String Theory 2007

Tutorial Sheet 3

Density of states and effective actions for massless modes

The following list of problems is intended to make the course attendants familiar with the notions of dimensional regularisation and renormalisation in non-linear sigma models, asymptotic density of states for generic strings and the correspondence principle between highly excited string states and black holes.

Problem 3.1. Following [1] prove that the one-loop beta functional for the nonlinear sigma model describing a 26-dimensional bosonic string in an arbitrary curved background $g_{\mu\nu}(X)$ is given by

$$\beta_{\mu\nu}(X) = -\frac{1}{2\pi} R_{\mu\nu}(X) \,.$$

Problem 3.2. Consider a bosonic open string. We want to analyse the asymptotic behaviour of the level density for very highly excited states. Equivalently, we want to know the total number of open string states with mass $\alpha' M^2 = n-1$ for large n. Let us denote this number by d_n and introduce the generating function

$$G(\omega) = \sum_{n=0}^{\infty} d_n\, \omega^n = \mathrm{Tr}\, \omega^N \,,$$

where the trace is taking over the Hilbert space of physical states and

$$N = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \,,$$

is the number operator and where in the lightcone gauge the oscillator α_n^i only run over the 24 transverse directions.

a. Prove that

$$G(\omega) = \left[f(\omega) \right]^{-24} \quad \text{where} \quad f(\omega) = \prod_{n=1}^{\infty} (1-\omega^n) \,.$$

b. Using $\omega = e^{2\pi i \tau}$, notice that $f(\omega)$ is very closely related to the Dedekind eta function $\eta(\tau)$ defined by

$$\eta(\tau) = e^{i\pi\tau/12} \prod_{n=1}^{\infty} \left(1 - e^{2\pi i n \tau} \right) ,$$

and satisfying the modular transformation formula

$$\eta(-1/\tau) = (-i\tau)^{1/2}\, \eta(\tau) \,.$$

Derive that $f(\omega)$ satisfies

$$f(\omega) = \left(\frac{-2\pi}{\log \omega} \right)^{1/2} \omega^{-1/24}\, q^{1/2} f(q^2) \,, \quad \text{where} \quad q \equiv e^{2\pi^2/\log\omega} \,.$$

c. Derive the asymptotic formula for $\omega \to 1$ (or $q \to 0$)

$$f(\omega) \sim (1-\omega)^{-1/2}\, e^{-\frac{\pi^2}{6(1-\omega)}}\,.$$

d. The coefficients d_n can be determined by a contour integral on a small circle about the origin

$$d_n = \frac{1}{2\pi i}\int \frac{G(\omega)}{\omega^{n+1}}\, d\omega\,.$$

Evaluate this integral for large n. This can be estimated by saddle point evaluation, giving

$$d_n \sim n^{-27/4}\, e^{4\pi\sqrt{n}}\,.$$

e. Study this asymptotic behaviour for the superstring. Details can be found in [2, §5.3].

Problem 3.3. Prove that the Horowitz-Polchinski correspondence principle [3] between highly excited string states and Schwarzschild black holes works in any spatial dimension d. Write, in your own words, what the correspondence implies, and what its assumptions are.
(*Hint*: The $(d+1)$-dimensional Schwarzschild black-hole solution is given by

$$ds^2 = -V(r)dt^2 + V(r)^{-1}dr^2 + r^2\, d\Omega_{d-1}\,,$$

where $V(r) = 1-(r_0/r)^{d-2}$, and the $(d+1)$-dimensional Newton's constant G is always proportional to g_s^2.)

References

[1] L. Alvarez-Gaumé, D. Z. Freedman, and S. Mukhi, "The background field method and the ultraviolet structure of the supersymmetric nonlinear sigma model," *Ann. Phys.* **134** (1981) 85–109.

[2] M. Green, J. Schwarz, and E. Witten, *Superstring Theory*. Cambridge University Press, Cambridge, UK, 1987. 2 vols.

[3] G. T. Horowitz and J. Polchinski, "A correspondence principle for black holes and strings," *Phys. Rev.* **D55** (1997) 6189–6197, `hep-th/9612146`.