$$

proving that F is horizontal. Equivariancy only uses property 1 and is left as exercise. ∎

Remember that $F = \underbrace{R}_{\text{curvature of EC } \omega} + T$ ⟶ torsion of EC ω, where $R = d\omega + \frac{1}{2}[\omega,\omega]$ and $T = \underbrace{dV + \omega \wedge V}_{d^\omega V}$.

<u>FACT 2:</u> The Bianchi Identities $d^\omega T = R \wedge V$, $d^\omega R = 0$ hold.

Pf. Let us define $d^{\mathcal{N}} = d + \mathcal{N}\wedge$ and take for granted for now that $d^{\mathcal{N}}F = 0$. Then

$$0 = dF + \mathcal{N} \wedge F = dR + dT + \omega \wedge R + \omega \wedge T + V \wedge R + \cancel{V \wedge T}$$

$= 0$ since $[\mathfrak{m},\mathfrak{m}] = 0$ in Poincaré algebra

$$= \underbrace{dR + \omega \wedge R}_{d^\omega R} + \underbrace{dT + \omega \wedge T + V \wedge R}_{d^\omega T + V \wedge R}$$

and the two terms vanish separately. To prove $d^{\mathcal{N}}F = 0$, we proceed as follows:

$$\begin{aligned} d^{\mathcal{N}}F &= dF + \mathcal{N} \wedge F = dF + [\mathcal{N},F] = d\left(d\mathcal{N} + \tfrac{1}{2}[\mathcal{N},\mathcal{N}]\right) + [\mathcal{N},F] \\ &= \tfrac{1}{2}[d\mathcal{N},\mathcal{N}] - \tfrac{1}{2}[\mathcal{N},d\mathcal{N}] + [\mathcal{N},F] = [d\mathcal{N},\mathcal{N}] + \underbrace{[\tfrac{1}{2}[\mathcal{N},\mathcal{N}],\mathcal{N}]}_{=0 \text{ by Jacobi Identity in Lie superalgebra } \Omega^\bullet(P,\mathfrak{g})} + [\mathcal{N},F] \\ &= [F,\mathcal{N}] + [\mathcal{N},F] = 0 \end{aligned}$$

Note that we didn't use properties 1 and 2 of Cartan connections anywhere..... ∎

Remember that μ satisfies properties 1 and 2, so we may fixing any local trivializing section of P, and safely interpret ω and V as (locally defined, vector valued) 1-forms on M: $V^a = V^a_\mu \, dx^\mu$ and $\omega^{ab} = \omega^{ab}{}_\mu \, dx^\mu$. I think D'Auria calls this the factorization hypothesis: "...which breaks from the very beginning the symmetry of the group, since it factorizes in a trivial way the dependence of the gauge fields ω, V from the coordinates y^{μ} of the Lorentz group." He then moves on to say: "if we do not assume factorization, the dependence on the coordinates $y^{\mu\nu}$ must be dictated by the field equations....It is then natural to try to construct gravity directly on the soft group manifold....This implies that ω and V will depend not only on x^μ but also on $y^{\mu\nu}$."
Finally he says: ".....the Lorentz invariance can then be retrieved from the equations of motion as a result of the action principle, even if the D-form Lagrangian has to be integrated on a soft group manifold whose dimensionality $\dim \mathcal{G}$ is bigger than D.
We will see that the integration of a D-form as a submanifold can be consistently performed as a result of its invariance under the GCTG, and the resulting equations of motion give horizontality of the curvatures in the Lorentz directions, leading to the factorization of the $y^{\mu\nu}$."

Main message: on a soft group manifold, Lorentz factorization is not assumed a priori but it is obtained from the field equations.

My understanding: a soft group manifold is a principal bundle but μ is relaxed so that it does not satisfy properties 1 and 2 from scratch....(Note that we can still use Bianchi Identities!)

We will then consider also N=1 pure supergravity in 4-dimensions. In this case:

$$\{\mu^A\} = \{\omega^{ab}, V^a, \psi^\alpha\} \quad 1 \le A \le 14, \; 0 \le a,b \le 3, \; 1 \le \alpha \le 4$$

$$\omega \in \Omega^1(P, \mathfrak{so}(1,3)), \; V \in \Omega^1(P, \mathbb{R}^{1,3}) \text{ and } \psi \in \Omega^1(P, S)$$

$\downarrow$ 4-dimensional module of Majorana spinors

Coordinates on P are now $\underbrace{x^\mu, y^{\mu\nu}}_{\text{Bosonic coordinates}}$ and $\underbrace{\theta^\alpha}_{\text{Fermionic coordinates}}$. "In this case the field equations give Lorentz factorization (i.e, horizontality of the supercurvatures in the Lorentz directions), leaving us on (4|4)-dimensional superspace, but they also give constraints on the curvatures in the fermionic directions, which allow to restrict the theory to M only. It is this property, dubbed rheonomy, which allows the interpretation of supersymmetry transformations on space-time as superspace diffeomorphisms."

