N=1 4-dimensional simple supergravity

"We will show that the invariance of the supergravity Lagrangian is due to the behavior of the curvatures along the fermionic components. Now the dependence of the fields on odd coordinates θ^α does not imply factorization, rather one finds that fermionic components of the curvature can be expressed linearly in terms of bosonic components, namely in terms of the $V^a \wedge V^b$. Inserting this result into the Lie derivative formula $\mathcal{L}_\varepsilon \mathcal{N}^A = i_\varepsilon F^A + d^{\mathcal{N}} \varepsilon^A$, we obtain a geometrical interpretation of local supersymmetry transformations."

$\pi: P \xrightarrow{H} M$ — Still $SO(1,3)$ w/t coordinates $\{y^\mu\}$

(4|4)-dimensional superspace w/t coordinates $\{x^\mu, \theta^\alpha\}$. Its underlying bosonic mnfd is our Lorentzian 4-mnfd M_0

$\mathcal{N} = \omega + \Psi + V \in \Omega^1(P, \mathfrak{g})$ where now

$\mathfrak{g} = so(1,3) \oplus S \oplus \mathbb{R}^{1,3}$ is Poincaré superalgebra

$\{\mathcal{N}^A\} = \{\omega^{ab}, \Psi^\alpha, V^a\}$ $\quad 1 \le A \le 14 \quad 0 \le a, b \le 3, \; 1 \le \alpha \le 4$

Brackets in $\mathfrak{g}$:

$[so(1,3), so(1,3)] \subseteq so(1,3)$ obvious commutator

$[so(1,3), S] \subseteq S$ spin representation

$[so(1,3), \mathbb{R}^{1,3}] \subseteq \mathbb{R}^{1,3}$ defining representation

$[S, S] \subseteq V$ Dirac current (i.e. $[e_\alpha, e_\beta] = (\bar{e}_\alpha \gamma^a e_\beta) e_a$ in a basis of S and V) — gamma matrices

(Note that D'Auria writes the Poincaré superalgebra as $\overline{osp(1|4)}$, since it is a IW contraction of the orthosymplectic Lie superalgebra. Recall that $sp_4(\mathbb{R}) \cong so(2,3)$.)

Let us decompose the curvature $F = d\mathcal{N} + \frac{1}{2}[\mathcal{N}, \mathcal{N}]$ into $so(1,3)$, V and S components:

$$F = R + \hat{T} + \rho \; .$$

explicitly given by (here and in the following we will mostly omit the spinor index α and sometimes omit the wedge product as well)

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$

$$\hat{T}^a = \underbrace{dV^a + \omega^a{}_b \wedge V^b}_{T^a} - \frac{i}{2} \bar{\Psi} \gamma^a \Psi$$

$$\rho = \underbrace{d\Psi + \frac{1}{4} \omega^{ab} \gamma_{ab} \Psi}_{d^\omega \Psi}$$

The curvatures satisfy the Bianchi identities $d^{\mathcal{N}} F = 0$, which expands into

$$d^\omega R^{ab} = 0$$

$$d^\omega \hat{T}^a - R^a{}_b V^b - i \bar{\Psi} \gamma^a \rho = 0$$

$$d^\omega \rho - \frac{1}{4} R^{ab} \gamma_{ab} \Psi = 0$$

D'Auria writes: "In order to write down the Lagrangian we require that it is geometric. For the sake of clarity let us repeat here what it amounts to:

- It must be constructed using only differential forms $\mathcal{N}$ and F, wedge products among them, and the exterior differential d;
- It must not contain the Hodge duality operator;
- As the Einstein term, which must be always present, scales as [L^2], all the terms must scale in the same way;
- If all the curvatures are zero, the equations of motion must vanish identically."

