String Theory 2007

Tutorial Sheet 1

Problems for lecture 1: Introduction and lightcone gauge

The following list of problems is intended to make the course attendants familiar with some basic notions of classical string propagation.

Problem 1.1. Consider the propagation of a bosonic string in D dimensions.

a. Prove that the Polyakov action

$$S = -\frac{T}{2}\int d^2\sigma\, \sqrt{\gamma}\,\gamma^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\,\eta_{\mu\nu}$$

describing such propagation in Minkowski spacetime is classically equivalent to the Nambu-Goto action

$$S_{NG} = -T\int d^2\sigma\,\sqrt{-\det\left(\partial_\alpha X^\mu\,\partial_\beta X^\nu\,\eta_{\mu\nu}\right)}\,.$$

b. Prove that the Polyakov action is invariant under the transformations:

$$\delta X^\mu = \left(\mathscr{L}_\xi X\right)^\mu + a^\mu + \omega^{\mu\nu}\,\eta_{\nu\rho}\,X^\rho\,,$$
$$\delta\gamma^{\alpha\beta} = \Delta(\sigma)\,\gamma^{\alpha\beta} + \left(\mathscr{L}_\xi\gamma\right)^{\alpha\beta}\,,$$

where $\mathscr{L}_\xi$ stands for the Lie derivative along an arbitrary worldsheet vector field ξ, and $\omega^{\mu\nu} = -\omega^{\mu\nu}$.

c. Prove that the Polyakov action

$$S = -\frac{T}{2}\int d^2\sigma\,\sqrt{\gamma}\,\gamma^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\,g_{\mu\nu}(X)$$

in an arbitrary spacetime with metric $g_{\mu\nu}(X)$ is invariant under the isometries of the metric $g_{\mu\nu}(X)$; i.e., under the transformations

$$\delta X^\mu = k^\mu(X) \quad \text{where} \quad \mathscr{L}_k g = 0\,.$$

Problem 1.2. Consider the gauge-fixed worldsheet action :

$$S = -\frac{T}{2}\int d^2\sigma\,\left(\partial_\alpha X^\mu\,\partial^\alpha X_\mu - i\overline{\psi}^\mu\,\rho^\alpha\partial_\alpha\psi_\mu\right)\,,$$

where ψ^μ are a set of two-dimensional Majorana fermions transforming in the vector representation of $SO(1, D-1)$ satisfying $\overline{\psi}^\mu\chi_\mu = \rho^0_{AB}\psi^\mu_A\chi_{B\,\mu} = \overline{\chi}^\mu\,\psi_\mu$ and where

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

is a set of two-dimensional imaginary Dirac matrices.

a. Derive the Euler–Lagrange equations of motion for the fields $X^\mu(\sigma)$ and $\psi^\mu(\sigma)$ and discuss the allowed boundary conditions for closed and open strings under which the latter hold.

b. Prove that S is invariant under the (worldsheet) supersymmetry transformations:

$$\delta_\epsilon X^\mu = \bar{\epsilon}\,\psi^\mu\,, \quad \delta_\epsilon \psi^\mu = -i\,\rho^\alpha \partial_\alpha X^\mu\,\epsilon\,,$$

and use Noether's Theorem to construct its associated conserved current J_α.

c. Construct the Noether currents associated with the Poincaré symmetries

$$\delta X^\mu = \omega^\mu{}_\nu X^\nu + a^\mu\,, \quad \delta\psi^\mu = \omega^\mu{}_\nu\,\psi^\nu\,,$$

and the energy-momentum tensor $T_{\alpha\beta}$ from the invariance of the action under constant worldsheet translations $\delta\sigma^\alpha = \xi^\alpha$.

d. Rewrite all previous currents in worldsheet lightlike coordinates $\sigma^\pm = \tau \pm \sigma$.