Review of Palatini formalism

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \qquad\qquad d^\omega R^{ab} = 0$$

$$T^a = dV^a + \omega^a{}_b \wedge V^b \qquad\qquad d^\omega T^a - R^a{}_b \wedge V^b = 0$$

(In the paper, the exterior covariant derivative d^ω is denoted by D and called the Lorentz covariant derivative.) The Lagrangian in the Palatini formalism, with internal indices in the fake tangent bundle Υ written explicitly is:

$$\mathcal{L} = \underset{\substack{\text{Hodge star}\\ \text{operator on}\\ \Lambda^0\Upsilon}}{\star}\left(R \wedge V \wedge V\right) = \underbrace{R^{ab} \wedge V^c \wedge V^d\, \varepsilon_{abcd}}_{\substack{\text{4-form on } M \text{ with values}\\ \text{in } \star\Lambda^4\Upsilon = \Lambda^0\Upsilon}},$$

where $R \in \Omega^2(M; so(\Upsilon)) \cong \Omega^2(M; \Lambda^2\Upsilon)$ and $V \in \Omega^1(M; \Lambda^1\Upsilon) \Longrightarrow$ the action is $\mathcal{A} = \int_M R^{ab} \wedge V^c \wedge V^d\, \varepsilon_{abcd}$. Since the Lagrangian $\mathcal{L}$ is built only in in terms of differential forms and wedge products, we automatically have the invariance under GCTG (general coordinate transformations group). It is a first-order formalism, i.e. the fields ω and V are independent; varying them yields:

$$T \wedge V = 0 \text{ as element of } \Omega^3(M; \Lambda^2\Upsilon) \to \text{zero torsion}$$

$$R \wedge V = 0 \text{ as element of } \Omega^3(M; \Lambda^3\Upsilon) \to \text{Einstein eqs.}$$

Quick Mathematical Detour

$\pi : P \xrightarrow{H} M$ principal bundle

$$0 \to \underbrace{\underbrace{\mathfrak{X}(P)^H_{\text{vert}}}_{\substack{\text{Infinitesimal}\\ \text{gauge transf.}}}} \to \underbrace{\mathfrak{X}(P)^H}_{\substack{\text{Infinitesimal}\\ \text{automorphisms}\\ \text{of } P}} \to \underbrace{\mathfrak{X}(M)}_{\substack{\text{vector fields}\\ \text{on } M}} \to 0$$

Short exact sequence of $C^\infty(M)$-modules (Use that $C^\infty(M) \cong C^\infty(P)^H$ to define the module structure for first two spaces.)

Section of $C^\infty(M)$-modules of this exact sequence is an EC on P. It is the horizontal lift of vector fields on M!

Since ν is absolute parallelism, any v.f. $\varepsilon \in \mathfrak{X}(P)$ may be uniquely written as $\varepsilon = \varepsilon^A T_A$, where T_A are "constant" v.f. on P for any $1 \leq A \leq 10$ (i.e. $\nu(T_A)$ is a constant element of Poincaré algebra $\mathfrak{g}$). So equivalently $\varepsilon \in C^\infty(P; \mathfrak{g})$ and it can be easily seen that $\varepsilon \in \mathfrak{X}(P)^H$ iff it corresponds to an element of $C^\infty_H(P; \mathfrak{g}) \cong C^\infty(M; P \times_H \mathfrak{g})$. Fixing a local trivializing section of P, we have a locally defined function $\varepsilon = \{\varepsilon^A\} = \{\varepsilon^{ab}, \varepsilon^a\}$ on M with values in $\mathfrak{g} = h \ltimes m$.

Def. (D'Auria, page 6)

The infinitesimal gauge variations associated with $\varepsilon : M \supseteq \mathcal{U} \to \mathfrak{p}$ are $\delta_\varepsilon \mathcal{N}^A := (d^{\mathcal{N}} \varepsilon)^A$,
where $d^{\mathcal{N}} = d + \mathcal{N}\wedge$ is the Poincaré exterior covariant derivative. (D'Auria uses the notation ∇.)
Explicitly:

$$\delta_\varepsilon \omega^{ab} = d^\omega \varepsilon^{ab} \qquad \in \Omega^1(\mathcal{U}; \mathfrak{h})$$

$$\delta_\varepsilon V^a = d^\omega \varepsilon^a + \varepsilon^{ab} V_b \qquad \in \Omega^1(\mathcal{U}; \mathfrak{m})$$

where ε^{ab} are parameters of infinitesimal Lorentz gauge transformations and ε^a of infinitesimal translation gauge transformations.

