String Theory 2007

Tutorial Sheet 4

Supergravity backgrounds

The following problems deal with the geometry of supergravity backgrounds.

Problem 4.1 Let (M, g) be a riemannian spin manifold and let ∇ be the Levi–Cività connection. Let ε be a nonzero spinor field which is parallel relative to ∇; that is, $\nabla\varepsilon = 0$. Show that (M, g) is Ricci-flat. For extra credit, what can you conclude if (M, g) is lorentzian?

Problem 4.2 Let (M, g) be a six-dimensional riemannian spin manifold admitting a parallel spinor. Show that it is Kähler.
(*Hint*: One way is to construct the Kähler form explicitly from the parallel spinor. Another way is to use the holonomy principle to show that the holonomy must be contained in $SU(3) < SO(6)$ and to *understand* that this means that (M, g) is Kähler and Ricci-flat.)

Problem 4.3 Consider an r-form potential A in a D-dimensional oriented lorentzian manifold (M, g) with action given by the squared norm of the field strength $F = dA$:

$$S = -\tfrac{1}{2}\int_M F \wedge \star F \,.$$

Compute the energy-momentum tensor T_F associated to F. For which values of D and r will T_F be traceless? Show that T_F is traceless precisely when there are dyonic objects; that is, objects which can be both electrically and magnetically charged with respect to A. In which dimensions $D \leq 10$ are there dyonic objects and which types of objects are they? For extra credit, explain why should the existence of dyonic objects have anything to do with conformal invariance of the "electromagnetic" action.

Problem 4.4 Let $a > 0$ be a real parameter. Show that the lorentzian metric

$$ds^2 = a^2\frac{dr^2}{r^2} + \frac{r^2}{a^2}ds^2(\mathbb{R}^{1,p})$$

has constant negative sectional curvature. What is the scalar curvature as a function of a?

Problem 4.5 Let us consider the following ansatz for a dyonic 3-brane in $d = 10$ charged with respect to a self-dual 5-form flux:

$$\begin{aligned} g &= a(r)ds^2(\mathbb{R}^{1,3}) + b(r)\left(dr^2 + r^2 ds^2(S^5)\right) \\ F &= dA + \star dA \,, \end{aligned}$$

where $A = c(r)\,\mathrm{dvol}(\mathbb{R}^{1,3})$, where a, b, c are functions of the transverse radius and going to 1 as $r \to \infty$ in order to ensure that this background is asymptotic to the

Minkowski background. The supergravity field equations for this background become

$$dF = 0$$
$$R_{\mu\nu} = \kappa F_{\mu\rho\sigma\tau\chi} F_{\nu}{}^{\rho\sigma\tau\chi} ,$$

for some constant κ. Plugging the above ansatz into the equations, derive the ODEs which a, b, c must satisfy. Show that there is a solution where a, b, c are powers of a harmonic function $H(r) = \alpha + (\beta/r)^4$ in the transverse euclidean space, where α, β are constants. (You have to determine the powers.)