Thus we obtain (omitting wedge products):

$$\mathcal{L} = \underbrace{R^{ab} V^c V^d \varepsilon_{abcd}}_{\text{Classical Palatini action}} + \lambda \underbrace{\overline{\Psi} \gamma^5 \gamma_a \rho V^a}_{\text{Rarita-Schwinger Kinetic term}}$$

$$\mathcal{A} = \int_{M \subseteq P} \mathcal{L}$$

where the coefficient $\lambda \in \mathbb{R}$ will be fixed in a moment. The equations of motion obtained by varying ω^{ab}, V^a, Ψ^α are:

$$\varepsilon_{abcd} \left(T^a + \frac{i}{8} \lambda \overline{\Psi} \gamma^a \Psi \right) \wedge V^d = 0 \qquad (1')$$

$$2R^{ab} \wedge V^c \varepsilon_{abcd} - \lambda \overline{\Psi} \gamma^5 \gamma_d \rho = 0 \qquad (2)$$

$$2\gamma^5 \gamma_a \rho \wedge V^a - \gamma^5 \gamma_a \Psi \wedge \hat{T}^a = 0 \qquad (3)$$

As the equations of motion have to vanish identically when all curvatures are zero, we see that we must set $\lambda = -4$ in the left hand side of $(1')$, which then becomes

$$\varepsilon_{abcd} \hat{T}^a \wedge V^d = 0 \qquad (1)$$

Next time, we will expand (1)-(3) along the absolute parallelism $\mathcal{N}$ and see what we get....

Solving the equations of motion

To analyze the content of (1)-(3), we expect the curvatures along the $\mathcal{N}^A \wedge \mathcal{N}^B$:

- Along $\omega^{ab}\omega^{cd}$, $\omega^{ab}V^c$, $\omega^{ab}\Psi$ we get horizontality of F, i.e. of $R, \hat{T}, \rho$. (D'Auria calls this fact the factorization of the Lorentz coordinates.)

- Let us introduce the following notation. We denote by $F^A_{(p,q)}$ the components of the curvature along p bosonic vielbein V^a and q fermionic vielbein Ψ^α. Moreover we call "outer" all the components s.t. $q \neq 0$ while when $q=0$, they are called "inner".

$$R^{ab} = R^{ab}_{(2,0)} + R^{ab}_{(1,1)} + \cancel{R^{ab}_{(0,2)}}$$

Here we will see an instance of rheonomy!

$$\hat{T}^a = \cancel{\hat{T}^a_{(2,0)}} + \cancel{\hat{T}^a_{(1,1)}} + \cancel{\hat{T}^a_{(0,2)}}$$

$$\rho^\alpha = \rho^\alpha_{(2,0)} + \cancel{\rho^\alpha_{(1,1)}} + \cancel{\rho^\alpha_{(0,2)}}$$

- Equation (1) tells us that $\hat{T}^a_{(1,1)} = \hat{T}^a_{(0,2)} = 0$, so we are left with $\hat{T}^a_{(2,0)}$, which satisfies an equation of the same form as in the classical case $\Rightarrow \hat{T}^a_{(2,0)} = 0$. (In this case solving for the spin connection with the usual procedure tells us that ω depends not only from V and its first derivatives, but also from gravitino Ψ.)

- Since the torsion is zero, equation (3) takes the form $\gamma_a \rho \wedge V^a = 0$. This implies $\rho^\alpha_{(1,1)} = \rho^\alpha_{(0,2)} = 0$.

$\Downarrow$

This component comprises the geometric data defining off-shell Killing spinor eqs: scalar, vector and pseudo scalar auxiliary fields.

- Equation (2) tells us that $R^{ab}_{(0,2)} = 0$ but $R^{ab}_{(1,1)} = \bar{\Theta}^{ab}_c \Psi \wedge V^c$ with $\bar{\Theta}^{ab}_c \in C^\infty(P; S^*)$ that is given by:

$$\bar{\Theta}^{ab}_c = -\epsilon^{abrs}\bar{\rho}_{rs}\gamma^5\gamma_c - \delta^{[a}_c \epsilon^{b]mst}\bar{\rho}_{st}\gamma^5\gamma_m$$

Note that the outer component $R^{ab}_{(1,1)}$ of Lorentz field strenght is written linearly, on-shell, in terms of the inner component of gravitino field strenght. (This is an example of rheonomy. Physically, it means that no new degrees of freedom are introduced in the theory other than those already present on space-time $\mathcal{M}$.)