Fact 3: the Palatini action is invariant only w.r.t. infinitesimal Lorentz gauge transformations. Indeed performing an infinitesimal translation $\varepsilon = \{\varepsilon^a\}$ on $\mathcal{A}$, using Bianchi Identities and integrating by parts yields

$$\delta_\varepsilon \int_M R^{ab} V^c V^d \varepsilon_{abcd} = 2 \int_M R^{ab} \, d^{\mathcal{N}} \varepsilon^c \, V^d \varepsilon_{abcd} = -2 \int_M \varepsilon^c R^{ab} T^d \varepsilon_{abcd} \neq 0$$

Extending the theory from M to P

Now $\{\mu^A\} = \{\omega^{ab}, V^a\}$ live on P so they depend both on coordinates x^μ and $y^{\mu\nu}$. I think axioms 1 and 2 of a Cartan connection are relaxed and we then speak of "soft group manifold". D'Auria notices that we cannot integrate

$$\mathcal{L} = \underbrace{R^{ab} \wedge V^c \wedge V^d \, \epsilon_{abcd}}_{\text{4-form on } P}$$

on P, since the latter is 10-dimensional. We may not project to M either, since we dot not have horizontality and equivariancy to rely on. D'Auria states:

> "A simple way out would be to embed M as a 4-dimensional hypersurface of P. However, new fields would enter in the action, corresponding to the embedding functions. The crucial observation is that one can safely ignore the variation of the embedding, since any such variation can be compensated by a change of coordinates $\{x^\mu, y^{\mu\nu}\}$ of P, under which $\mathcal{L}$, built only in terms of differential forms and wedge products among them, is invariant. (The presence of the Hodge operator on P, instead, would not allow for invariance under GCTG of P.)"

So we may write the action still as $\mathcal{A} = \int_{M \subseteq P} R^{ab} \wedge V^c \wedge V^d \epsilon_{abcd}$. The equation of motion are

$$T \wedge V = 0 \text{ as element of } \Lambda^3(P; \Lambda^2_m)$$
$$R \wedge V = 0 \text{ as element of } \Lambda^3(P; \Lambda^3_m)$$

and we need to expand R and T not only in terms of $V^a \wedge V^b$ but also of $\omega^{ab} \wedge V^c$ and $\omega^{ab} \wedge \omega^{cd}$. The projection along $V^a \wedge V^b$ leads to the same equations considered before. On the other hand, expanding along the other 2-forms yields horizontality of R and T. In D'Auria words:

> "the soft group manifold has acquired dynamically the structure of a fiber bundle.........and factorization of the Lorentz coordinates and the Lorentz invariance of the Lagrangian (see below) are a result of the field equations."

Gauge transformations reloaded (D'Auria page 11)

Now $\varepsilon = \varepsilon^A T_A \in \mathfrak{X}(P) \cong C^\infty(P;\mathfrak{g})$ so that

$$\mathcal{L}_\varepsilon \mathcal{N}^A = (i_\varepsilon d + d i_\varepsilon)\mathcal{N}^A = i_\varepsilon d\mathcal{N}^A + d\varepsilon^A$$

$$= i_\varepsilon\left(d\mathcal{N}^A + \frac{1}{2}\underbrace{C^A{}_{BC}}_{\substack{\text{Structure}\\ \text{constants}\\ \text{of } \mathfrak{g}}}\mathcal{N}^B \wedge \mathcal{N}^C\right) - \varepsilon^B C^A{}_{BC}\mathcal{N}^C + d\varepsilon^A$$

$$= i_\varepsilon F^A + d^{\mathcal{N}}\varepsilon^A$$

where we used skew-symmetry of C^A_{BC} in the lower indices. Hence, if $\varepsilon = \{\varepsilon^{ab}\}$ and the curvature is horizontal, then $\mathcal{L}_\varepsilon \mathcal{N}^A = d^{\mathcal{N}}\varepsilon^A = \delta_\varepsilon \mathcal{N}^A$, i.e., the Lie derivative along ε and the infinitesimal Lorentz gauge transformation coincide.