String Theory 2007

Tutorial Sheet 2

Spectrum of strings propagating in Minkowski spacetime

The following problems deal with several aspects of the lightcone quantisation of strings propagating in Minkowski spacetime. No problem is hard, but some might be computationally involved. Those which are, are adorned thus: .

Problem 2.1 In this problem you will derive the value of the critical dimension of bosonic strings propagating in Minkowski spacetime by ensuring that the Hilbert space of the quantum theory carries a representation of the Lorentz algebra.
The Lorentz generators for the open bosonic string are given by

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i \sum_{n=1}^{\infty} \tfrac{1}{n} \left(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n\right)$$

whereas for the closed string there is an extra term with the $\tilde{\alpha}$ oscillators. Notice that there are no ordering ambiguities in the above expressions because of the skewsymmetry in $\mu \leftrightarrow \nu$. For open strings in the lightcone gauge, $\alpha^+_n = 0$ for all $n \neq 0$ and the constraints allow us to solve for α^-_n as follows:

$$\alpha^-_n = \frac{1}{2\alpha^+_0} \left(\sum_{m=1}^{\infty} :\alpha^\perp_{n-m} \cdot \alpha^\perp_m: - \delta_{n0}\, a \right) ,$$

with a similar expression for $\tilde{\alpha}^-_n$ for the closed string.

a. Define $L^\perp_n := \alpha^+_0 \alpha^-_n$ and similarly for $\tilde{L}^\perp_n$ and show that

$$[L^\perp_m, L^\perp_n] = (m-n)L^\perp_{m+n} + \left(\tfrac{D-2}{12}(m^3 - m) + 2am\right)\delta_{m+n,0} ,$$

and similarly for $\tilde{L}^\perp_n$.

b. Compute the commutator $[J^{-i}, J^{-j}]$ and show that it vanishes (as it should) provided that $a = 1$ and $D = 26$. To do this argue that there are three possible types of terms: quartic, quadratic and of zeroth order in oscillators. The quartic terms vanish because these are impervious to the ordering ambiguities whereas the zeroth order terms have to vanish by $SO(D-2)$-covariance: there are no (nonzero) $SO(V)$-invariants in $\Lambda^2 V$ for $\dim V > 2$. This leaves the quadratic terms, which you should find to be proportional to

$$\sum_{m=1}^{\infty} \Delta_m \left(\alpha^i_{-m}\alpha^j_m - \alpha^j_{-m}\alpha^i_m \right) ,$$

where

$$\Delta_m = m\frac{26-D}{12} + \left(\tfrac{D-26}{12} + 2(1-a)\right) .$$

Problem 2.2 Work out the spacetime field content of the first couple of massive levels in the open and closed bosonic strings propagating in Minkowski spacetime.

Problem 2.3 The strings we have been discussing in lecture are *oriented*: the orientation is given by the direction of increasing σ. Consider the orientation-reversal map taking $\sigma \mapsto \pi - \sigma$, which takes the interval $[0,\pi]$ to itself but reverses the orientation. This map is realised in the quantum Hilbert space of the string by an operator Ω. Since the orientation reversal map is involutive, we have that $\Omega^2 = 1$, whence the Hilbert space will decompose into two eigenspaces of Ω with eigenvalues ± 1. Keeping only the $+1$-eigenspace defines the *unoriented* strings.

a. Work out the effect of the orientation reversal on the modes of open and closed strings; that is, show that for open strings $\Omega\alpha_n = (-1)^n \alpha_n \Omega$, whereas for closed strings $\Omega\alpha_n = \tilde{\alpha}_n \Omega$.
(*Hint*: See what happens to $X(\sigma,\tau)$ under orientation reversal.)

b. Defining the ground states to have eigenvalue 1 with respect to Ω, work out the first few levels of the spectrum of the unoriented open and closed strings. Which massless fields remain?

Problem 2.4 Determine the first massive level of the (GSO-projected) type I superstring. Do the same for the (GSO-projected) types IIA/B superstring in Minkowski spacetime in terms of spacetime fields. Show that there is no difference at this level between types IIA and IIB. Prove that this persists to all massive levels.

Problem 2.5 In this problem you are asked to compute the partition function for open and closed, oriented and unoriented strings. Let d_n denote the number[1] of states in the spectrum of the string theory in question at level n, for $n = 0, 1, \ldots$ and consider the partition function

$$Z(q) = \sum_{n=1}^{\infty} d_n q^n ,$$

where q is a formal variable or, if you insist, you can take q to be a complex number for which the above sum converges.

a. Show that $Z_{oo}(q) = \operatorname{Tr} q^N$ for oriented open strings and $Z_{oc}(q) = \operatorname{Tr} q^{N+\tilde{N}}$ for oriented closed strings, where

$$N = \sum_{n=1}^{\infty} \alpha^{\perp}_{-n} \cdot \alpha^{\perp}_{n} \quad \text{and} \quad \tilde{N} = \sum_{n=1}^{\infty} \tilde{\alpha}^{\perp}_{-n} \cdot \tilde{\alpha}^{\perp}_{n} ,$$

and the traces are taken over the relevant Hilbert spaces.

b. For unoriented strings, show that $Z_{uo}(q) = \frac{1}{2} \operatorname{Tr}(1+\Omega) q^N$ for open strings and $Z_{uc}(q) = \frac{1}{2} \operatorname{Tr}(1+\Omega) q^{N+\tilde{N}}$ for closed strings.

c. Derive infinite-product expressions for each of these partition functions.

d. Do the same for the partition functions of the (oriented, GSO-projected) types I, IIA and IIB superstrings.

[1] Strictly speaking this is the number of states at a fixed value of the centre-of-mass momentum in the corresponding mass-shell.