In conclusion, we get the constraints

$$\left.\begin{aligned}
&R^{ab} = R^{ab}_{(2,0)} + R^{ab}_{(1,1)} = R^{ab}_{cd}\, V^c \wedge V^d + \bar{\Theta}^{ab}_c\, \psi \wedge V^c \\
&\hat{T}^a = 0 \\
&\rho = \rho_{(2,0)} = \rho_{ab}\, V^a \wedge V^b
\end{aligned}\right\} \quad (*)$$

with $\bar{\Theta}^{ab}_c$ given as above by rheonomy. Inserting these parametrizations in (2) and (3) and looking at $(3,0)$-components, we get supergravity Einstein and Rarita Schwinger eqs:

$$R^{am}_{bm} - \frac{1}{2}\delta^a_b R^{mn}_{mn} = 0$$

$$\varepsilon^{mnrs}\gamma^5\gamma_s\,\rho_{mn} = 0$$

Be aware that the space-time components of the Lorentz curvature expanded along the differentials of the coordinates, namely $R^{ab}_{\mu\nu}$, do not coincide with the components along $V^c \wedge V^d$. Indeed writing $R^{ab}_{cd}\, V^c_\mu \wedge V^d_\nu = R^{ab}_{\mu\nu} - \bar{\Theta}^{ab}_c\, \psi_\mu \wedge V^c_\nu$, we see that the Einstein equations comprise an energy-momentum tensor of the gravitino field.

Supersymmetry invariance of the Lagrangian

The supersymmetry variation of the Lagrangian is expressed by the Lie derivative of the Lagrangian along $\varepsilon = \varepsilon^{\alpha} T_{\alpha} \in \mathfrak{X}(P)$:

$$\mathcal{L}_{\varepsilon}\, \mathcal{L} = d i_{\varepsilon} \mathcal{L} + i_{\varepsilon} d\mathcal{L} \cong i_{\varepsilon} d\mathcal{L} \,,$$

where we have discarded the total derivative term and other possible exact 4-forms on the R.H.S. as we are assuming that the fields vanish at infinite. A computation using Bianchi Identities and Fierz rearrangements leads to

$$d\mathcal{L} = R^{ab}\, \hat{T}^{c}\, V^{d}\, \epsilon_{abcd} + \bar{\rho}\, \gamma^{5} \gamma_{a}\, \rho\, V^{a} - 4\, \bar{\psi}\, \gamma^{5} \gamma_{a}\, \rho\, \hat{T}^{a}$$

and contracting by ε we get

$$i_{\varepsilon} d\mathcal{L} \cong 2\left(i_{\varepsilon} R^{ab}\right) \hat{T}^{c} V^{d} \epsilon_{abcd} + 2R^{ab} \cancel{\left(i_{\varepsilon} \hat{T}^{c}\right)} V^{d} \epsilon_{abcd} + 8\, \cancel{\left(i_{\varepsilon} \bar{\rho}\right)} \gamma^{5} \gamma_{a}\, \rho\, V^{a}$$
$$- 4\, \bar{\varepsilon}\, \gamma^{5} \gamma_{a}\, \rho\, \hat{T}^{a} - 4\, \bar{\psi}\, \gamma^{5} \gamma_{a} \cancel{\left(i_{\varepsilon} \rho\right)} \hat{T}^{a} - 4\, \bar{\psi}\, \gamma^{5} \gamma_{a}\, \rho \cancel{\left(i_{\varepsilon} \hat{T}^{a}\right)}$$

If we require the following constraints on the components of the curvatures

$$2\left(i_{\varepsilon} R^{ab}\right) V^{d} \epsilon_{abcd} + 4\, \bar{\varepsilon}\, \gamma^{5} \gamma_{c}\, \rho = 0$$

$$i_{\varepsilon} \hat{T}^{a} = 0$$

$$i_{\varepsilon} \rho = 0$$

we get invariance of the Lagrangian under supersymmetry. In particular, one can check that the constraints $(*)$ are enough to get invariance.

We conclude that the Lagrangian is invariant under supersymmetry transformations when the curvatures are restricted as in $(*)$. However, this restriction implies that supersymmetry transformations preserving the Lagrangian do not form a closed algebra, unless one uses the equations of motion.

Explicit form of supersymmetry transformations

Let us apply our formula $\mathcal{L}_\varepsilon \nu^A = d^N \varepsilon^A + i_\varepsilon \mathcal{F}^A$ with $\varepsilon = \varepsilon^A T_A \in \mathfrak{X}(P)$ under the constraints $(*)$:

$$\mathcal{L}_\varepsilon \omega^{ab} = d^N \varepsilon^{ab} + \varepsilon^r V^s R^{ab}_{\ \ rs} + \varepsilon^r \overline{\Theta}^{ab}_r \Psi + \overline{\Theta}^{ab}_r \varepsilon V^r$$

The spinorial index α has been suppressed.

$$\mathcal{L}_\varepsilon V^a = d^N \varepsilon^a$$

$$\mathcal{L}_\varepsilon \Psi^\alpha = d^N \varepsilon^\alpha + \varepsilon^r \rho^\alpha_{rs} V^s$$

where

$$d^N \varepsilon^{ab} = d^\omega \varepsilon^{ab}$$

$$d^N \varepsilon^a = d^\omega \varepsilon^a + \varepsilon^{ab} V_b - i \overline{\Psi} \gamma^a \varepsilon$$

index α again omitted

$$d^N \varepsilon^\alpha = d^\omega \varepsilon^\alpha - \frac{1}{4} \varepsilon^{ab} \gamma_{ab} \Psi$$

In particular, specializing to the case of a supersymmetry transformation (i.e., we set $\varepsilon^{ab} = \varepsilon^a = 0$) yields

$$\mathcal{L}_\varepsilon \omega^{ab} = d^N \varepsilon^{ab} + \overline{\Theta}^{ab}_r \varepsilon V^r = \overline{\Theta}^{ab}_r \varepsilon V^r$$

$$\mathcal{L}_\varepsilon V^a = d^N \varepsilon^a = -i \overline{\Psi} \gamma^a \varepsilon$$

$$\mathcal{L}_\varepsilon \Psi^\alpha = d^N \varepsilon^\alpha = d^\omega \varepsilon^\alpha$$

Rem 1. Let's recall the completely general fact that $[T_B, T_C] = (\underbrace{C^A_{BC}}_{\text{Structure constants}} - F^A_{BC})\, T_A$, which holds also for Cartan connections not satisfying axioms 1-2.

In our case, we also have the constraints (*) in force and the algebra of supersymmetry transformations leaving the Lagrangian invariant can only close on-shell.

Rem 2. D'Auria says that the presence of a non-vanishing component along Ψ in the on-shell value of the curvature is sufficient to exclude factorization of the fermionic coordinates $\{\theta^\alpha\}$ and that this is what makes supersymmetry transformations act as infinitesimal diffeomorphisms instead of gauge transformations.

A.S.: Should we treat also the infinitesimal translations as Lie derivatives $\mathcal{L}_{\varepsilon^a T_a}$ instead of gauge transformations $d^N \varepsilon^a$?

Rem 3. Consider rheonomy in full generality, i.e.,

$$F^A_{\alpha C} = \underbrace{C^{A\,ab}_{\alpha CB}}_{\substack{\text{Proportionality} \\ \text{Superfunctions}}} F^B_{ab}$$

and insert it in our formula $\mathcal{L}_\varepsilon N^A = d^N \varepsilon^A + i_\varepsilon F^A$ with $\varepsilon = \varepsilon^\alpha T_\alpha$ to get

$$\mathcal{L}_\varepsilon N^A = d^N \varepsilon^A + \varepsilon^\alpha C^{A\,ab}_{\alpha CB} F^B_{ab} N^C$$

In particular the supersymmetry transformation depends only on the $F^A_{(2,0)}$.

Rem 4. At the end, I am inclined to say that the definition of soft group manifold used in this paper is the following: manifold P with an absolute parallelism $N : TP \longrightarrow \mathfrak{g}$ taking values in a Lie algebra $\mathfrak{g}$.